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ASYMPTOTIC EXPANSIONS PERTAINING TO THE LOGARITHMIC SERIES AND RELATED TRIGONOMETRIC SUMS

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Abstract. The partial sum of the Maclaurin series of $-\ln(1-z)$ is $f_n(z) \equiv \sum_{k=1}^{n-1} z^k/k$. We find concise closed-form expressions, involving Eulerian polynomials, for the full asymptotic expansion of $f_n(z)$ as $n \to \infty$. We then use our expressions to find large-n compound asymptotic expansions, involving real quantities only, for $c_n(\theta) \equiv \sum_{k=1}^{n-1} \cos k\theta/k$, $s_n(\theta) \equiv \sum_{k=1}^{n-1} \sin k\theta/k$, $r_n(\theta) \equiv \sum_{k=0}^{n-1} (-1)^k \cos[(2k+1)\theta]/(2k+1)$, and a number of other trigonometric sums. Many of these sums are ubiquitous in the literature on the Gibbs phenomenon in the context of Fourier series.

1. Introduction

Let $f_n(z)$ be the sum defined by

$$f_n(z) \equiv \sum_{k=1}^{n-1} \frac{z^k}{k}, \quad z \in \mathbb{C}, \quad n = 2,3,4,\ldots \tag{1}$$

and recently encountered in an investigation related to superdirective-type effects arising in certain numerical solutions of electromagnetic scattering problems [1, 16]. For the case where z is real with $z > 1$, ref. [16] deals with the full asymptotic expansion of $f_n(z)$ as $n \to \infty$. While [16] shows how to determine as many terms as one desires, the expansion therein does not have a simple and explicit form. It is the purpose of the present paper to derive such an expansion for $z \in \mathbb{C}$ (i.e., not only for $z \in (1,+\infty)$). We then derive similar expansions for a number of related trigonometric sums. Our explicit expansions involve Eulerian polynomials, which often arise in combinatorial problems.

For $|z| \leqslant 1$ and $z \neq 1$, $f_n(z)$ is the partial sum, with $n-1$ terms, of the convergent series

$$f_\infty(z) \equiv \sum_{k=1}^{\infty} \frac{z^k}{k} = -\ln(1-z), \quad |z| \leqslant 1 \text{ and } z \neq 1 \tag{2}$$

[33, §4.6.1], which is the so-called Mercator series or Newton-Mercator series or logarithmic series. In (2), and throughout this paper, $\ln$ denotes the principal value (also called principal branch) of the logarithm function. While $f_\infty(z)$ converges under the conditions stated in (2), it diverges otherwise.

The specific *first goal* of this paper (Sections 3 and 4) is to determine closed-form asymptotic expansions of the following two quantities:

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(i) of $f_n(z)$ in the divergent case $|z| > 1$; and

(ii) of the remainder $f_n(z) - f_\infty(z)$ in the convergent case. We will denote this remainder by $g_n(z)$,

$$g_n(z) \equiv f_n(z) - f_\infty(z) = f_n(z) + \ln(1-z) = -\sum_{k=n}^{\infty} \frac{z^k}{k}, \quad |z| \leqslant 1 \text{ and } z \neq 1. \tag{3}$$

It is to be expected that $f_n(z)$ is exponentially large for $|z| > 1$ and that, for $|z| < 1$, the remainder $g_n(z)$ is exponentially small. It is also logical to expect a more subtle behaviour for $|z| = 1$.

We do not consider $z = 1$ because the asymptotic expansion of the quantity $f_n(1) = \psi(n) + \gamma$ (where ψ is the psi function and γ is Euler's constant) is well known: the said expansion is due to Euler and was stated by Ramanujan in his notebooks [3, pp. 150–151, p. 182]. Moreover, the quantity $f_n(1)$ was frequently used by Ramanujan to express results related to analogues of the gamma function [3, p. 181].

It is apparent from (3) that

$$g_n(z) = -z^n \Phi(z, 1, n), \quad |z| \leqslant 1 \text{ and } z \neq 1, \tag{4}$$

where

$$\Phi(z, s, \alpha) \equiv \sum_{k=0}^{\infty} \frac{z^k}{(k+\alpha)^s} \tag{5}$$

is the Hurwitz-Lerch zeta-function or Lerch's transcendent [35, p. 27], [33, §25.14], [25, 31]. In Sections 3 and 4, we will point out further connections between f_n and Φ.

In (1), set $z = e^{i\theta}$, where $\theta \in \mathbb{R}$, and separate the real and imaginary parts to obtain the sums

$$c_n(\theta) \equiv \operatorname{Re}\left\{ f_n\left(e^{i\theta}\right)\right\} = \sum_{k=1}^{n-1} \frac{\cos k\theta}{k}, \quad n = 2, 3, 4, \dots, \tag{6}$$

$$s_n(\theta) \equiv \operatorname{Im}\left\{ f_n\left(e^{i\theta}\right)\right\} = \sum_{k=1}^{n-1} \frac{\sin k\theta}{k}, \quad n = 2, 3, 4, \dots. \tag{7}$$

In the literature [10], $s_n(\theta)$ has been called the Fejér-Jackson sum.

In (2), set $z = e^{i\theta}$ where $\theta \in (0, 2\pi)$, and separate the real and imaginary parts to show that the sums in (6) and (7) are partial sums of the convergent series

$$c_\infty(\theta) \equiv \operatorname{Re}\left\{ f_\infty\left(e^{i\theta}\right)\right\} = \sum_{k=1}^{\infty} \frac{\cos k\theta}{k} = -\ln\left(2\sin\frac{\theta}{2}\right), \quad 0 < \theta < 2\pi, \tag{8}$$

$$s_\infty(\theta) \equiv \operatorname{Im}\left\{ f_\infty\left(e^{i\theta}\right)\right\} = \sum_{k=1}^{\infty} \frac{\sin k\theta}{k} = \frac{\pi - \theta}{2}, \quad 0 < \theta < 2\pi. \tag{9}$$

The infinite summations in (8) and (9) are well known see, e.g., [18, eqs. 1.441.1, 1.441.4]; in fact, the series in (9) was summed by Euler [21].

The *second goal* of this paper is to obtain asymptotic series for the remainders $c_n(\theta) - c_\infty(\theta)$ and $s_n(\theta) - s_\infty(\theta)$. This amounts to separating the real and imaginary parts of $f_n(e^{i\theta}) - f_\infty(e^{i\theta})$, something that turns out to be a nontrivial exercise (Section 5) within the context of Eulerian polynomials.

For $\theta \in \mathbb{R}$, the sum

$$r_n(\theta) \equiv \sum_{k=0}^{n-1} (-1)^k \frac{\cos(2k+1)\theta}{2k+1}, \quad n = 1,2,3,\ldots \tag{10}$$

can be found from $s_n(\theta)$ through

$$r_n(\theta) = \frac{1}{2}\left[s_{2n}\left(\frac{\pi}{2} - \theta\right) + s_{2n}\left(\frac{\pi}{2} + \theta\right)\right].$$

Taking the limit and using (9) and $r_n(\theta) = r_n(-\theta) = -r_n(\pi - \theta)$, we see that $r_n(\theta)$ is the partial sum of the convergent series

$$\begin{aligned} r_\infty(\theta) &\equiv \sum_{k=0}^{\infty} (-1)^k \frac{\cos(2k+1)\theta}{2k+1} = \frac{1}{2}\left[s_\infty\left(\frac{\pi}{2} - \theta\right) + s_\infty\left(\frac{\pi}{2} + \theta\right)\right] \\ &= \begin{cases} \frac{\pi}{4}, & |\theta| < \frac{\pi}{2}, \\ -\frac{\pi}{4}, & \frac{\pi}{2} < |\theta| < \pi. \end{cases} \end{aligned} \tag{11}$$

Equation (11), like (8) and (9), is well known [18, eq. 1.442.4]. From the above discussions, it follows that the asymptotic expansion of $r_n(\theta) - r_\infty(\theta)$ can be found from the expansion of $s_n(\theta) - s_\infty(\theta)$.

The *third goal* of this paper (Section 6) is to provide the former expansion, and to show how to similarly determine the expansions of a number of other trigonometric sums.

The starting points of our discussions on $f_n(z)$ are two relations that can be found in [16]. First, differentiation of (1) and use of $f_n(0) = 0$ easily lead to the integral representation

$$f_n(z) = \int_0^z \frac{t^{n-1} - 1}{t - 1} dt, \quad z \in \mathbb{C}. \tag{12}$$

Our second starting point is an asymptotic relation, shown in [16], that concerns the case $z \in (1, +\infty)$:

$$f_n(z) \sim z^n \int_0^y \frac{e^{-nu}}{ze^{-u} - 1} du, \quad \text{as } n \to \infty \quad (z \in (1, +\infty)), \quad 0 < y < \ln z. \tag{13}$$

In (13), y is an arbitrary positive number that is independent of n and smaller than $\ln z$ (note that $u = \ln z$ is a simple pole of the integrand).

The quantity multiplying e^{-nu} in the integrand of (13) admits a Maclaurin expansion of the form

$$\frac{1}{ze^{-u} - 1} = \sum_{k=0}^{\infty} a_k(z) u^k. \tag{14}$$

After giving explicit expressions for the first few terms in (14), [16] applies Watson's lemma to (13). That is, (14) is substituted into (13) and then formally integrated term by term with the upper integration limit replaced by $+\infty$, thus yielding the first few terms of the asymptotic expansion of $f_n(z)$ for the case $z \in (1,+\infty)$. In Section 2, we first demonstrate how the $a_k(z)$ in (14) can be expressed in terms of Eulerian polynomials. This allows us to obtain (in Section 3) a closed-form representation of the full asymptotic expansion of $f_n(z)$ (still for the case $z \in (1,+\infty)$), and then to treat the case $z \notin [1,+\infty)$ in a similar manner.

2. Eulerian numbers, Eulerian polynomials

The Eulerian polynomials $A_k(z)$ are given by [11]

$$A_k(z) = \sum_{m=0}^{k} \left\langle {k \atop m} \right\rangle z^{k-m}, \quad k = 0,1,2,\dots, \tag{15}$$

in which $\left\langle {k \atop m} \right\rangle$ are the Eulerian numbers given by

$$\left\langle {k \atop m} \right\rangle = \sum_{p=0}^{m} (-1)^p \binom{k+1}{p} (m+1-p)^k, \quad m = 0,1,\dots,k; \quad k = 0,1,2,\dots, \tag{16}$$

where we use the usual notation for the binomial coefficients. The first few Eulerian polynomials are

$$\begin{aligned} A_0(z) &= 1, \\ A_1(z) &= z, \\ A_2(z) &= z^2 + z, \\ A_3(z) &= z^3 + 4z^2 + z, \\ A_4(z) &= z^4 + 11z^3 + 11z^2 + z, \\ A_5(z) &= z^5 + 26z^4 + 66z^3 + 26z^2 + z. \end{aligned} \tag{17}$$

Closely related to the $A_k(z)$ are what have recently been called the "Apostol-Bernoulli numbers" $\beta_1(z), \beta_2(z), \dots$ [29, 7] (in [31], they are called the "Hurwitz-Lerch Bernoulli numbers"). These quantities (which are actually rational functions of z rather than numbers) are [7, eq. (4.6)]

$$\beta_k(z) = (-1)^{k+1} k \frac{A_{k-1}(z)}{(z-1)^k}, \quad k = 1,2,\dots. \tag{18}$$

The aforementioned references further define $\beta_0(z) = 0$, but we will not use $\beta_0(z)$. The main result to be used herein is the closed-form expression for all terms of the series in (14):

$$\frac{1}{ze^{-u}-1} = \sum_{k=0}^{\infty} \frac{A_k(z)}{k!(z-1)^{k+1}} u^k = \sum_{k=0}^{\infty} (-1)^k \frac{\beta_{k+1}(z)}{(k+1)!} u^k. \tag{19}$$

While (16) can be found in the aforementioned references, it is in error in Comtet's well-known book [11], so we take the opportunity to derive (16) and correct the error in Appendix A.

We will additionally employ the following properties of the Eulerian numbers [33, §26.14]

$$\left\langle {0 \atop 0} \right\rangle = 1; \; \left\langle {k \atop k} \right\rangle = 0; \; \left\langle {k \atop 1} \right\rangle = 2^k - k - 1; \; \left\langle {k \atop m} \right\rangle = \left\langle {k \atop k-1-m} \right\rangle, \quad k = 1, 2, \ldots, \quad m = 0, 1, \ldots, k-1. \tag{20}$$

We close this section with a note on terminology and notation. We use the term "Eulerian numbers" consistently with the NIST Digital Library of Functions [33, §26.14] and the book of Graham, Knuth, and Patashnik [19, §6.2], but in a slightly different context from Comtet [11]: the "Eulerian number" $A(n,k)$ of [11] equals our $\left\langle {n \atop k-1} \right\rangle$. Nevertheless, Comtet's "Eulerian polynomial" $A_k(z)$ [11] is the same as ours. Berndt's edition of Ramanujan's notebooks [3] retains Ramanujan's original terminology and notation regarding Eulerian numbers and polynomials; specifically, the "Eulerian polynomial" $\psi_n(p)$ of [3, p. 109, p. 116] is the polynomial in p of degree $n-1$ given by $(-p)^n A_n(-1/p)$. Ramanujan produced many results pertaining to Eulerian numbers and polynomials (most of which were not new) [3, p. 109]. In particular, (16) can be viewed as a straightforward corollary of Entry 3 of [3, p. 113].

3. Asymptotic expansion of $f_n(z)$: The case $z \in (1, +\infty)$

Substitution of (19) into (13) and application of Watson's lemma immediately yields the full asymptotic expansion of $f_n(z)$:

$$f_n(z) \sim z^n \sum_{k=0}^{\infty} \frac{A_k(z)}{(z-1)^{k+1}} \frac{1}{n^{k+1}}, \quad \text{as } n \to \infty \quad (z \in (1, +\infty)). \tag{21}$$

Equation (17) allows us to write down the leading terms of (21); these agree with the results of [16] (which, as already mentioned, gives explicit expressions only for the first few terms).

4. Asymptotic expansion of $f_n(z)$: The case $z \in \mathbb{C} \setminus [1, +\infty)$

Equation (19) is also useful when $z \in \mathbb{C} \setminus [1, +\infty)$, but we cannot proceed from (13) as it stands. Instead, we choose an integration path in (12) that does not intersect $[1, +\infty)$ and then split the resulting integral to get

$$f_n(z) = -\ln(1-z) + g_n(z), \quad z \in \mathbb{C} \setminus [1, +\infty), \tag{22}$$

where, as already mentioned, $\ln$ denotes the principal value of the logarithm and

$$g_n(z) \equiv \int_0^z \frac{t^{n-1}}{t-1} dt, \quad z \in \mathbb{C} \setminus [1, +\infty). \tag{23}$$

In the cut z-plane (the cut being the line $[1,+\infty)$), the integral in (23) equals the "remainder" $g_n(z)$ we defined in (3), or its analytic continuation. An additional integral representation for $g_n(z)$ is obtained by setting $u=\ln(z/t)$ in (23); this gives

$$g_n(z)=z^n\int_0^{\infty}\frac{e^{-nu}}{ze^{-u}-1}du, \quad z\in\mathbb{C}\setminus[1,+\infty). \tag{24}$$

Equation (23) provides yet another integral representation for $g_n(z)$ if one sets $u=(1-z)t/[z(1-t)]$; that result forms Entry 4.1.7.8 of the work [34] by Prudnikov, Brychkov, and Marichev.[1]

Equation (24) is most useful for our purposes because it is suitable for the application of Watson's lemma: Substitution of (19) into (24) followed by term-by-term integration yields

$$g_n(z)\sim z^n\sum_{k=0}^{\infty}\frac{A_k(z)}{(z-1)^{k+1}}\frac{1}{n^{k+1}}, \quad \text{as } n\to\infty \quad (z\in\mathbb{C}\setminus[1,+\infty)). \tag{25}$$

In the convergent case, (25) is an asymptotic expansion for the remainder:

$$g_n(z)=f_n(z)-f_\infty(z)=f_n(z)+\ln(1-z)\sim z^n\sum_{k=0}^{\infty}\frac{A_k(z)}{(z-1)^{k+1}}\frac{1}{n^{k+1}}, \quad \text{as } n\to\infty \quad (|z|\leqslant 1 \text{ and } z\neq 1). \tag{26}$$

In the divergent case $|z|>1$, (25) shows that the term $-\ln(1-z)$ in (22) is negligible compared to $g_n(z)$ so that (22) and (25) yield the asymptotic expansion

$$f_n(z)\sim z^n\sum_{k=0}^{\infty}\frac{A_k(z)}{(z-1)^{k+1}}\frac{1}{n^{k+1}}, \quad \text{as } n\to\infty \quad (|z|>1 \text{ and } z\in\mathbb{C}\setminus[1,+\infty)). \tag{27}$$

It can be shown that our (25) is consistent with the asymptotic expansion of Lerch's transcendent in [15, Theorem 1]. However, the final result in [15] is not expressed in terms of Eulerian polynomials. Furthermore, our (26) is consistent with the result in [31, Theorem 7], which concerns what [31] calls the Hurwitz-Lerch digamma function. Equation (26) is derived in a different way in [5]; compare [5, eqs. (5.35) and (3.18)] to (26). The result in [5, eq. (5.35)] is expressed in terms of the so-called geometric polynomials, which are closely related to the Eulerian polynomials. A related expression is also derived by still different methods in [37]; that expression does not involve Eulerian polynomials. In the special case $|z|=1$ with $z\neq 1$, our (26) is consistent with a result in the literature [26] concerning so-called Lerch zeta-function

$$\phi(\xi,s,\alpha)\equiv\Phi\left(e^{i2\pi\xi},s,\alpha\right)=\sum_{k=0}^{\infty}\frac{e^{2\pi ik\xi}}{(k+\alpha)^s}, \quad \xi\in\mathbb{R}. \tag{28}$$

[1] We note that all three integral representations of $g_n(z)$ (namely, (23), (24), and [34, Entry 4.1.7.8] possess straightforward generalizations that represent $\Phi(z,s,\alpha)$. The generalization of (24), which is the simplest of the three, is well known; it forms Entry 1.11.3 of [35] and Entry 25.14.5 of [33], and can also be found in [25].

We note that one often finds alternative notations for the function in (28) in the literature; for example, the second and third arguments are sometimes reversed [4, p. 259], [25]. It is easy to show that $g_n(z)$ is related to the hypergeometric function ${}_2F_1(n,1;n+1;z)$. Very general results on the asymptotic expansions of ${}_2F_1(a,b;c;z)$ for large values of the parameters $a,b,$ and c can be found in [28, 32, 36, 12, 13, 14]. None of these references utilizes Eulerian polynomials.

The special case $z=-1$ of (26) is interesting because it is related to a formula of Ramanujan and, also, because two related formulas in the literature are erroneous. We discuss these issues in Appendix B.

5. Large-n expansions of $c_n(\theta)$ and $s_n(\theta)$

In this section, we apply our previous results to find asymptotic expansions, involving real quantities only, for $c_n(\theta)$ and $s_n(\theta)$. Setting $z=e^{i\theta}$ in (22) and (24), and comparing with (6) and (7), we obtain

$$c_n(\theta)-c_\infty(\theta)=\lambda_n(\theta)\cos n\theta-\mu_n(\theta)\sin n\theta, \quad 0<\theta<2\pi, \tag{29}$$

$$s_n(\theta)-s_\infty(\theta)=\lambda_n(\theta)\sin n\theta+\mu_n(\theta)\cos n\theta, \quad 0<\theta<2\pi, \tag{30}$$

where $c_\infty(\theta)$ and $s_\infty(\theta)$ are defined in (8) and (9) and where $\lambda_n(\theta)$ and $\mu_n(\theta)$ are the real-valued integrals

$$\lambda_n(\theta)\equiv\int_0^\infty e^{-nu}\,\mathrm{Re}\left\{\frac{1}{e^{i\theta}e^{-u}-1}\right\}du=\int_0^\infty e^{-nu}\frac{\cos\theta e^{-u}-1}{e^{-2u}-2\cos\theta e^{-u}+1}du, \tag{31}$$

$$\mu_n(\theta)\equiv\int_0^\infty e^{-nu}\,\mathrm{Im}\left\{\frac{1}{e^{i\theta}e^{-u}-1}\right\}du=\int_0^\infty e^{-nu}\frac{-\sin\theta e^{-u}}{e^{-2u}-2\cos\theta e^{-u}+1}du. \tag{32}$$

Asymptotic expansions for $\lambda_n(\theta)$ and $\mu_n(\theta)$ can be found via the first integral expressions in (31) and (32) by setting $z=e^{i\theta}$ in (19) and applying Watson's lemma. We now proceed with the only nontrivial task involved in this procedure, which is the determination of the real and imaginary parts of the Apostol-Bernoulli number $\beta_k(e^{i\theta})$.

We do this by obtaining new (to the best of our knowledge), explicit expressions for the quantities $\beta_k(e^{i\theta})$ $(k=1,2,\ldots)$ and $A_k(e^{i\theta})$ $(k=0,1,\ldots)$, where $\theta\in\mathbb{R}$. By [7, eq. (6.10)], $\beta_k(e^{i\theta})$ can be expressed via $P_{k-1}(\cot(\theta/2))$, where $P_k(z)$ is the "derivative polynomial for the tangent." However, the only explicit expression for $P_k(z)$ that we are aware of [6, eq. (2.11)] involves complex quantities, even if z is real. The final formulas to be developed here, on the other hand, exclusively involve quantities that are real.

From (15) and (20) we have

$$A_k(e^{i\theta})=\sum_{m=0}^{k-1}\left\langle {k \atop m}\right\rangle e^{i(k-m)\theta}, \quad k=1,2,\ldots. \tag{33}$$

Let us first consider the case $k=2,4,6,\ldots$: Split the sum in (33) as $\sum_{m=0}^{k/2-1}+\sum_{m=k/2}^{k-1}$. In the second sum, set $p=k-1-m$ and apply (20) to the resulting Eulerian number;

the transformed second sum thus has identical summation limits (i.e., 0 and $k/2-1$) as does the first sum, as well as identical Eulerian numbers. Now combine the two sums to obtain

$$A_k(e^{i\theta}) = \sum_{m=0}^{k/2-1} \left\langle {k \atop m} \right\rangle \left[e^{i(k-m)\theta} + e^{i(m+1)\theta} \right], \quad k = 2,4,6,\ldots, \tag{34}$$

which in turn gives

$$A_k(e^{i\theta}) = 2\exp\left[i(k+1)\frac{\theta}{2} \right] \sum_{m=0}^{k/2-1} \left\langle {k \atop m} \right\rangle \cos\left[(k-1-2m)\frac{\theta}{2} \right], \quad k = 2,4,6,\ldots. \tag{35}$$

For the case $k = 1,3,5,\ldots$ it can be shown in a similar manner that (33) yields

$$A_k(e^{i\theta}) = \exp\left[i(k+1)\frac{\theta}{2} \right] \left\{ \left\langle {k \atop \frac{k-1}{2}} \right\rangle + 2\sum_{m=0}^{(k-3)/2} \left\langle {k \atop m} \right\rangle \cos\left[(k-1-2m)\frac{\theta}{2} \right] \right\}, \quad k = 1,3,5,\ldots, \tag{36}$$

in which the first term inside the braces corresponds to the middle term in (33) (i.e., the term with $m = (k-1)/2$), which we wrote separately. We point out, in passing, that (35) and (36) are convenient for the calculation of the modulus and phase of $A_k(e^{i\theta})$, and may thus be of independent interest within the context of Eulerian polynomials.

Equations (36) and (18) now give

$$\beta_k(e^{i\theta}) = \frac{k(-1)^{\frac{k}{2}+1}}{2^{k-1}} \csc^k\frac{\theta}{2} \left\{ \frac{1}{2} \left\langle {k-1 \atop \frac{k}{2}-1} \right\rangle + \sum_{m=0}^{k/2-2} \left\langle {k-1 \atop m} \right\rangle \cos\left[\left(\frac{k}{2} - m - 1 \right)\theta \right] \right\}, \quad k = 2,4,6,\ldots. \tag{37}$$

Similarly, from (35) and (18) we obtain

$$\beta_k(e^{i\theta}) = \frac{ik(-1)^{\frac{k+1}{2}}}{2^{k-1}} \csc^k\frac{\theta}{2} \sum_{m=0}^{(k-3)/2} \left\langle {k-1 \atop m} \right\rangle \cos\left[(k-2m-2)\frac{\theta}{2} \right], \quad k = 3,5,7,\ldots. \tag{38}$$

When evaluated on the unit circle, the Apostol-Bernoulli number $\beta_k(z)$ is thus real when $k = 2,4,6,\ldots$ and imaginary when $k = 3,5,7,\ldots$. The case $k = 1$ is exceptional because $\beta_1(e^{i\theta})$ is complex, with

$$\beta_1(e^{i\theta}) = \frac{A_0(e^{i\theta})}{e^{i\theta}-1} = -\frac{1}{2} - i\frac{1}{2}\cot\frac{\theta}{2}. \tag{39}$$

We thus set $z = e^{i\theta}$ in (19) and transform (19) as follows.

(i) We define the *real* quantities

$$\sigma_k(\theta) = \frac{1}{2k}\beta_{2k}\left(e^{i\theta}\right), \quad k = 1,2,3,\ldots, \tag{40}$$

$$\tau_k(\theta) = \frac{1}{i}\frac{1}{2k+1}\beta_{2k+1}\left(e^{i\theta}\right), \quad k = 1,2,3,\ldots, \tag{41}$$

where the factors $1/(2k)$ and $1/(2k+1)$ are chosen for later convenience.

(ii) We separately write the first term in (19) – i.e., the term involving the complex quantity $\beta_1(e^{i\theta})$ – and substitute $\beta_1(e^{i\theta})$ using the expression in (39).

(iii) We then separate the real and imaginary parts of (19) to obtain

$$\mathrm{Re}\left\{\frac{1}{e^{i\theta}e^{-u}-1}\right\} = -\frac{1}{2} - \sum_{k=1}^{\infty}\frac{1}{(2k-1)!}\sigma_k(\theta)u^{2k-1}, \tag{42}$$

$$\mathrm{Im}\left\{\frac{1}{e^{i\theta}e^{-u}-1}\right\} = -\frac{1}{2}\cot\frac{\theta}{2} + \sum_{k=1}^{\infty}\frac{1}{(2k)!}\tau_k(\theta)u^{2k}. \tag{43}$$

From (40) and (37),

$$\sigma_k(\theta) = \frac{(-1)^{k+1}}{2^{2k-1}}\csc^{2k}\frac{\theta}{2}\left\{\frac{1}{2}\left\langle{2k-1 \atop k-1}\right\rangle + \sum_{m=0}^{k-2}\left\langle{2k-1 \atop m}\right\rangle\cos\left[(k-m-1)\theta\right]\right\},$$
$$k = 1,2,3,\ldots. \tag{44}$$

From (41) and (38),

$$\tau_k(\theta) = \frac{(-1)^{k+1}}{2^{2k}}\csc^{2k+1}\frac{\theta}{2}\sum_{m=0}^{k-1}\left\langle{2k \atop m}\right\rangle\cos\left[(2k-2m-1)\frac{\theta}{2}\right], \quad k = 1,2,3,\ldots. \tag{45}$$

Finally, substitution of (42) and (43) in (31) and (32) and application of Watson's lemma gives

$$\lambda_n(\theta) \sim -\frac{1}{2n} - \sum_{k=1}^{\infty}\sigma_k(\theta)\frac{1}{n^{2k}}, \quad \text{as } n\to\infty, \tag{46}$$

$$\mu_n(\theta) \sim -\frac{1}{2n}\cot\frac{\theta}{2} + \sum_{k=1}^{\infty}\tau_k(\theta)\frac{1}{n^{2k+1}}, \quad \text{as } n\to\infty. \tag{47}$$

By (44), (45), and (20), the first few terms in (46) and (47) are

$$\lambda_n(\theta) = -\frac{1}{2n} - \csc^2\frac{\theta}{2}\frac{1}{4n^2} + (2+\cos\theta)\csc^4\frac{\theta}{2}\frac{1}{8n^4} + O\left(\frac{1}{n^6}\right), \quad \text{as } n\to\infty, \tag{48}$$

$$\mu_n(\theta) = -\cot\frac{\theta}{2}\frac{1}{2n} + \cot\frac{\theta}{2}\csc^2\frac{\theta}{2}\frac{1}{4n^3} + O\left(\frac{1}{n^5}\right), \quad \text{as } n\to\infty. \tag{49}$$

To summarize, in (29) and (30) we have expressed, exactly, the oscillatory quantities $c_n(\theta)-c_\infty(\theta)$ and $s_n(\theta)-s_\infty(\theta)$ in terms of sines and cosines and the auxiliary functions $\lambda_n(\theta)$ and $\mu_n(\theta)$. These two functions are non-oscillatory (in n) and possess the asymptotic power series (Poincaré asymptotic expansions) provided in (46) and (47). Together, equations (29), (30), (46), and (47) form a "compound asymptotic expansion" in the sense discussed in [38] and [16]. (Familiar examples of compound asymptotic expansions are those of the Airy functions $\mathrm{Ai}(z)$ and $\mathrm{Bi}(z)$ as $z\to-\infty$ [2] and the large-z asymptotic expansions of the Sine and Cosine Integrals $\mathrm{Si}(z)$ and $\mathrm{Ci}(z)$ [33, §6.12(ii)], [16].)

6. Large-n expansions of related trigonometric sums; large-n expansion of $r_n(\theta)$

If a trigonometric sum has a sufficiently simple expression in terms of $c_n(\theta)$ and $s_n(\theta)$, its asymptotic expansion can be determined using our previous results. Six such sums are provided in Table 1 (this list is, of course, not exhaustive).

Table 1: *Trigonometric sums easily expressible in terms of $c_n(\theta)$ and $s_n(\theta)$.*

Row No.	Finite sum	Expression in terms of $c_n(\theta)$ and $s_n(\theta)$	Corresponding infinite series
1	$\sum_{k=1}^{n-1}(-1)^{k+1}\frac{\cos k\theta}{k}$	$-c_n(\pi-\theta)$	$\sum_{k=1}^{\infty}(-1)^{k+1}\frac{\cos k\theta}{k}=\ln\left(2\cos\frac{\theta}{2}\right),\ \lvert\theta\rvert<\pi$
2	$s_n'(\theta)\equiv\sum_{k=1}^{n-1}(-1)^{k+1}\frac{\sin k\theta}{k}$	$s_n(\pi-\theta)$	$s_\infty'(\theta)\equiv\sum_{k=1}^{\infty}(-1)^{k+1}\frac{\sin k\theta}{k}=\frac{\theta}{2},\ \lvert\theta\rvert<\pi$
3	$\sum_{k=0}^{n-1}\frac{\cos(2k+1)\theta}{2k+1}$	$\frac{1}{2}\left[c_{2n}(\theta)-c_{2n}(\pi-\theta)\right]$	$\sum_{k=0}^{\infty}\frac{\cos(2k+1)\theta}{2k+1}=\frac{1}{2}\ln\cot\frac{\theta}{2},\ 0<\theta<\pi$
4	$\sum_{k=0}^{n-1}\frac{\sin(2k+1)\theta}{2k+1}$	$\frac{1}{2}\left[s_{2n}(\theta)+s_{2n}(\pi-\theta)\right]$	$\sum_{k=0}^{\infty}\frac{\sin(2k+1)\theta}{2k+1}=\frac{\pi}{4},\ 0<\theta<\pi$
5	$r_n(\theta)\equiv\sum_{k=0}^{n-1}(-1)^k\frac{\cos(2k+1)\theta}{2k+1}$	$\frac{1}{2}\left[s_{2n}\left(\frac{\pi}{2}-\theta\right)+s_{2n}\left(\frac{\pi}{2}+\theta\right)\right]$	$r_\infty(\theta)\equiv\sum_{k=0}^{\infty}(-1)^k\frac{\cos(2k+1)\theta}{2k+1}=\frac{\pi}{4},\ \lvert\theta\rvert<\frac{\pi}{2}$
6	$\sum_{k=0}^{n-1}(-1)^k\frac{\sin(2k+1)\theta}{2k+1}$	$\frac{1}{2}\left[c_{2n}\left(\frac{\pi}{2}-\theta\right)-c_{2n}\left(\frac{\pi}{2}+\theta\right)\right]$	$\sum_{k=0}^{\infty}(-1)^k\frac{\sin(2k+1)\theta}{2k+1}=\frac{1}{2}\ln(\sec\theta+\tan\theta),\ \lvert\theta\rvert<\frac{\pi}{2}$

The middle column in Table 1 is a trivial consequence of the first column and the definitions of $c_n(\theta)$ and $s_n(\theta)$, eqs. (6) and (7). The last column is easily obtained by finding the limit of the middle column using (8) and (9). All infinite series in the last column can also be found in [24]. Interestingly, the infinite series in Rows 2 and 5 appear in [3, pp. 96–97], as examples corresponding to two general, nonrigorous formulas due to Ramanujan (the book [3] further provides rigorous versions of both those formulas). Let us also note that the summable series $\sum_{k=1}^{\infty}(-1)^{k+1}\cos k\theta/k$ in the last column of Row 1 is referred to as the lowest non-Bernoulli Lanczos-Krylov function, or Clausen function [8].

We illustrate the use of Table 1 for the sum $r_n(\theta)$ (see fifth row of Table 1 and (10)): To obtain the large-n asymptotic expansion of $r_n(\theta)$, express $r_n(\theta)-r_\infty(\theta)$ in terms of $\lambda_n(\theta)$ and $\mu_n(\theta)$ using the middle column of Table 1 together with (29) and

(30). The result is

$$r_n(\theta) - r_\infty(\theta) = \frac{(-1)^n}{2} \left\{ \begin{array}{l} \left[-\lambda_{2n}\left(\frac{\pi}{2}-\theta\right)+\lambda_{2n}\left(\frac{\pi}{2}+\theta\right)\right]\sin 2n\theta \\ +\left[\mu_{2n}\left(\frac{\pi}{2}-\theta\right)+\mu_{2n}\left(\frac{\pi}{2}+\theta\right)\right]\cos 2n\theta \end{array} \right\}, \quad |\theta| < \pi/2, \tag{50}$$

where $r_\infty(\theta)$ is found from the last column of Table 1. Equation (50) together with (44)–(47) provide the desired compound asymptotic expansion of $r_n(\theta)$. Equation (50) can be extended to all real, fixed θ except $\theta = \pm\pi/2, \pm 3\pi/2, \pm 5\pi/2, \ldots$ using $r_n(\theta) = r_n(-\theta) = -r_n(\pi-\theta)$ and the 2π-periodicity of $r_n(\theta)$.

7. Summary and future work

This paper dealt with large-n expansions of the sums $f_n(z)$, $c_n(\theta)$, $s_n(\theta)$, and $r_n(\theta)$ defined in (1), (6), (7), and (10). The main results of this paper are the following.

For the case of $f_n(z)$, the expansions are provided in (26), (27), and (21), where $A_k(z)$ are the Eulerian polynomials of (15). Equation (B.9), which is a special case of (26), serves as a correction to certain asymptotic formulas in the literature.

The trigonometric sums $c_n(\theta)$, $s_n(\theta)$, and $r_n(\theta)$ are exactly given in terms of auxiliary functions $\lambda_n(\theta)$ and $\mu_n(\theta)$ in (29), (30), and (50), where the large-n limits $c_\infty(\theta)$, $s_\infty(\theta)$, and $r_\infty(\theta)$ are given by (8), (9), and (11); $\lambda_n(\theta)$ and $\mu_n(\theta)$ possess asymptotic power series, the first few terms of which are given in (48) and (49). Their full asymptotic expansions are provided in (44)-(47), where $\left\langle {k \atop m} \right\rangle$ stand for the Eulerian numbers of (16).

The aforementioned equations constitute the compound asymptotic expansions of the three trigonometric sums. Formulas (35)-(38), which are intermediate steps in the relevant derivations, may be of independent interest within the context of Eulerian polynomials and Apostol-Bernoulli numbers.

Table 1 enables one to obtain similar compound asymptotic expansions for some further trigonometric sums.

For reasons obvious from (9), (11), and the second row of Table 1, the 2π-periodic extensions of the infinite series $s_\infty(\theta)$, $r_\infty(\theta)$, and $s'_\infty(\theta)$, which are discontinuous, are often called the "sawtooth-like," "square wave," and "sawtooth" functions, respectively. The corresponding finite sums $s_n(\theta)$, $r_n(\theta)$, and $s'_n(\theta)$ are ubiquitous in the literature [33, §6.16(i)], [21, 27, 39, 20, 22, 23, 17] dealing with the Gibbs phenomenon (also called the Gibbs-Wilbraham phenomenon). Specifically, numerically-obtained graphs of these sums are frequently used to introduce and illustrate the Gibbs phenomenon in the context of Fourier series. Furthermore, the above three sums are the most usual test cases for methods – such as the method of Fejér averaging – aiming to overcome the Gibbs phenomenon; in other words, the effectiveness of such methods is judged by application to the three sums. We are currently working on the study and development of such methods with the aid of the compound asymptotic expansions developed herein.

Particularly useful for this purpose are the expansions' first few terms, which constitute simple asymptotic approximations.

APPENDICES

A. Derivation of (19); correction to an equation in [11]

The first equality in (19) is essentially eq. [5k] of p. 244 of [11], which, however, is in error. For this reason, we correct that equation starting from eq. [5i] of p. 244 of [11], which is

$$\frac{1-u}{1-ue^{t(1-u)}}=\sum_{n=0}^{\infty}A_n(u)\frac{t^n}{n!}.$$

Following [11], we replace t by $t/(u-1)$ to obtain

$$\frac{1-u}{1-ue^{-t}}=\sum_{n=0}^{\infty}\frac{A_n(u)}{(u-1)^n}\frac{t^n}{n!}. \tag{A.1}$$

Equation (A.1) is a corrected version of eq. [5k] of p. 244 of [11] and immediately yields the first equality in (19); the second one follows from (18).

B. On certain formulas in the literature connected to $f_n(-1)$

In this appendix, we further discuss (26) for the special case $z=-1$: The value of the Eulerian polynomial for $z=-1$ can be written as

$$A_k(-1)=-\frac{2^{k+1}(2^{k+1}-1)}{k+1}B_{k+1},\quad k=0,1,2,\ldots, \tag{B.1}$$

where B_k is the Bernoulli number [33, §24.2]. Equation (B.1) can be readily deduced from [33, §26.14.11] with the aid of (15), (16), and the values $B_1=-1/2$ and $0=B_3=B_5=B_7=\ldots$ of the Bernoulli numbers. Note that a particular case of (B.1) is $0=A_2(-1)=A_4(-1)=A_6(-1)=\ldots$.

Equations (26) and (B.1) give

$$f_n(-1)\sim-\ln 2-\frac{(-1)^n}{2n}-(-1)^n\sum_{k=1}^{\infty}\frac{2^{2k}-1}{2k}B_{2k}\frac{1}{n^{2k}},\quad \text{as } n\to\infty.$$

Adding the term $(-1)^n/n$ and using the definition (1) of $f_n(-1)$ we obtain the slightly different form

$$\sum_{k=1}^{n}\frac{(-1)^{k+1}}{k}\sim\ln 2-\frac{(-1)^n}{2n}+(-1)^n\sum_{k=1}^{\infty}\frac{2^{2k}-1}{2k}B_{2k}\frac{1}{n^{2k}},\quad \text{as } n\to\infty. \tag{B.2}$$

Equation (B.2) can, alternatively, easily be obtained from a formula [3, p. 145, Example 1] of Ramanujan. (That formula is derived in [3] by using Boole's summation formula.)

For the special case $n =$ even, a formula similar to (B.2) appears in Mangulis's handbook as [30, Part III, Sect. 1D, eq. (17)], but that formula is incorrect. The related formula [30, Part III, Sect. 1D, eq. (20)] is also incorrect. We thus proceed to correct both formulas, demonstrate their consistency with our (26), and discuss connections to a relevant formula in Bromwich's book [9]. Before doing this, let us stress that our symbol B_k (which is the same as the B_k of [3]) differs from the corresponding symbols in [30] and [9]: The B_k in [30] and [9] equals our $(-1)^{k+1}B_{2k}$.

We first find the asymptotic expansion of the sum

$$\alpha_n \equiv \sum_{m=1}^{n} \frac{1}{2m-1}. \tag{B.3}$$

From [33, §5.4.15], we have the exact expression

$$\alpha_n = \frac{\gamma}{2} + \ln 2 + \frac{1}{2}\psi\left(n+\frac{1}{2}\right), \tag{B.4}$$

where ψ is the psi function and γ is Euler's constant. With the duplication formula for the psi function [33, §5.5.8], eq. (B.4) yields

$$\alpha_n = \frac{\gamma}{2} + \psi(2n) - \frac{1}{2}\psi(n). \tag{B.5}$$

The asymptotic expansion of the Psi function is [33, §5.11.2]

$$\psi(z) \sim \ln z - \frac{1}{2z} - \sum_{k=1}^{\infty} \frac{B_{2k}}{2k}\frac{1}{z^{2k}}, \quad \text{as } z \to \infty \quad (|\text{ph}\, z| < \pi). \tag{B.6}$$

Equations (B.5) and (B.6) give

$$\alpha_n \sim \frac{\gamma}{2} + \ln 2 + \frac{1}{2}\ln n + \sum_{k=1}^{\infty} \frac{B_{2k}}{2k}\left(2^{2k-1}-1\right)\frac{1}{(2n)^{2k}}, \quad \text{as } n \to \infty. \tag{B.7}$$

Equation (B.7) is a corrected version of [30, Part III, Sect. 1D, eq. (20)].

To obtain a corrected version of [30, Part III, Sect. 1D, eq. (17)], we start from the well-known asymptotic formula

$$\delta_n \equiv \sum_{m=1}^{n} \frac{1}{m} \sim \gamma + \ln n + \frac{1}{2n} - \sum_{k=1}^{\infty} \frac{B_{2k}}{2kn^{2k}}, \tag{B.8}$$

which, as mentioned in our Introduction, is due to Euler and stated by Ramanujan in his notebooks. Equation (B.8) correctly appears in Mangulis's book (allowing for the differences in notation in the Bernoulli numbers) as [30, Part III, Sect. 1D, eq. (16)]. By (B.3) and the definition of δ_n in (B.8), we have

$$\sum_{m=1}^{2n} \frac{(-1)^{m+1}}{m} = 2\alpha_n - \delta_{2n}.$$

Equation (B.7) and the asymptotic expansion in (B.8) therefore yield

$$\sum_{m=1}^{2n} \frac{(-1)^{m+1}}{m} \sim \ln 2 - \frac{1}{4n} + \sum_{k=1}^{\infty} \frac{2^{2k}-1}{2k} B_{2k} \frac{1}{(2n)^{2k}}, \quad \text{as } n \to \infty. \tag{B.9}$$

Equation (B.9), which is a special case of (26), is a corrected version of [30, Part III, Sect. 1D, eq. (17)].

Let us finally note that [30], which has no derivations, cites Bromwich's book [9] as the source of both equations corrected in the present appendix. Ref. [9] does include a formula corresponding to (B.7). While correct, that formula only contains the first few terms of the full asymptotic expansion given in (B.7).

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