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REFINEMENTS OF THE MAJORIZATION THEOREMS VIA FINK IDENTITY AND RELATED RESULTS

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Abstract. The well known majorization theorem (see [5, p. 11] and [7, p. 320]) plays an important role in our paper. By using A. M. Fink's identity in the majorization difference, we obtain an interesting identity and with the help of this useful identity, we obtain many significant results. We investigate the bounds for this identity, by using Grüss-type inequalities and we also present some results relating to the Ostrowski-type inequality.

1. Introduction and preliminaries

The theory of convex functions has experienced a rapid development. This can be attributed to several causes: firstly, so many areas in modern analysis directly or indirectly involve the application of convex functions; secondly, convex functions are closely related to the theory of inequalities and many important inequalities are consequences of the applications of convex functions (see [7]).

DEFINITION 1. A function $f: I \to \mathbb{R}$ is convex on I if

$$(x_3 - x_2) f(x_1) + (x_1 - x_3) f(x_2) + (x_2 - x_1) f(x_3) \geqslant 0 \tag{1}$$

holds for all $x_1, x_2, x_3 \in I$ such that $x_1 < x_2 < x_3$.

An important characterization of convex function is stated in [7, p. 2].

THEOREM 1.1. *If f is a convex function defined on I and if $x_1 \leqslant y_1$, $x_2 \leqslant y_2$, $x_1 \neq x_2$, $y_1 \neq y_2$, then the following inequality is valid*

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \leqslant \frac{f(y_2) - f(y_1)}{y_2 - y_1}.$$

If the function f is concave, then the inequality reverses.

The following definition is given in [4].

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DEFINITION 2. A function $f : I \to (0,\infty)$ is said to be log-convex in the Jensen sense if for all $x,y \in I$, the inequality

$$f^2\left(\frac{x+y}{2}\right) \leqslant f(x)f(y)$$

holds.

REMARK 1.2. It is easy to see that a function $f : I \to (0,\infty)$ is log-convex in the Jensen sense if and only if the relation

$$\alpha^2 f(x) + 2\alpha\beta f\left(\frac{x+y}{2}\right) + \beta^2 f(y) \geqslant 0$$

holds for every $\alpha, \beta \in \mathbb{R}$ and $x,y \in I$.

A log-convex function is defined as follows (see [7, p. 7]):

DEFINITION 3. A function $f : I \to (0,\infty)$ is said to be log-convex or multiplicatively convex if $\log f$ is convex. Equivalently, f is log-convex if for all $x,y \in I$ and for all $\lambda \in [0,1]$, the inequality

$$f(\lambda x + (1-\lambda)y) \leqslant f^{\lambda}(x) f^{(1-\lambda)}(y)$$

holds. If the inequality reverses, then f is said to be log-concave.

REMARK 1.3. If f is continuous, then a log-convex function in the Jensen sense is log-convex.

Divided difference of a function is defined as follows (see [7, p. 14]):

DEFINITION 4. The nth-order divided difference of a function $f : [a,b] \to \mathbb{R}$ at mutually distinct points $x_0,\ldots,x_n \in [a,b]$ is defined recursively by

$$\begin{aligned} [x_i; f] &= f(x_i), \quad i = 0,\ldots,n, \\ [x_0,\ldots,x_n; f] &= \frac{[x_1,\ldots,x_n; f] - [x_0,\ldots,x_{n-1}; f]}{x_n - x_0}. \end{aligned} \tag{2}$$

It is easy to see that (2) is equivalent to

$$[x_0,\ldots,x_n; f] = \sum_{i=0}^{n} \frac{f(x_i)}{q'(x_i)}, \quad \text{where} \quad q(x) = \prod_{j=0}^{n} (x - x_j).$$

The definition of a real-valued convex function is characterized by the nth-order divided difference (see [7, p. 15]).

DEFINITION 5. A function $f:[a,b]\to\mathbb{R}$ is said to be n-convex $(n\geqslant 0)$ if and only if for all choices of $(n+1)$ distinct points $x_0,\ldots,x_n\in[a,b]$, $[x_0,\ldots,x_n;f]\geqslant 0$ holds.

If this inequality is reversed, then f is said to be n-concave. If the inequality is strict, then f is said to be a strictly n-convex (n-concave) function.

REMARK 1.4. Note that 0-convex functions are non-negative functions, 1-convex functions are increasing functions and 2-convex functions are simply the convex functions.

The following theorem gives an important criteria to examine the n-convexity of a function f (see [7, p. 16]).

THEOREM 1.5. *If $f^{(n)}$ exists, then f is n-convex if and only if $f^{(n)}\geqslant 0$.*

The notion of majorization arose as a measure of the diversity of the components of an m-dimensional vector (an m-tuple) and is closely related to convexity. It is treated most comprehensively by A. W. Marshall, I. Olkin and B. C. Arnold in [5] (see also [7]).

Let $\mathbf{x}=(x_1,\ldots,x_m)$ and $\mathbf{y}=(y_1,\ldots,y_m)$ be two real m-tuples for fixed $m\geqslant 2$ and let

$$\begin{aligned} x_{[1]}\geqslant x_{[2]}\geqslant\ldots\geqslant x_{[m]},\quad & y_{[1]}\geqslant y_{[2]}\geqslant\ldots\geqslant y_{[m]},\\ x_{(1)}\leqslant x_{(2)}\ldots\leqslant x_{(m)},\quad & y_{(1)}\leqslant y_{(2)}\ldots\leqslant y_{(m)},\end{aligned}$$

be their ordered components.

$\mathbf{x}$ is said to majorize $\mathbf{y}$ or $\mathbf{y}$ is said to be majorized by $\mathbf{x}$ (mathematically $\mathbf{x}\succ\mathbf{y}$) if

$$\begin{cases}\sum_{i=1}^{k}x_{[i]}\geqslant\sum_{i=1}^{k}y_{[i]},\ k=1,\ldots,m-1,\\ \sum_{i=1}^{m}x_i=\sum_{i=1}^{m}y_i,\end{cases}\tag{3}$$

holds. The inequality in (3) is equivalent to

$$\sum_{i=m-k+1}^{m}x_{(i)}\geqslant\sum_{i=m-k+1}^{m}y_{(i)},\quad k=1,\ldots,m-1.$$

The well known majorization theorem is given in [5, p. 14] (see also [7, p. 320]).

THEOREM 1.6. *Let $\mathbf{x}=(x_1,\ldots,x_m)$ and $\mathbf{y}=(y_1,\ldots,y_m)$ be two real m-tuples such that $x_i,y_i\in[a,b]$, where $i=1,\ldots,m$. Then for every continuous convex function $\vartheta:[a,b]\to\mathbb{R}$, the inequality*

$$\sum_{i=1}^{m}\vartheta(x_i)\geqslant\sum_{i=1}^{m}\vartheta(y_i)\tag{4}$$

holds if and only if $\mathbf{x}\succ\mathbf{y}$.

If the function ϑ is concave, then the inequality reverses.

A weighted version, which is in fact the generalization of Theorem 1.6, was proved by L. Fuchs in [3] (see also [7, p. 323]).

THEOREM 1.7. *Let $\mathbf{p} = (p_1, \ldots, p_m)$ be a real m-tuple and $\mathbf{x} = (x_1, \ldots, x_m)$, $\mathbf{y} = (y_1, \ldots, y_m)$ be two decreasing real m-tuples such that*

$$\sum_{i=1}^{k} p_i x_i \geqslant \sum_{i=1}^{k} p_i y_i, \quad k = 1, \ldots, m-1, \tag{5}$$

and

$$\sum_{i=1}^{m} p_i x_i = \sum_{i=1}^{m} p_i y_i \tag{6}$$

hold. Then for every continuous convex function $\vartheta : I \to \mathbb{R}$, we have

$$\sum_{i=1}^{m} p_i \vartheta(x_i) \geqslant \sum_{i=1}^{m} p_i \vartheta(y_i). \tag{7}$$

If ϑ is concave, then opposite inequality holds in (7).

The following proposition represents an integral majorization result which is in fact a consequence of Theorem 1 given in [6].

PROPOSITION 1.8. *Let $p : [c,d] \to \mathbb{R}$ be a continuous function and $\varphi, \psi : [c,d] \to [a,b]$ be two decreasing continuous functions such that*

$$\int_c^u p(z)\varphi(z)\,dz \geqslant \int_c^u p(z)\psi(z)\,dz, \qquad \forall\, u \in [a,b], \tag{8}$$

and

$$\int_c^d p(z)\varphi(z)\,dz = \int_c^d p(z)\psi(z)\,dz \tag{9}$$

hold. Then for every continuous convex function $\vartheta : [a,b] \to \mathbb{R}$, we have

$$\int_c^d p(z)\vartheta(\varphi(z))\,dz \geqslant \int_c^d p(z)\vartheta(\psi(z))\,dz. \tag{10}$$

If ϑ is concave, then opposite inequality holds in (10).

In our paper, we use A. M. Fink's identity and prove many interesting results. The following theorem is proved by A. M. Fink in [2].

THEOREM 1.9. *Let $a, b \in \mathbb{R}$, $f : [a,b] \to \mathbb{R}$, $n \geqslant 1$ and $f^{(n-1)}$ is absolutely continuous on $[a,b]$. Then*

$$\begin{aligned} f(x) = {} & \frac{n}{b-a}\int_a^b f(t)\,dt \\ & -\sum_{k=1}^{n-1}\left(\frac{n-k}{k!}\right)\left(\frac{f^{(k-1)}(a)(x-a)^k - f^{(k-1)}(b)(x-b)^k}{b-a}\right) \\ & +\frac{1}{(n-1)!\,(b-a)}\int_a^b (x-t)^{n-1} k^{[a,b]}(t,x) f^{(n)}(t)\,dt, \end{aligned} \tag{11}$$

where

$$k^{[a,b]}(t,x)=\begin{cases} t-a, & a\leqslant t\leqslant x\leqslant b,\\ t-b, & a\leqslant x<t\leqslant b.\end{cases} \tag{12}$$

The organization of the paper is the following: in Section 2, we present some interesting results by using A. M. Fink's identity combined together with the n-convexity of the function f. We present a refinement of the weighted majorization-type inequality for the two decreasing m-tuples $\mathbf{x}$ and $\mathbf{y}$ as well as a refinement of the integral majorization-type inequality for the two decreasing functions φ and ψ. We also present a refinement of the majorization-type inequality for the two majorized m-tuples $\mathbf{x}$ and $\mathbf{y}$. We study the functionals defined as the difference between the right-hand and the left-hand side of the generalized inequalities. In Section 3, we present some interesting results by using Čebyšev functional and Grüss-type inequalities along with some results relating to the Ostrowski-type inequality. In Section 4, our objective is to study the properties of functionals, such as n-exponential and logarithmic convexity. Furthermore, we prove monotonicity property of the generalized Cauchy means obtained via these functionals. Finally, in Section 5 we give several examples of the families of functions for which the obtained results can be applied.

2. Refinements of the Majorization Theorems Via A. M. Fink's Identity

Our first main result of this section states that:

THEOREM 2.1. *Let $f:[a,b]\to\mathbb{R}$ be such that for $n\geqslant 1$, $f^{(n-1)}$ is absolutely continuous. Let $x_i,y_i\in[a,b]$, $p_i\in\mathbb{R}$ $(i=1,\ldots,m)$ and let $k^{[a,b]}(t,x)$ be the same as defined in (12). Then we have*

$$\begin{aligned}
&\sum_{i=1}^{m}p_if(x_i)-\sum_{i=1}^{m}p_if(y_i)=\\
&\sum_{k=1}^{n-1}\left(\frac{n-k}{k!\,(b-a)}\right)\left[f^{(k-1)}(a)\left(\sum_{i=1}^{m}p_i(y_i-a)^k-\sum_{i=1}^{m}p_i(x_i-a)^k\right)\right.\\
&\qquad\left.-f^{(k-1)}(b)\left(\sum_{i=1}^{m}p_i(y_i-b)^k-\sum_{i=1}^{m}p_i(x_i-b)^k\right)\right]\\
&+\frac{1}{(n-1)!\,(b-a)}\times\\
&\int_a^b f^{(n)}(t)\left(\sum_{i=1}^{m}p_i(x_i-t)^{n-1}k^{[a,b]}(t,x_i)-\sum_{i=1}^{m}p_i(y_i-t)^{n-1}k^{[a,b]}(t,y_i)\right)dt.
\end{aligned} \tag{13}$$

Proof. By using (11) for $x = x_i$ and $y = y_i$ in the majorization difference, we have

$$\sum_{i=1}^{m} p_i f(x_i) - \sum_{i=1}^{m} p_i f(y_i) = \sum_{i=1}^{m} p_i \sum_{k=1}^{n-1} \left(\frac{n-k}{k!}\right) \times$$
$$\left(\frac{f^{(k-1)}(a)\left((y_i-a)^k-(x_i-a)^k\right) - f^{(k-1)}(b)\left((y_i-b)^k-(x_i-b)^k\right)}{b-a}\right)$$
$$-\sum_{i=1}^{m} p_i \left(\frac{\int_a^b f^{(n)}(t)\left((y_i-t)^{n-1}k^{[a,b]}(t,y_i)-(x_i-t)^{n-1}k^{[a,b]}(t,x_i)\right)dt}{(n-1)!(b-a)}\right).$$

Now apply Fubini's theorem, we have (13). □

The following theorem is the integral version of Theorem 2.1.

THEOREM 2.2. *Let $f:[a,b]\to\mathbb{R}$ be such that for $n \geqslant 1$, $f^{(n-1)}$ is absolutely continuous on $[a,b]$ and let $k^{[a,b]}(t,x)$ be the same as defined in (12). Let $p:[c,d]\to\mathbb{R}$ and $\varphi,\psi:[c,d]\to[a,b]$ be continuous functions. Then we have*

$$\int_c^d p(z) f(\varphi(z))\,dz - \int_c^d p(z) f(\psi(z))\,dz = \sum_{k=1}^{n-1}\left(\frac{n-k}{k!(b-a)}\right)\times \tag{14}$$
$$\Bigg[f^{(k-1)}(a)\left(\int_c^d p(z)(\psi(z)-a)^k dz - \int_c^d p(z)(\varphi(z)-a)^k dz\right)$$
$$- f^{(k-1)}(b)\left(\int_c^d p(z)(\psi(z)-b)^k dz - \int_c^d p(z)(\varphi(z)-b)^k dz\right)\Bigg]$$
$$+\frac{1}{(n-1)!(b-a)}\int_a^b f^{(n)}(t)\Bigg[\int_c^d p(z)(\varphi(z)-t)^{n-1}k^{[a,b]}(t,\varphi(z))\,dz$$
$$-\int_c^d p(z)(\psi(z)-t)^{n-1}k^{[a,b]}(t,\psi(z))\,dz\Bigg]dt.$$

Proof. By using (11) for $x=\varphi(z)$ and $y=\psi(z)$ in the integral majorization difference $\int_c^d p(z) f(\varphi(z))\,dz - \int_c^d p(z) f(\psi(z))\,dz$, and after simplification we have (14). □

The following theorem is our second main result of this section:

THEOREM 2.3. *Let all the assumptions of Theorem 2.1 be satisfied and let for $n \geqslant 1$*

$$\sum_{i=1}^{m} p_i (x_i-t)^{n-1} k^{[a,b]}(t,x_i) \geqslant \sum_{i=1}^{m} p_i (y_i-t)^{n-1} k^{[a,b]}(t,y_i) \tag{15}$$

holds. If f is n-convex, then we have

$$\sum_{i=1}^{m} p_i f(x_i) - \sum_{i=1}^{m} p_i f(y_i) \geqslant \tag{16}$$
$$\sum_{k=1}^{n-1} \left(\frac{n-k}{k!\,(b-a)}\right) \left[f^{(k-1)}(a) \left(\sum_{i=1}^{m} p_i (y_i - a)^k - \sum_{i=1}^{m} p_i (x_i - a)^k \right) \right.$$
$$\left. - f^{(k-1)}(b) \left(\sum_{i=1}^{m} p_i (y_i - b)^k - \sum_{i=1}^{m} p_i (x_i - b)^k \right) \right].$$

If opposite inequality holds in (15)*, then* (16) *holds in the reverse direction.*

Proof. Since $f^{(n-1)}$ is absolutely continuous on $[a,b]$, $f^{(n)}$ exists almost everywhere. As f is n-convex, applying Theorem 1.5, we have, $f^{(n)}(x) \geqslant 0$ for all $x \in [a,b]$. Now by using $f^{(n)} \geqslant 0$ and (15) in (13), we have (16). □

An integral version of our second main result states that:

THEOREM 2.4. *Let all the assumptions of Theorem 2.2 be satisfied and let for $n \geqslant 1$*

$$\int_c^d p(z)(\varphi(z) - t)^{n-1} k^{[a,b]}(t, \varphi(z))\, dz$$
$$\geqslant \int_c^d p(z)(\psi(z) - t)^{n-1} k^{[a,b]}(t, \psi(z))\, dz \tag{17}$$

holds. If f is n-convex, then we have

$$\int_c^d p(z) f(\varphi(z))\, dz - \int_c^d p(z) f(\psi(z))\, dz \geqslant \sum_{k=1}^{n-1} \left(\frac{n-k}{k!\,(b-a)}\right) \times \tag{18}$$
$$\left[f^{(k-1)}(a) \left(\int_c^d p(z)(\psi(z) - a)^k\, dz - \int_c^d p(z)(\varphi(z) - a)^k \right) dz \right.$$
$$\left. - f^{(k-1)}(b) \left(\int_c^d p(z)(\psi(z) - b)^k\, dz - \int_c^d p(z)(\varphi(z) - b)^k \right) dz \right].$$

If opposite inequality holds in (17)*, then* (18) *holds in the reverse direction.*

Proof. The idea of the proof is the same as that of the proof of Theorem 2.3. □

The following corollary presents a refinement of the weighted majorization-type inequality for the two decreasing m-tuples $\mathbf{x}$ and $\mathbf{y}$.

COROLLARY 2.5. *Let all the assumptions of Theorem 2.1 be satisfied and let $\mathbf{x} = (x_1, \ldots, x_m)$ and $\mathbf{y} = (y_1, \ldots, y_m)$ be two decreasing real m-tuples such that* (5) *and* (6) *hold.*

(i) *Let n be even and $n \geqslant 2$. If the function $f : [a,b] \to \mathbb{R}$ is n-convex, then* (16) *holds.*

(ii) *Let the inequality* (16) *be satisfied and let $F : [a,b] \to \mathbb{R}$ be a function defined by*

$$F(x) = \sum_{k=1}^{n-1} \left(\frac{n-k}{k!\,(b-a)} \right) \left((x-b)^k f^{(k-1)}(b) - (x-a)^k f^{(k-1)}(a) \right). \quad (19)$$

If F is a convex function, then the right hand side of (16) *is non-negative and we have*

$$\sum_{i=1}^{m} p_i f(x_i) \geqslant \sum_{i=1}^{m} p_i f(y_i). \quad (20)$$

Proof.
(i) For

$$\eta(x) := (x-t)^{n-1} k^{[a,b]}(t,x) = \begin{cases} (x-t)^{n-1}(t-a), \ a \leqslant t \leqslant x \leqslant b, \\ (x-t)^{n-1}(t-b), \ a \leqslant x < t \leqslant b, \end{cases}$$

we have,

$$\eta''(x) := \begin{cases} (n-1)(n-2)(x-t)^{n-3}(t-a), \ a \leqslant t \leqslant x \leqslant b, \\ (n-1)(n-2)(x-t)^{n-3}(t-b), \ a \leqslant x < t \leqslant b, \end{cases}$$

showing that η is convex for even n, where $n \geqslant 2$. As $\mathbf{x}$ and $\mathbf{y}$ are decreasing real m-tuples such that (5) and (6) hold, by using the convex function $\eta(x) := (x-t)^{n-1} k^{[a,b]}(t,x)$ in (7), we obtain (15) for even n, where $n \geqslant 2$. Now as f is n-convex for even n, by applying Theorem 2.3, we have (16).

(ii) It is easy to see that (16) is equivalent to

$$\sum_{i=1}^{m} p_i f(x_i) - \sum_{i=1}^{m} p_i f(y_i) \geqslant \sum_{i=1}^{m} p_i F(x_i) - \sum_{i=1}^{m} p_i F(y_i).$$

As (5) and (6) hold, by replacing the convex function F by the convex function ϑ in Theorem 1.7 (7), the non-negativity of the right hand side of (16) is immediate and we have (20). □

An integral version of Corollary 2.5, provides a refinement of the integral majorization-type inequality for the two decreasing functions φ and ψ as follows:

COROLLARY 2.6. *Let all the assumptions of Theorem 2.2 be satisfied and let $\varphi, \psi : [c,d] \to [a,b]$ be two decreasing functions such that* (8) *and* (9) *hold.*

(i) *Let n be even and $n \geqslant 2$. If the function $f : [a,b] \to \mathbb{R}$ is n-convex, then* (18) *holds.*

(ii) *Let the inequality* (18) *be satisfied and let* F *be the same as defined in* (19)*. If* F *is a convex function, then the right hand side of* (18) *is non-negative and we have*

$$\int_c^d p(z)\, f(\varphi(z))\, dz \geqslant \int_c^d p(z)\, f(\psi(z))\, dz.$$

Proof. It is easy to see that (18) is equivalent to

$$\int_c^d p(z)\, f(\varphi(z))\, dz - \int_c^d p(z)\, f(\psi(z))\, dz$$
$$\geqslant \int_c^d p(z)\, F(\varphi(z))\, dz - \int_c^d p(z)\, F(\psi(z))\, dz.$$

The proof is analogous to the proof of Corollary 2.5 but we apply Proposition 1.8 and Theorem 2.4 instead of Theorem 1.7 and Theorem 2.3. □

For the two m-tuples $\mathbf{x}$ and $\mathbf{y}$ such that $\mathbf{x} \succ \mathbf{y}$, the following corollary presents a refinement of the majorization-type inequality.

COROLLARY 2.7. *Let all the assumptions of Theorem* 2.1 *be satisfied and let* $\mathbf{x} = (x_1, \ldots, x_m)$ *and* $\mathbf{y} = (y_1, \ldots, y_m)$ *be two real* m*-tuples such that* $\mathbf{x} \succ \mathbf{y}$.

(i) *Let* n *be even and* $n \geqslant 2$*. If the function* $f : [a,b] \to \mathbb{R}$ *is* n*-convex, then we have*

$$\sum_{i=1}^m f(x_i) - \sum_{i=1}^m f(y_i) \geqslant \tag{21}$$
$$\sum_{k=1}^{n-1} \left(\frac{n-k}{k!\,(b-a)} \right) \left[f^{(k-1)}(a) \left(\sum_{i=1}^m (y_i - a)^k - \sum_{i=1}^m (x_i - a)^k \right) \right.$$
$$\left. - f^{(k-1)}(b) \left(\sum_{i=1}^m (y_i - b)^k - \sum_{i=1}^m (x_i - b)^k \right) \right].$$

(ii) *Let the inequality* (21) *be satisfied and let* F *be the same as defined in* (19)*. If* F *is a convex function, then the right hand side of* (21) *is non-negative and we have the following inequality*

$$\sum_{i=1}^m f(x_i) \geqslant \sum_{i=1}^m f(y_i). \tag{22}$$

Proof. (i) As $\mathbf{x} = (x_1, \ldots, x_m)$ and $\mathbf{y} = (y_1, \ldots, y_m)$ be two real m-tuples such that $\mathbf{x} \succ \mathbf{y}$ and as $\eta(x)$ is convex for even n, where $n \geqslant 2$, by applying Theorem 1.6 (4) for the convex function $\eta(x)$, we have

$$\sum_{i=1}^m (x_i - t)^{n-1} k^{[a,b]}(t, x_i) \geqslant \sum_{i=1}^m (y_i - t)^{n-1} k^{[a,b]}(t, y_i),$$

which is equivalent to (15) for each $p_i = 1$ $(i = 1,\ldots,m)$. Now as f is n-convex for even n, where $n \geqslant 2$, we apply Theorem 2.3 for each $p_i = 1$ $(i = 1,\ldots,m)$ and (21) is immediate.

(ii) It is easy to see that (21) is equivalent to

$$\sum_{i=1}^{m} f(x_i) - \sum_{i=1}^{m} f(y_i) \geqslant \sum_{i=1}^{m} F(x_i) - \sum_{i=1}^{m} F(y_i).$$

As $\mathbf{x} \succ \mathbf{y}$, by replacing the convex function F by the convex function ϑ in (4), the non-negativity of the right hand side of (21) is immediate and we have (22). $\square$

Consider the inequalities (16) and (18) and define linear functionals

$$\begin{aligned}\Phi_1(f) = {} & \sum_{i=1}^{m} p_i f(x_i) - \sum_{i=1}^{m} p_i f(y_i) \\ & - \sum_{k=1}^{n-1} \left(\frac{n-k}{k!(b-a)} \right) \left[f^{(k-1)}(a) \left(\sum_{i=1}^{m} p_i (y_i - a)^k - \sum_{i=1}^{m} p_i (x_i - a)^k \right) \right. \\ & \left. - f^{(k-1)}(b) \left(\sum_{i=1}^{m} p_i (y_i - b)^k - \sum_{i=1}^{m} p_i (x_i - b)^k \right) \right],\end{aligned} \tag{23}$$

and

$$\begin{aligned}\Phi_2(f) = {} & \int_c^d p(z) f(\varphi(z))\,dz - \int_c^d p(z) f(\psi(z))\,dz - \sum_{k=1}^{n-1} \left(\frac{n-k}{k!(b-a)} \right) \\ & \times \left[f^{(k-1)}(a) \left(\int_c^d p(z)(\psi(z) - a)^k dz - \int_c^d p(z)(\varphi(z) - a)^k \right) dz \right. \\ & \left. - f^{(k-1)}(b) \left(\int_c^d p(z)(\psi(z) - b)^k dz - \int_c^d p(z)(\varphi(z) - b)^k \right) dz \right],\end{aligned} \tag{24}$$

where $f : [a,b] \to \mathbb{R}$ is such that for $n \geqslant 1$, $f^{(n-1)}$ is absolutely continuous, $x_i, y_i \in [a,b]$, $p_i \in \mathbb{R}$ $(i = 1,\ldots,m)$; and $\varphi, \psi : [c,d] \to [a,b]$ and $p : [c,d] \to \mathbb{R}$ are continuous functions. If the function f is n-convex defined on $[a,b]$, then by the assumptions of Theorems 2.3 and 2.4, we have $\Phi_i(f) \geqslant 0$, where $i = 1,2$.

Now, we give mean value theorems for the functionals Φ_i, where $i = 1,2$. These theorems enable us to define various classes of means that can be expressed in terms of linear functionals.

First, we state the Lagrange-type mean value theorem related to the functionals Φ_i, where $i = 1,2$.

THEOREM 2.8. *Let $f : [a,b] \to \mathbb{R}$ be such that for $n \geqslant 1$, $f^{(n-1)}$ is absolutely continuous. Let $x_i, y_i \in [a,b]$, $p_i \in \mathbb{R}$ $(i = 1,\ldots,m)$ and let $\varphi, \psi : [c,d] \to [a,b]$ and $p : [c,d] \to \mathbb{R}$ be continuous functions. Suppose that for $n \geqslant 1$, (15) and (17) hold,* *where* $k^{[a,b]}(t,x)$ *is the same as defined in* (12). *If* $f \in C^n([a,b])$ *and if* Φ_1 *and* Φ_2 *are linear functionals as defined in* (23) *and* (24) *respectively, then there exists* $\xi_1, \xi_2 \in [a,b]$ *such that*

$$\Phi_i(f) = f^{(n)}(\xi_i)\Phi_i(f_0), \qquad i = 1,2,$$

holds, where $f_0(x) = \frac{x^n}{n!}$.

Proof. Analogous to the proof of Theorem 2.2 in [8]. □

The following theorem is a new analogue of the classical Cauchy mean value theorem, related to the functionals Φ_i $(i = 1,2)$ and it can be proven by following the proof of Theorem 2.4 in [8].

THEOREM 2.9. *Let all the assumptions of Theorem 2.8 be satisfied and let* $f,k \in C^n([a,b])$. *Then there exist* $\xi_i \in [a,b]$ *such that*

$$\frac{\Phi_i(f)}{\Phi_i(k)} = \frac{f^{(n)}(\xi_i)}{k^{(n)}(\xi_i)}, \qquad i = 1,2, \tag{25}$$

holds, provided that the denominators are non-zero.

REMARK 2.10. (i) By taking $f(x) = x^s$ and $k(x) = x^q$ in (25), where $s,q \in \mathbb{R} \setminus \{0,1,\ldots,n-1\}$ are such that $s \neq q$, we have

$$\xi_i^{s-q} = \frac{q(q-1)\ldots(q-(n-1))\Phi_i(x^s)}{s(s-1)\ldots(s-(n-1))\Phi_i(x^q)}, \qquad i = 1,2.$$

(ii) If the inverse of the function $f^{(n)}/k^{(n)}$ exists, then (25) gives

$$\xi_i = \left(\frac{f^{(n)}}{k^{(n)}}\right)^{-1}\left(\frac{\Phi_i(f)}{\Phi_i(k)}\right), \qquad i = 1,2.$$

3. Čebyšev-Grüss Type Inequalities Via A. M. Fink's Identity and Ostrowski-Type Inequalities

In this section we present some interesting results by using Čebyšev functional and Grüss-type inequalities.

Consider the Čebyšev functional for the two Lebesgue integrable functions $g,h : [a,b] \to \mathbb{R}$,

$$F(g,h) := \frac{1}{b-a}\int_a^b g(t)h(t)\,dt - \frac{1}{b-a}\int_a^b g(t)\,dt \cdot \frac{1}{b-a}\int_a^b h(t)\,dt. \tag{26}$$

The following Grüss-type inequalities are given in [1].

THEOREM 3.1. *Let $g,h:[a,b]\to\mathbb{R}$ be two absolutely continuous functions with $(\cdot-a)(b-\cdot)(h')^2\in L[a,b]$. Then we have*

$$|F(g,h)|\leqslant\frac{1}{\sqrt{2}}[F(g,g)]^{\frac{1}{2}}\frac{1}{\sqrt{b-a}}\left(\int_a^b(t-a)(b-t)\left(h'(t)\right)^2dt\right)^{\frac{1}{2}}. \tag{27}$$

The constant $\frac{1}{\sqrt{2}}$ is the best possible in (27).

THEOREM 3.2. *Let $g:[a,b]\to\mathbb{R}$ be an absolutely continuous function with $g'\in L_\infty[a,b]$ and let $h:[a,b]\to\mathbb{R}$ be a monotonic non-decreasing function. Then we have*

$$|F(g,h)|\leqslant\frac{1}{2(b-a)}\|g'\|_\infty\int_a^b(t-a)(b-t)\,dh(t). \tag{28}$$

The constant $\frac{1}{2}$ is the best possible in (28).

Before presenting our first main result of this section, let us denote

$$\zeta(t)=\sum_{i=1}^m p_i(x_i-t)^{n-1}k^{[a,b]}(t,x_i)-\sum_{i=1}^m p_i(y_i-t)^{n-1}k^{[a,b]}(t,y_i), \tag{29}$$

and

$$\begin{aligned}\hat{\zeta}(t)\ &=\int_c^d p(z)(\varphi(z)-t)^{n-1}k^{[a,b]}(t,\varphi(z))\,dz\\ &\quad-\int_c^d p(z)(\psi(z)-t)^{n-1}k^{[a,b]}(t,\psi(z))\,dz,\end{aligned} \tag{30}$$

where $x_i,y_i,t\in[a,b]$, $p_i\in\mathbb{R}$ $(i=1,\ldots,m)$, $\varphi,\psi:[c,d]\to[a,b]$ and $p:[c,d]\to\mathbb{R}$ are continuous functions and $k^{[a,b]}(t,.)$ is the same as defined in (12).

THEOREM 3.3. *Let $f:[a,b]\to\mathbb{R}$ be such that for $n\geqslant1$, $f^{(n)}$ is absolutely continuous with $(\cdot-a)(b-\cdot)\left(f^{(n+1)}\right)^2\in L[a,b]$. Let $x_i,y_i\in[a,b]$ and $p_i\in\mathbb{R}$ $(i=1,\ldots,m)$. If F and ζ are the same as defined in (26) and (29) respectively, then we have*

$$\begin{aligned}&\sum_{i=1}^m p_if(x_i)-\sum_{i=1}^m p_if(y_i)=\\ &\sum_{k=1}^{n-1}\left(\frac{n-k}{k!(b-a)}\right)\left[f^{(k-1)}(a)\left(\sum_{i=1}^m p_i(y_i-a)^k-\sum_{i=1}^m p_i(x_i-a)^k\right)\right.\\ &\qquad\left.-f^{(k-1)}(b)\left(\sum_{i=1}^m p_i(y_i-b)^k-\sum_{i=1}^m p_i(x_i-b)^k\right)\right]\\ &+\frac{1}{(n-1)!(b-a)}\left[f^{(n-1)};a,b\right]\int_a^b\zeta(t)\,dt+G_n(f;a,b),\end{aligned} \tag{31}$$

where

$$\left[f^{(n-1)};a,b\right]=\frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{b-a}, \tag{32}$$

is the divided difference and the remainder $G_n(f;a,b)$ *satisfies the estimation*

$$|G_n(f;a,b)|\leqslant\frac{[F(\zeta(t),\zeta(t))]^{\frac{1}{2}}}{(n-1)!\sqrt{2}}\cdot\frac{1}{\sqrt{b-a}}\left(\int_a^b(t-a)(b-t)\left(f^{(n+1)}(t)\right)^2dt\right)^{\frac{1}{2}}. \tag{33}$$

Proof. By applying Theorem 3.1 for $g\to\zeta$ and $h\to f^{(n)}$, we have

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b\zeta(t)f^{(n)}(t)\,dt-\frac{1}{b-a}\int_a^b\zeta(t)\,dt\cdot\frac{1}{b-a}\int_a^b f^{(n)}(t)\,dt\right| \\ &\leqslant\frac{1}{\sqrt{2}}[F(\zeta(t),\zeta(t))]^{\frac{1}{2}}\cdot\frac{1}{\sqrt{b-a}}\left(\int_a^b(t-a)(b-t)\left(f^{(n+1)}(t)\right)^2dt\right)^{\frac{1}{2}}.\end{aligned} \tag{34}$$

Divide both sides of (34) by $(n-1)!$, we have

$$\begin{aligned}&\left|\frac{1}{(n-1)!(b-a)}\int_a^b\zeta(t)f^{(n)}(t)\,dt-\frac{1}{(n-1)!(b-a)}\int_a^b\zeta(t)\,dt\cdot\left[f^{(n-1)};a,b\right]\right| \\ &\leqslant\frac{1}{(n-1)!\sqrt{2}}\cdot[F(\zeta(t),\zeta(t))]^{\frac{1}{2}}\cdot\frac{1}{\sqrt{b-a}}\left(\int_a^b(t-a)(b-t)\left(f^{(n+1)}(t)\right)^2dt\right)^{\frac{1}{2}}.\end{aligned} \tag{35}$$

By denoting

$$\begin{aligned}G_n(f;a,b)=&\ \frac{1}{(n-1)!(b-a)}\int_a^b\zeta(t)f^{(n)}(t)\,dt \\ &-\frac{1}{(n-1)!(b-a)}\int_a^b\zeta(t)\,dt\cdot\left[f^{(n-1)};a,b\right]\end{aligned} \tag{36}$$

in (35), we have (33). Now take the value of $\frac{1}{(n-1)!(b-a)}\int_a^b\zeta(t)f^{(n)}(t)dt$ from (36) and substitute in (13), we have (31). □

The following theorem is the integral version of Theorem 3.3.

THEOREM 3.4. *Let* $f:[a,b]\to\mathbb{R}$ *be such that for* $n\geqslant 1$, $f^{(n)}$ *is absolutely continuous with* $(\cdot-a)(b-\cdot)\left(f^{(n+1)}\right)^2\in L[a,b]$. *Let* $p:[c,d]\to\mathbb{R}$ *and* $\varphi,\psi:[c,d]\to[a,b]$ *be continuous functions. If* F *and* $\hat{\zeta}$ *are the same as defined in* (26) *and* (30) *respectively, then we have*

$$\int_c^d p(z) f(\varphi(z))\,dz - \int_c^d p(z) f(\psi(z))\,dz = \sum_{k=1}^{n-1} \left(\frac{n-k}{k!\,(b-a)} \right) \times$$
$$\left[f^{(k-1)}(a) \left(\int_c^d p(z)(\psi(z)-a)^k dz - \int_c^d p(z)(\varphi(z)-a)^k dz \right) \right.$$
$$\left. - f^{(k-1)}(b) \left(\int_c^d p(z)(\psi(z)-b)^k dz - \int_c^d p(z)(\varphi(z)-b)^k dz \right) \right]$$
$$+ \frac{1}{(n-1)!\,(b-a)} \left[f^{(n-1)}; a,b \right] \int_a^b \hat{\zeta}(t)\,dt + \hat{G}_n(f;a,b), \tag{37}$$

where $\left[f^{(n-1)};a,b\right]$ *is the same as defined in* (32) *and the remainder* $\hat{G}_n(f;a,b)$ *satisfies the estimation*

$$\left|\hat{G}_n(f;a,b)\right| \leqslant \frac{\left[F\left(\hat{\zeta}(t),\hat{\zeta}(t)\right)\right]^{\frac{1}{2}}}{(n-1)!\sqrt{2}} \cdot \frac{1}{\sqrt{b-a}} \left(\int_a^b (t-a)(b-t) \left(f^{(n+1)}(t)\right)^2 dt \right)^{\frac{1}{2}}.$$

Proof. The proof is analogous to the proof of Theorem 3.3. We apply Theorem 3.1 for $g \to \hat{\zeta}$ and $h \to f^{(n)}$ and get the desired results. □

The second main result of this section states that:

THEOREM 3.5. *Let* $f : [a,b] \to \mathbb{R}$ *be such that for* $n \geqslant 1$, $f^{(n)}$ *is absolutely continuous and let* $f^{(n+1)} \geqslant 0$ *on* $[a,b]$*. Let* F *and* ζ *be the same as defined in* (26) *and* (29) *respectively. Then we have the representation* (31) *and the remainder* $G_n(f;a,b)$ *satisfies the estimation*

$$|G_n(f;a,b)| \leqslant \frac{\|\zeta'(t)\|_\infty}{(n-1)!} \left(\frac{f^{(n-1)}(a) + f^{(n-1)}(b)}{2} - \left[f^{(n-2)};a,b\right] \right). \tag{38}$$

Proof. By applying Theorem 3.2 for $g \to \zeta$ and $h \to f^{(n)}$, we have

$$\left| \frac{1}{b-a} \int_a^b \zeta(t) f^{(n)}(t)\,dt - \frac{1}{b-a} \int_a^b \zeta(t)\,dt \cdot \frac{1}{b-a} \int_a^b f^{(n)}(t)\,dt \right|$$
$$\leqslant \frac{1}{2(b-a)} \|\zeta'(t)\|_\infty \left(\int_a^b (t-a)(b-t) f^{(n+1)}(t)\,dt \right). \tag{39}$$

Now dividing both sides of (39) by $(n-1)!$ and using the fact that

$$\int_a^b (t-a)(b-t) f^{(n+1)}(t)\,dt = \int_a^b (2t-(a+b)) f^{(n)}(t)\,dt$$
$$= (b-a)\left(f^{(n-1)}(a) + f^{(n-1)}(b)\right) - 2\left(f^{(n-2)}(b) - f^{(n-2)}(a)\right),$$

we have

$$\left|\frac{1}{(n-1)!\,(b-a)}\int_a^b \zeta(t)\,f^{(n)}(t)\,dt-\frac{1}{(n-1)!\,(b-a)}\int_a^b \zeta(t)\,dt\cdot\left[f^{(n-1)};a,b\right]\right|$$
$$\leqslant \frac{1}{2\,(b-a)}\cdot\frac{\|\zeta'(t)\|_\infty}{(n-1)!}\times$$
$$\left((b-a)\left(f^{(n-1)}(a)+f^{(n-1)}(b)\right)-2\left(f^{(n-2)}(b)-f^{(n-2)}(a)\right)\right). \tag{40}$$

By substituting the value of $\frac{1}{(n-1)!(b-a)}\int_a^b \zeta(t)\,f^{(n)}(t)\,dt$ from

$$G_n(f;a,b):=\ \frac{1}{(n-1)!\,(b-a)}\int_a^b \zeta(t)\,f^{(n)}(t)\,dt$$
$$-\frac{1}{(n-1)!\,(b-a)}\int_a^b \zeta(t)\,dt\cdot\left[f^{(n-1)};a,b\right]$$

into (13), we have (31). After simplification, (40) reduces to

$$|G_n(f;a,b)|\leqslant\frac{\|\zeta'(t)\|_\infty}{(n-1)!}\left(\frac{f^{(n-1)}(a)+f^{(n-1)}(b)}{2}-\frac{f^{(n-2)}(b)-f^{(n-2)}(a)}{b-a}\right),$$

which is equivalent to (38). □

An integral version of Theorem 3.5 states that:

THEOREM 3.6. *Let $f:[a,b]\to\mathbb{R}$ be such that for $n\geqslant 1$, $f^{(n)}$ is absolutely continuous and let $f^{(n+1)}\geqslant 0$ on $[a,b]$. Let F and $\hat{\zeta}$ be the same as defined in (26) and (30) respectively. Then we have the representation (37) and the remainder $\hat{G}_n(f;a,b)$ satisfies the estimation*

$$\left|\hat{G}_n(f;a,b)\right|\leqslant\frac{\|\hat{\zeta}'(t)\|_\infty}{(n-1)!}\left(\frac{f^{(n-1)}(a)+f^{(n-1)}(b)}{2}-\left[f^{(n-2)};a,b\right]\right).$$

Proof. The idea of the proof is the same as that of the proof of Theorem 3.5. We apply Theorem 3.2 for $g\to\hat{\zeta}$ and $h\to f^{(n)}$ and get the desired results. □

An Ostrowski-type inequality related to the generalization of the majorization inequality states that:

THEOREM 3.7. *Let all the assumptions of Theorem 2.1 be satisfied. Let (p,q) be a pair of conjugate exponents, that is, $p,q\in[1,\infty]$ such that $\frac{1}{p}+\frac{1}{q}=1$. Let $|f^{(n)}|^p$:*

$[a,b] \to \mathbb{R}$ *be an* $\mathbb{R}$*-integrable function for some* $n \geqslant 2$*. Then we have*

$$\left| \sum_{i=1}^{m} p_i f(x_i) - \sum_{i=1}^{m} p_i f(y_i) - \sum_{k=1}^{n-1} \left(\frac{n-k}{k!(b-a)} \right) \times \right.$$
$$\left[f^{(k-1)}(a) \left(\sum_{i=1}^{m} p_i (y_i - a)^k - \sum_{i=1}^{m} p_i (x_i - a)^k \right) \right.$$
$$\left.\left. - f^{(k-1)}(b) \left(\sum_{i=1}^{m} p_i (y_i - b)^k - \sum_{i=1}^{m} p_i (x_i - b)^k \right) \right] \right|$$
$$\leqslant \left(\int_a^b \left| f^{(n)}(t) \right|^p dt \right)^{\frac{1}{p}} \left(\int_a^b \left| \bar{\zeta}(t) \right|^q dt \right)^{\frac{1}{q}}, \tag{41}$$

where,

$$\bar{\zeta}(t) := \frac{\sum_{i=1}^{m} p_i (x_i - t)^{n-1} k^{[a,b]}(t, x_i) - \sum_{i=1}^{m} p_i (y_i - t)^{n-1} k^{[a,b]}(t, y_i)}{(n-1)!(b-a)}.$$

The constant $\left(\int_a^b \left| \bar{\zeta}(t) \right|^q dt \right)^{\frac{1}{q}}$ *is sharp for* $1 < p \leqslant \infty$ *and best possible for* $p = 1$*.*

Proof. From identity (13), we have

$$\left| \sum_{i=1}^{m} p_i f(x_i) - \sum_{i=1}^{m} p_i f(y_i) - \sum_{k=1}^{n-1} \left(\frac{n-k}{k!(b-a)} \right) \times \right.$$
$$\left[f^{(k-1)}(a) \left(\sum_{i=1}^{m} p_i (y_i - a)^k - \sum_{i=1}^{m} p_i (x_i - a)^k \right) \right.$$
$$\left.\left. - f^{(k-1)}(b) \left(\sum_{i=1}^{m} p_i (y_i - b)^k - \sum_{i=1}^{m} p_i (x_i - b)^k \right) \right] \right|$$
$$= \left| \int_a^b f^{(n)}(t) \bar{\zeta}(t) dt \right|. \tag{42}$$

Apply Hölder's inequality for integrals on the right hand side of (42), we have

$$\left| \int_a^b f^{(n)}(t) \bar{\zeta}(t) dt \right| \leqslant \left(\int_a^b \left| f^{(n)}(t) \right|^p dt \right)^{\frac{1}{p}} \left(\int_a^b \left| \bar{\zeta}(t) \right|^q dt \right)^{\frac{1}{q}}, \tag{43}$$

which combined together with (42) gives (41).

In order to prove the sharpness of the constant $\left(\int_a^b \left| \bar{\zeta}(t) \right|^q dt \right)^{\frac{1}{q}}$, we define a function

$$f^{(n)}(t) = \begin{cases} \operatorname{sgn}\bar{\zeta}(t) \left| \bar{\zeta}(t) \right|^{\frac{1}{p-1}}, & 1 < p < \infty, \\ \operatorname{sgn}\bar{\zeta}(t), & p = \infty, \end{cases}$$

such that the equality in (43) holds.

For $p=1$, we will prove that

$$\left|\int_a^b f^{(n)}(t)\bar{\zeta}(t)\,dt\right| \leqslant \max_{t\in[a,b]}\left|\bar{\zeta}(t)\right|\left(\int_a^b\left|f^{(n)}(t)\right|dt\right) \tag{44}$$

is the best possible inequality. Suppose that $\left|\bar{\zeta}(t)\right|$ attains its maximum at $t_0\in[a,b]$. First assume that $\bar{\zeta}(t_0)>0$. For ε (small enough), if we define

$$f_\varepsilon(t)=\begin{cases}0, & a\leqslant t\leqslant t_0,\\ \frac{1}{\varepsilon n!}(t-t_0)^n, & t_0\leqslant t\leqslant t_0+\varepsilon,\\ \frac{1}{(n-1)!}(t-t_0)^{n-1}, & t_0+\varepsilon\leqslant t\leqslant b,\end{cases}$$

then it is easy to see that

$$\left|\int_a^b f_\varepsilon^{(n)}(t)\bar{\zeta}(t)\,dt\right| = \left|\int_{t_0}^{t_0+\varepsilon}\frac{1}{\varepsilon}\bar{\zeta}(t)\,dt\right| = \frac{1}{\varepsilon}\int_{t_0}^{t_0+\varepsilon}\bar{\zeta}(t)\,dt,$$

and

$$\int_a^b\left|f_\varepsilon^{(n)}(t)\right|dt=\int_{t_0}^{t_0+\varepsilon}\frac{1}{\varepsilon}dt=1.$$

Now using the above two results in (44) and as $\left|\bar{\zeta}(t)\right|$ attains its maximum at $t_0\in[a,b]$, we have

$$\frac{1}{\varepsilon}\int_{t_0}^{t_0+\varepsilon}\bar{\zeta}(t)\,dt\leqslant\bar{\zeta}(t_0)\cdot 1=\bar{\zeta}(t_0).$$

As

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{t_0}^{t_0+\varepsilon}\bar{\zeta}(t)\,dt=\bar{\zeta}(t_0),$$

the statement follows. For the case $\bar{\zeta}(t_0)<0$, define

$$f_\varepsilon(t)=\begin{cases}\frac{1}{(n-1)!}(t-t_0-\varepsilon)^{n-1}, & a\leqslant t\leqslant t_0,\\ -\frac{1}{\varepsilon n!}(t-t_0-\varepsilon)^n, & t_0\leqslant t\leqslant t_0+\varepsilon,\\ 0, & t_0+\varepsilon\leqslant t\leqslant b,\end{cases}$$

and the remaining part is the same as above. □

The following theorem is the integral version of Theorem 3.7.

THEOREM 3.8. *Let all the assumptions of Theorem 2.2 be satisfied. Let (p,q) be a pair of conjugate exponents, that is, $p,q\in[1,\infty]$ such that $\frac{1}{p}+\frac{1}{q}=1$. Let $|f^{(n)}|^p:$*

$[a,b] \to \mathbb{R}$ be an $\mathbb{R}$-integrable function for some $n \geqslant 2$. Then we have

$$\left| \begin{array}{c} \int_c^d p(z) f(\varphi(z))\, dz - \int_c^d p(z) f(\psi(z))\, dz - \sum_{k=1}^{n-1} \left(\frac{n-k}{k!(b-a)}\right) \times \\ \left(\begin{array}{c} f^{(k-1)}(a) \left(\int_c^d p(z)(\psi(z)-a)^k dz - \int_c^d p(z)(\varphi(z)-a)^k dz\right) \\ -f^{(k-1)}(b) \left(\int_c^d p(z)(\psi(z)-b)^k dz - \int_c^d p(z)(\varphi(z)-b)^k dz\right) \end{array} \right) \end{array} \right|$$
$$\leqslant \left(\int_a^b \left|f^{(n)}(t)\right|^p dt\right)^{\frac{1}{p}} \left(\int_a^b \left|\ddot{\zeta}(t)\right|^q dt\right)^{\frac{1}{q}},$$

where,

$$\ddot{\zeta}(t) := \frac{\int_c^d p(z)(\varphi(z)-t)^{n-1} k^{[a,b]}(t,\varphi(z))\, dz - \int_c^d p(z)(\psi(z)-t)^{n-1} k^{[a,b]}(t,\psi(z))\, dz}{(n-1)!\,(b-a)}.$$

The constant $\left(\int_a^b \left|\ddot{\zeta}(t)\right|^q dt\right)^{\frac{1}{q}}$ is sharp for $1 < p \leqslant \infty$ and best possible for $p = 1$.

Proof. The proof is analogous to the proof of Theorem 3.7 but we use identity (14) instead of using (13). □

4. n-Exponential Convexity and Log-Convexity

We begin this section by recollecting definitions and properties which are going to be explored here and we also study some useful characterizations of these properties. In the sequel, let I be an open interval in $\mathbb{R}$.
The following definitions are given in [8].

DEFINITION 6. A function $f : I \to \mathbb{R}$ is n-exponentially convex in the Jensen sense if

$$\sum_{i,j=1}^{n} \varsigma_i \varsigma_j f\left(\frac{x_i + x_j}{2}\right) \geqslant 0$$

holds for every $\varsigma_i \in \mathbb{R}$ and $x_i \in I$ $(i = 1, \ldots, n)$.

DEFINITION 7. A function $f : I \to \mathbb{R}$ is n-exponentially convex if it is n-exponentially convex in the Jensen sense and continuous on I.

REMARK 4.1. From the above definition it is clear that 1-exponentially convex functions in the Jensen sense are non-negative functions. Also, n-exponentially convex functions in the Jensen sense are k-exponentially convex functions in the Jensen sense for all $k \in \mathbb{N}$, $k \leqslant n$.

By definition of positive semi-definite matrices and some basic linear algebra, we have the following proposition:

PROPOSITION 4.2. *If f is n-exponentially convex in the Jensen sense on I, then the matrix $\left[f\left(\frac{x_i+x_j}{2}\right)\right]_{i,j=1}^{k}$ is positive semi-definite for all $k \in \mathbb{N}$, $k \leqslant n$. Particularly,*

$$\det\left[f\left(\frac{x_i+x_j}{2}\right)\right]_{i,j=1}^{k} \geqslant 0 \quad \text{for every } k \in \mathbb{N},\ k \leqslant n,\ \ x_i \in I,\ \ i=1,\ldots,n.$$

DEFINITION 8. A function $f : I \to \mathbb{R}$ is exponentially convex in the Jensen sense if it is n-exponentially convex in the Jensen sense for all $n \in \mathbb{N}$.

DEFINITION 9. A function $f : I \to \mathbb{R}$ is exponentially convex if it is exponentially convex in the Jensen sense and continuous.

REMARK 4.3. It follows that a positive function is log-convex in the Jensen sense if and only if it is 2-exponentially convex in the Jensen sense. Also, by using basic convexity theory, a positive function is log-convex if and only if it is 2-exponentially convex.

Next, we study the n-exponential convexity and log-convexity of the functions associated with the linear functionals Φ_i $(i=1,2)$ as defined in (23) and (24).

THEOREM 4.4. *Let $\Omega = \{f_s : s \in I \subseteq \mathbb{R}\}$ be a family of functions defined on $[a,b]$ such that the function $s \mapsto [z_0,\ldots,z_n; f_s]$ is n-exponentially convex in the Jensen sense on I for every $(n+1)$ mutually distinct points $z_0,\ldots,z_n \in [a,b]$. Let Φ_i $(i=1,2)$ be linear functionals as defined in (23) and (24). Then the following statements hold:*

(i) *The function $s \mapsto \Phi_i(f_s)$ is n-exponentially convex in the Jensen sense on I and the matrix $\left[\Phi_i\left(f_{\frac{s_j+s_k}{2}}\right)\right]_{j,k=1}^{m}$ is positive semi-definite for all $m \in \mathbb{N}$, $m \leqslant n$ and $s_1,\ldots,s_m \in I$. Particularly,*

$$\det\left[\Phi_i\left(f_{\frac{s_j+s_k}{2}}\right)\right]_{j,k=1}^{m} \geqslant 0,\ \forall\ m \in \mathbb{N},\ m \leqslant n.$$

(ii) *If the function $s \mapsto \Phi_i(f_s)$ is continuous on I, then it is n-exponentially convex on I.*

Proof. The idea of the proof is the same as that of the proof of Theorem 3.1 in [8].

(i) Let $\varsigma_j \in \mathbb{R}$ $(j=1,\ldots,n)$ and consider the function

$$\Delta(z) = \sum_{j,k=1}^{n} \varsigma_j \varsigma_k f_{\frac{s_j+s_k}{2}}(z),$$

where $s_j \in I$ and $f_{\frac{s_j+s_k}{2}} \in \Omega$. Then

$$[z_0,\ldots,z_n;\Delta] = \sum_{j,k=1}^{n} \varsigma_j \varsigma_k \left[z_0,\ldots,z_n; f_{\frac{s_j+s_k}{2}}\right]$$

and since $\left[z_0,\ldots,z_n; f_{\frac{s_j+s_k}{2}}\right]$ is n-exponentially convex in the Jensen sense on I by assumption, it follows that

$$\begin{aligned}
[z_0,\ldots,z_n;\Delta] &= \sum_{j,k=1}^{n} \varsigma_j \varsigma_k \left[z_0,\ldots,z_n; f_{\frac{s_j+s_k}{2}}\right] \geqslant 0 \\
&= \left[z_0,\ldots,z_n; \sum_{j,k=1}^{n} \varsigma_j \varsigma_k f_{\frac{s_j+s_k}{2}}\right] \geqslant 0
\end{aligned}$$

and so by using Definition 5, we conclude that Δ is n-convex. Hence

$$\Phi_i(\Delta) \geqslant 0, \qquad i = 1,2,$$

which is equivalent to

$$\sum_{j,k=1}^{n} \varsigma_j \varsigma_k \Phi_i\left(f_{\frac{s_j+s_k}{2}}\right) \geqslant 0, \qquad i = 1,2,$$

and so we conclude that the function $s \mapsto \Phi_i(f_s)$ is n-exponentially convex in the Jensen sense on I.

The remaining part follows from Proposition 4.2.

(ii) If the function $s \mapsto \Phi_i(f_s)$ is continuous on I, then from (i) and by Definition 7 it follows that it is n-exponentially convex on I. □

The following corollary is an immediate consequence of the Theorem 4.4.

COROLLARY 4.5. *Let $\Omega = \{f_s : s \in I \subseteq \mathbb{R}\}$ be a family of functions defined on $[a,b]$ such that the function $s \mapsto [z_0,\ldots,z_n; f_s]$ is exponentially convex in the Jensen sense on I for every $(n+1)$ mutually distinct points $z_0,\ldots,z_n \in [a,b]$. Let Φ_i $(i = 1,2)$ be linear functionals as defined in (23) and (24). Then the following statements hold:*

(i) *The function $s \mapsto \Phi_i(f_s)$ is exponentially convex in the Jensen sense on I and the matrix $\left[\Phi_i\left(f_{\frac{s_j+s_k}{2}}\right)\right]_{j,k=1}^{m}$ is positive semi-definite for all $m \in \mathbb{N}$, $m \leqslant n$ and $s_1,\ldots,s_m \in I$. Particularly,*

$$\det\left[\Phi_i\left(f_{\frac{s_j+s_k}{2}}\right)\right]_{j,k=1}^{m} \geqslant 0, \ \forall\, m \in \mathbb{N},\ m \leqslant n.$$

(ii) *If the function $s \mapsto \Phi_i(f_s)$ is continuous on I, then it is exponentially convex on I.*

COROLLARY 4.6. *Let $\Omega = \{f_s : s \in I \subseteq \mathbb{R}\}$ be a family of functions defined on $[a,b]$ such that the function $s \mapsto [z_0,\ldots,z_n; f_s]$ is 2-exponentially convex in the Jensen sense on I for every $(n+1)$ mutually distinct points $z_0,\ldots,z_n \in [a,b]$. Let Φ_i $(i=1,2)$ be linear functionals as defined in (23) and (24). Further, assume that $\Phi_i(f_s)$ $(i=1,2)$ is strictly positive for $f_s \in \Omega$. Then the following statements hold:*

(i) *If the function $s \mapsto \Phi_i(f_s)$ is continuous on I, then it is 2-exponentially convex on I and so it is* log*-convex on I and for $r,s,t \in I$ such that $r<t<s$, we have*

$$[\Phi_i(f_t)]^{s-r} \leqslant [\Phi_i(f_r)]^{s-t}[\Phi_i(f_s)]^{t-r}, \qquad i=1,2, \tag{45}$$

known as Lyapunov's inequality. If $r<s<t$ or $t<r<s$, then opposite inequalities hold in (45).

(ii) *If the function $s \mapsto \Phi_i(f_s)$ is differentiable on I, then for every $s,q,u,v \in I$ such that $s \leqslant u$ and $q \leqslant v$, we have*

$$\mu_{s,q}(\Phi_i,\Omega) \leqslant \mu_{u,v}(\Phi_i,\Omega), \qquad i=1,2, \tag{46}$$

where

$$\mu_{s,q}(\Phi_i,\Omega) = \begin{cases} \left(\dfrac{\Phi_i(f_s)}{\Phi_i(f_q)}\right)^{\frac{1}{s-q}}, & s \neq q, \\ \exp\left(\dfrac{\frac{d}{ds}\Phi_i(f_s)}{\Phi_i(f_s)}\right), & s=q, \end{cases} \tag{47}$$

for $f_s, f_q \in \Omega$.

Proof. The idea of the proof is the same as that of the proof of Corollary 3.2 given in [8].

(i) The claim that the function $s \mapsto \Phi_i(f_s)$ is log-convex on I is an immediate consequence of Theorem 4.4 and Remark 4.3, and (45) can be obtained by replacing the convex function f with the convex function $f(z) = \log \Phi_i(f_z)$ for $z=r,s,t$ in (1), where $r,s,t \in I$ such that $r<t<s$.

(ii) Since by (i) the function $s \mapsto \Phi_i(f_s)$ is log-convex on I, that is, the function $s \mapsto \log\Phi_i(f_s)$ is convex on I. Applying Theorem 1.1 with setting $f(z) = \log\Phi_i(f_z)$, we have

$$\frac{\log\Phi_i(f_s) - \log\Phi_i(f_q)}{s-q} \leqslant \frac{\log\Phi_i(f_u) - \log\Phi_i(f_v)}{u-v}, \tag{48}$$

for $s \leqslant u$, $q \leqslant v$, $s \neq q, u \neq v$; and therefore, we conclude that

$$\mu_{s,q}(\Phi_i,\Omega) \leqslant \mu_{u,v}(\Phi_i,\Omega), \qquad i=1,2.$$

If $s=q$, we consider the limit when $q \to s$ in (48) and conclude that

$$\mu_{s,s}(\Phi_i,\Omega) \leqslant \mu_{u,v}(\Phi_i,\Omega), \qquad i=1,2.$$

The case $u=v$ can be treated similarly. $\square$

REMARK 4.7. Note that the results from Theorem 4.4, Corollary 4.5 and Corollary 4.6 still hold when two of the points $z_0,\ldots,z_n \in [a,b]$ coincide, say $z_1 = z_0$, for a family of differentiable functions f_s such that the function $s \mapsto [z_0,\ldots,z_n; f_s]$ is n-exponentially convex in the Jensen sense (exponentially convex in the Jensen sense, log-convex in the Jensen sense on I); and furthermore, they still hold when all $(n+1)$ points coincide for a family of n-differentiable functions with the same property.

5. Examples

In this section, we present several families of functions which fulfil the conditions of Theorem 4.4, Corollaries 4.5 and 4.6, and Remark 4.7 and so the results of these theorem and corollaries can be applied for them.

EXAMPLE 5.1. Consider the family of functions

$$\Omega_1 = \{f_s : (0,\infty) \to \mathbb{R} : s \in \mathbb{R}\}$$

defined by

$$f_s(x) = \begin{cases} \frac{x^s}{s(s-1)\ldots(s-(n-1))}, & s \notin \{0,1,\ldots,n-1\}, \\ \frac{x^j \ln x}{(-1)^{n-1-j} j!(n-1-j)!}, & s = j \in \{0,1,\ldots,n-1\}. \end{cases}$$

Here, $\frac{d^n}{dx^n} f_s(x) = x^{s-n} = e^{(s-n)\ln x} > 0$, which shows that f_s is n-convex for $x > 0$ and $s \mapsto \frac{d^n}{dx^n} f_s(x)$ is exponentially convex by definition.

In order to prove that the function $s \mapsto [z_0,\ldots,z_n; f_s]$ is exponentially convex, it is enough to show that

$$\sum_{j,k=1}^n \varsigma_j \varsigma_k \left[z_0,\ldots,z_n; f_{\frac{s_j+s_k}{2}}\right] = \left[z_0,\ldots,z_n; \sum_{j,k=1}^n \varsigma_j \varsigma_k f_{\frac{s_j+s_k}{2}}\right] \geqslant 0, \tag{49}$$

$\forall\ n \in \mathbb{N},\ \varsigma_j, s_j \in \mathbb{R},\ j = 1,\ldots,n$. By Definition 5, (49) will hold if

$$\Lambda(x) := \sum_{j,k=1}^n \varsigma_j \varsigma_k f_{\frac{s_j+s_k}{2}}(x)$$

is n-convex. Since $s \mapsto \frac{d^n}{dx^n} f_s(x)$ is exponentially convex, that is

$$\sum_{j,k=1}^n \varsigma_j \varsigma_k f^{(n)}_{\frac{s_j+s_k}{2}} \geqslant 0, \qquad \forall\, n \in \mathbb{N},\ \varsigma_j, s_j \in \mathbb{R},\ j = 1,\ldots,n,$$

showing the n-convexity of Λ and so (49) holds. Now as the function $s \mapsto [z_0,\ldots,z_n; f_s]$ is exponentially convex, $s \mapsto [z_0,\ldots,z_n; f_s]$ is exponentially convex in the Jensen sense and by using Corollary 4.5, we have $s \mapsto \Phi_i(f_s)$ $(i = 1,2)$ is exponentially convex in the Jensen sense. Since these mappings are continuous, so $s \mapsto \Phi_i(f_s)$ $(i = 1,2)$ is exponentially convex.

In this case, $\mu_{s,q}(\Phi_i,\Omega)$ $(i=1,2)$ defined in (47) becomes

$$\mu_{s,q}(\Phi_i,\Omega_1)=\begin{cases}\left(\frac{\Phi_i(f_s)}{\Phi_i(f_q)}\right)^{\frac{1}{s-q}}, & s\neq q,\\ \exp\left(\frac{(-1)^{n-1}(n-1)!\Phi_i(f_0f_s)}{\Phi_i(f_s)}+\sum_{k=0}^{n-1}\frac{1}{k-s}\right), & s=q\notin\{0,1,\dots,n-1\},\\ \exp\left(\frac{(-1)^{n-1}(n-1)!\Phi_i(f_0f_s)}{2\Phi_i(f_s)}+\sum_{\substack{k=0\\k\neq s}}^{n-1}\frac{1}{k-s}\right), & s=q\in\{0,1,\dots,n-1\}.\end{cases}$$

In particular for $i=1$, we have

$$\begin{aligned}\Phi_1(f_s)&=\sum_{i=1}^m p_i f_s(x_i)-\sum_{i=1}^m p_i f_s(y_i)\\&-\sum_{k=1}^{n-1}\left(\frac{n-k}{k!(b-a)}\right)\left[f_s^{(k-1)}(a)\left(\sum_{i=1}^m p_i(y_i-a)^k-\sum_{i=1}^m p_i(x_i-a)^k\right)\right.\\&\qquad\left.-f_s^{(k-1)}(b)\left(\sum_{i=1}^m p_i(y_i-b)^k-\sum_{i=1}^m p_i(x_i-b)^k\right)\right]\end{aligned}$$

and

$$\begin{aligned}\Phi_1(f_0f_s)A_s&=\sum_{i=1}^m p_i x_i^s\ln x_i-\sum_{i=1}^m p_i y_i^s\ln y_i\\&-\sum_{k=1}^{n-1}\left(\frac{n-k}{k!(b-a)}\right)\left[B_{k,s}(a)\left(\sum_{i=1}^m p_i(y_i-a)^k-\sum_{i=1}^m p_i(x_i-a)^k\right)\right.\\&\qquad\left.-B_{k,s}(b)\left(\sum_{i=1}^m p_i(y_i-b)^k-\sum_{i=1}^m p_i(x_i-b)^k\right)\right],\end{aligned}$$

where $A_s=(-1)^{n-1}(n-1)!\prod_{i=0}^{n-1}(s-i)$ such that $s\neq 0,1,\dots,n-1$ and

$$B_{k,s}(x)=x^{s-(k-1)}\left(\prod_{i=0}^{k-1}(s-i)\ln x+\sum_{i=0}^{k-1}\prod_{\substack{j=0\\j\neq i}}^{k-1}(s-j)\right).$$

If Φ_i $(i=1,2)$ is positive, then Theorem 2.9 applied for $f=f_s\in\Omega_1$ and $k=f_q\in\Omega_1$ yields that there exists $\xi_i\in[a,b]$ such that

$$\xi_i^{s-q}=\frac{\Phi_i(f_s)}{\Phi_i(f_q)},\qquad i=1,2.$$

Since the function $\xi_i\mapsto\xi_i^{s-q}$ is invertible for $s\neq q$, we have

$$a\leqslant\left(\frac{\Phi_i(f_s)}{\Phi_i(f_q)}\right)^{\frac{1}{s-q}}\leqslant b,\qquad i=1,2,$$

which together with the fact that $\mu_{s,q}(\Phi_i,\Omega_1)$ is continuous, symmetric and monotonous (by (46)), shows that $\mu_{s,q}(\Phi_i,\Omega_1)$ is a mean.

EXAMPLE 5.2. Consider the family of functions

$$\Omega_2 = \{g_s : \mathbb{R} \to [0, \infty) : s \in \mathbb{R}\}$$

defined by

$$g_s(x) = \begin{cases} \frac{e^{sx}}{s^n}, & s \neq 0, \\ \frac{x^n}{n!}, & s = 0. \end{cases}$$

We have $\frac{d^n}{dx^n} g_s(x) = e^{sx} > 0$, which shows that g_s is n-convex on $\mathbb{R}$ for every $s \in \mathbb{R}$ and $s \mapsto \frac{d^n}{dx^n} g_s(x)$ is exponentially convex by definition. It is easy to prove that the function $s \mapsto [z_0,\dots,z_n; g_s]$ is exponentially convex. Arguing as in Example 5.1, we have $s \mapsto \Phi_i(g_s)$ $(i = 1,2)$ is exponentially convex.

For this family of functions, $\mu_{s,q}(\Phi_i,\Omega)$ $(i = 1,2)$ from (47) becomes

$$\mu_{s,q}(\Phi_i,\Omega_2) = \begin{cases} \left(\frac{\Phi_i(g_s)}{\Phi_i(g_q)}\right)^{\frac{1}{s-q}}, & s \neq q, \\ \exp\left(\frac{\Phi_i(id \cdot g_s)}{\Phi_i(g_s)} - \frac{n}{s}\right), & s = q \neq 0, \\ \exp\left(\frac{\Phi_i(id \cdot g_0)}{(n+1)\Phi_i(g_0)}\right), & s = q = 0, \end{cases}$$

where id is the identity function.

By using Theorem 2.9, it can be seen that

$$M_{s,q}(\Phi_i,\Omega_2) = \log \mu_{s,q}(\Phi_i,\Omega_2), \qquad i = 1,2,$$

satisfy $a \leqslant M_{s,q}(\Phi_i,\Omega_2) \leqslant b$, which shows that $M_{s,q}(\Phi_i,\Omega_2)$ is a mean.

EXAMPLE 5.3. Consider the family of functions

$$\Omega_3 = \{h_s : (0,\infty) \to (0,\infty) : s \in (0,\infty)\}$$

defined by

$$h_s(x) = \begin{cases} \frac{s^{-x}}{(-\ln s)^n}, & s \neq 1, \\ \frac{x^n}{n!}, & s = 1. \end{cases}$$

We have $\frac{d^n}{dx^n} h_s(x) = s^{-x} > 0$, which shows that h_s is n-convex for all $s > 0$. Since $s \mapsto \frac{d^n}{dx^n} h_s(x) = s^{-x}$ is the Laplace transform of a non-negative function (see [9]), it is exponentially convex. It is easy to prove that the function $s \mapsto [z_0,\dots,z_n; h_s]$ is exponentially convex. Arguing as in Example 5.1, we have $s \mapsto \Phi_i(h_s)$ $(i = 1,2)$ is exponentially convex.

For this family of functions, $\mu_{s,q}(\Phi_i,\Omega)$ $(i=1,2)$ from (47) becomes

$$\mu_{s,q}(\Phi_i,\Omega_3)=\begin{cases}\left(\frac{\Phi_i(h_s)}{\Phi_i(h_q)}\right)^{\frac{1}{s-q}}, & s\neq q,\\ \exp\left(-\frac{\Phi_i(id\cdot h_s)}{s\Phi_i(h_s)}-\frac{n}{s\ln s}\right), & s=q\neq 1,\\ \exp\left(-\frac{\Phi_i(id\cdot h_1)}{(n+1)\Phi_i(h_1)}\right), & s=q=1.\end{cases}$$

By using Theorem 2.9, it follows that

$$M_{s,q}(\Phi_i,\Omega_3)=-L(s,q)\log\mu_{s,q}(\Phi_i,\Omega_3),\qquad i=1,2,$$

satisfy $a\leqslant M_{s,q}(\Phi_i,\Omega_3)\leqslant b$ and so $M_{s,q}(\Phi_i,\Omega_3)$ is a mean, where $L(s,q)$ is a logarithmic mean defined by

$$L(s,q)=\begin{cases}\frac{s-q}{\log s-\log q}, & s\neq q,\\ s, & s=q.\end{cases}$$

EXAMPLE 5.4. Consider the family of functions

$$\Omega_4=\{k_s:(0,\infty)\to(0,\infty):s\in(0,\infty)\}$$

defined by

$$k_s(x)=\frac{e^{-x\sqrt{s}}}{(-\sqrt{s})^n}.$$

Here, $\frac{d^n}{dx^n}k_s(x)=e^{-x\sqrt{s}}>0$, which shows that k_s is n-convex for all $s>0$. Since $s\mapsto\frac{d^n}{dx^n}k_s(x)=e^{-x\sqrt{s}}$ is the Laplace transform of a non-negative function (see [9]), it is exponentially convex. It is easy to prove that the function $s\mapsto[z_0,\ldots,z_n;k_s]$ is exponentially convex. Arguing as in Example 5.1, we have $s\mapsto\Phi_i(k_s)$ $(i=1,2)$ is exponentially convex.

In this case, $\mu_{s,q}(\Phi_i,\Omega)$ $(i=1,2)$ defined in (47), is of the form

$$\mu_{s,q}(\Phi_i,\Omega_4)=\begin{cases}\left(\frac{\Phi_i(k_s)}{\Phi_i(k_q)}\right)^{\frac{1}{s-q}}, & s\neq q,\\ \exp\left(-\frac{\Phi_i(id\cdot k_s)}{2\sqrt{s}\Phi_i(k_s)}-\frac{n}{2s}\right), & s=q.\end{cases}$$

By using Theorem 2.9, it is easy to see that

$$M_{s,q}(\Phi_i,\Omega_4)=-\left(\sqrt{s}+\sqrt{q}\right)\log\mu_{s,q}(\Phi_i,\Omega_4),\qquad i=1,2,$$

satisfy $a\leqslant M_{s,q}(\Phi_i,\Omega_4)\leqslant b$, showing that $M_{s,q}(\Phi_i,\Omega_4)$ is a mean.

REMARK 5.5. (i) From (46), it is clear that $\mu_{s,q}(\Phi_i,\Omega)$ $(i=1,2)$ for $\Omega=\Omega_2,\Omega_3$ and Ω_4 are monotonous functions in parameters s and q.

(ii) In Examples 5.2, 5.3 and 5.4, we can also give particular cases for Φ_i $(i=1,2)$ as given in Example 5.1.

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