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ON SUM AND PRODUCT THEOREMS RELATED TO RELATIVE L^*–TYPE AND RELATIVE L^*–WEAK TYPE OF ENTIRE FUNCTIONS

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Abstract. In this paper we would like to investigate some basic properties of relative L^*-type and relative L^*-weak type of entire functions.

1. Introduction

In the value distribution theory as introduced by Rolf Nevanlinna in 1926, the role of the growth indicators like *order* and *lower order* is very much significant in the study of comparative growth analysis of entire functions. The rate of growth of an entire function generally depends upon *order (lower order)* of it. The entire function with *higher order* is of faster growth than that of *lesser order*. But if orders of two entire functions are same, then it is impossible to detect the function with faster growth. In that case, it is necessary to compute another class of growth indicators of entire functions called their *types*. For further study on it, one may see [6]. However, if one is interested to compare the growth rates of any entire function with respect to another, the concepts of *relative growth* indicators will come. The most mordern treatment upon this area of research is the study of the same in terms of a slowly changing function $L(r)$ which means that $L(ar) \sim L(r)$ as $r \to \infty$ for every positive constant a i.e., $\lim\limits_{r\to\infty} \frac{L(ar)}{L(r)} = 1$ where $L \equiv L(r)$ is a positive continuous function.

In fact, in this paper we wish to prove some results related to the sum and product theorems of *relative* L^**-type* and *relative* L^**-weak type* of entire functions under somewhat different conditions.where L^* is nothing but a weaker assumption of L.

2. Definitions and Notations

The standard notations and definitions of the theory of entire functions frequently used in this paper are available in [9] and therefore we do not explain those in details.

Let $\mathbb{C}$ be the set of all finite complex numbers and f be an entire function defined on it. The Nevanlinna's characteristic function $T_f(r)$ and the maximum modulus function $M_f(r)$ of f are defined as $T_f(r) = \frac{1}{2\pi}\int\limits_0^{2\pi} \log^+ \left|f(re^{i\theta})\right| d\theta$ and $M_f(r) = \max\{|f(z)| : |z| = r\}$ respectively where $\log^+ x = \max(0, \log x)$ for $x > 0$. If f is

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non-constant then $M_f(r)$ is strictly increasing and continuous and its inverse $M_f^{-1}(r):(|f(0)|,\infty)\to(0,\infty)$ exists and is such that $\lim\limits_{s\to\infty} M_f^{-1}(s)=\infty$ and given two entire functions f and g the ratio $\frac{M_f(r)}{M_g(r)}$ as $r\to\infty$ is called the growth of f with respect to g in terms of their maximum moduli.

Somasundaram and Thamizharasi [8] introduced the notions of L-*order* for entire function f. The more generalised concept for L-*order* for entire function f is L^*-*order* which is as follows:

DEFINITION 1. [8] The L^*-*order* $\rho_f^{L^*}$ and the L^*-*lower order* $\lambda_f^{L^*}$ of an entire function f are defined as

$$\rho_f^{L^*}=\limsup_{r\to\infty}\frac{\log^{[2]}M_f(r)}{\log\left[re^{L(r)}\right]} \quad \text{and} \quad \lambda_f^{L^*}=\liminf_{r\to\infty}\frac{\log^{[2]}M_f(r)}{\log\left[re^{L(r)}\right]},$$

where $\log^{[k]}x=\log\left(\log^{[k-1]}x\right)$ for $k=1,2,3,....$ and $\log^{[0]}x=x$.

An entire function for which L^*-*order* and L^*-*lower order* are the same is said to be of *regular* L^*-*growth*. Functions which are not of *regular* L^*-*growth* are said to be of *irregular* L^*-*growth*.

DEFINITION 2. [8] The L^*-*type* $\sigma_f^{L^*}$ of an entire function f is defined as

$$\sigma_f^{L^*}=\limsup_{r\to\infty}\frac{\log M_f(r)}{\left[re^{L(r)}\right]^{\rho_f^{L^*}}},\quad 0<\rho_f^{L^*}<\infty.$$

Similarly one can define the L^*-*lower type* of an entire function f denoted by $\overline{\sigma}_f^{L^*}$ as follows:

$$\overline{\sigma}_f^{L^*}=\liminf_{r\to\infty}\frac{\log M_f(r)}{\left[re^{L(r)}\right]^{\rho_f^{L^*}}},\quad 0<\rho_f^{L^*}<\infty.$$

In order to determine the relative growth of two entire functions of same non zero finite L^*-*lower order* one may define the L^*-*weak type* in the following way:

DEFINITION 3. The L^*-*weak type* $\tau_f^{L^*}$ of an entire function f is defined as follows:

$$\tau_f^{L^*}=\liminf_{r\to\infty}\frac{\log M_f(r)}{\left[re^{L(r)}\right]^{\lambda_f^{L^*}}},\quad 0<\lambda_f^{L^*}<\infty.$$

Likewise one may also define the growth indicator $\overline{\tau}_f^{L^*}$ of an entire function f in the following way:

$$\overline{\tau}_f^{L^*}=\liminf_{r\to\infty}\frac{\log M_f(r)}{\left[re^{L(r)}\right]^{\lambda_f^{L^*}}},\quad 0<\lambda_f^{L^*}<\infty.$$

From Definition 1, we see that the L^**-order* of an entire function f which is generally used in computational purpose is defined in terms of the growth of f with respect to the exponential function as:

$$\rho_f^{L^*} = \limsup_{r\to\infty} \frac{\log^{[2]} M_f(r)}{\log^{[2]} M_{\exp z}\left[re^{L(r)}\right]} = \limsup_{r\to\infty} \frac{\log^{[2]} M_f(r)}{\log\left[re^{L(r)}\right]}.$$

In the line of Somasundaram and Thamizharasi [7], Datta and Biswas [2] introduced the definition of *relative* L^**-order* of entire functions in order to avoid comparing *growth* of the same just with $\exp z$ in the following way:

DEFINITION 4. [2] The *relative* L^**-order* of an entire function f with respect to another entire function g, denoted by $\rho_g^{L^*}(f)$ is defined in the following way

$$\rho_g^{L^*}(f) = \limsup_{r\to\infty} \frac{\log M_g^{-1} M_f(r)}{\log\left[re^{L(r)}\right]}.$$

Similarly, one can define the *relative* L^**-lower order* of f with respect to g denoted by $\lambda_g^{L^*}(f)$ as follows:

$$\lambda_g^{L^*}(f) = \liminf_{r\to\infty} \frac{\log M_g^{-1} M_f(r)}{\log\left[re^{L(r)}\right]}.$$

The definitions coincide with the classical one if $g(z) = \exp z$.

It is to be mentioned that an entire function f is said to be of *regular relative* L^**-growth* with respect to g if its *relative* L^**-order* with respect to g coincides with its *relative* L^**-lower order* with respect to g.

To compare the *relative* L^**-growth* of two entire functions having same non zero finite *relative* L^**-order* with respect to another entire function, Datta, Biswas and Bhattacharyya [3] recently introduced the notion of *relative* L^**-type* of two entire functions in the following manner:

DEFINITION 5. [3] Let f and g be any two entire functions such that $0 < \rho_g^{L^*}(f) < \infty$. Then the *relative* L^**-type* $\sigma_g^{L^*}(f)$ of f with respect to g is defined as:

$$\begin{aligned}
&\sigma_g^{L^*}(f) \\
&= \inf \left\{ \begin{array}{c} k > 0 : M_f(r) < M_g\left(k\left[re^{L(r)}\right]^{\rho_g^{L^*}(f)}\right) \\ \text{for all sufficiently large values of } r \end{array} \right\} \\
&= \limsup_{r\to\infty} \frac{M_g^{-1} M_f(r)}{\left[re^{L(r)}\right]^{\rho_g^{L^*}(f)}}.
\end{aligned}$$

Likewise one can define the *relative L^*-lower type* of an entire function f with respect to an entire function g denoted by $\overline{\sigma}_g^{L^*}(f)$ as follows:

$$\overline{\sigma}_g^{L^*}(f) = \liminf_{r\to\infty} \frac{M_g^{-1}M_f(r)}{\left[re^{L(r)}\right]^{\rho_g^{L^*}(f)}}, \quad 0 < \rho_g^{L^*}(f) < \infty.$$

Analogously to determine the relative growth of two entire functions having same non zero finite *relative L^*-lower order* with respect to another entire function, one may introduced the definition of *relative L^*-weak type* of an entire function f with respect to another entire function g of finite positive *relative L^*-lower order* $\lambda_g^{L^*}(f)$ in the following way:

DEFINITION 6. [3] The *relative L^*-weak type* $\tau_g^{L^*}(f)$ of an entire function f with respect to another entire function g having finite positive *relative L^*-lower order* $\lambda_g^{L^*}(f)$ is defined as:

$$\tau_g^{L^*}(f) = \liminf_{r\to\infty} \frac{M_g^{-1}M_f(r)}{\left[re^{L(r)}\right]^{\lambda_g^{L^*}(f)}}.$$

Also one may define the growth indicator $\overline{\tau}_g^{L^*}(f)$ of an entire function f with respect to an entire function g in the following way:

$$\overline{\tau}_g^{L^*}(f) = \limsup_{r\to\infty} \frac{M_g^{-1}M_f(r)}{\left[re^{L(r)}\right]^{\lambda_g^{L^*}(f)}}, \quad 0 < \lambda_g^{L^*}(f) < \infty.$$

Considering $g = \exp z$, one may easily verify that Definition 5 and Definition 6 coincide with the classical L^*-*type* (L^*-*lower type*) and L^*-*weak type* respectively.

In this connection the following definition is relevant:

DEFINITION 7. [1] A non-constant entire function f is said have the *Property (A)* if for any $\sigma > 1$ and for all large r, $\left[M_f(r)\right]^2 \leqslant M_f(r^\sigma)$ holds. For examples of functions with or without the *Property (A)*, one may see [1].

Throughout the paper we consider $\sigma_{f_i}^{L^*}(g_k)$, $\overline{\sigma}_{f_i}^{L^*}(g_k)$, $\tau_{f_i}^{L^*}(g_k)$ and $\overline{\tau}_{f_i}^{L^*}(g_k)$ for entire functions $f_i \mid i = 1,2$ and $g_k \mid k = 1,2$ are all non-zero finite.

3. Theorems

First of all, we recall some related properties of *relative L^*-order* and *relative L^*-lower order* of entire functions as proved by Datta et al. [4] which will be needed in order to prove our main results, as we see in the following four theorems:

THEOREM A. [4] *Let f_1, f_2, g_1 and g_2 be any four entire functions. Then*
(i)

$$\rho_{f_1}^{L^*}(g_1 \pm g_2) \leqslant \rho_{f_1}^{L^*}(g_i)$$

where $\rho_{f_1}^{L^*}(g_i) = \max\left\{\rho_{f_1}^{L^*}(g_k) \mid k = i = 1,2\right\}$. *The sign of equality holds when* $\rho_{f_1}^{L^*}(g_1) \neq \rho_{f_1}^{L^*}(g_2)$*;*

(ii)

$$\rho_{f_1 \pm f_2}^{L^*}(g_1) \geqslant \rho_{f_i}^{L^*}(g_1)$$

where $\rho_{f_i}^{L^*}(g_1) = \min\left\{\rho_{f_k}^{L^*}(g_1) \mid k = i = 1,2\right\}$ *and* g_1 *is of regular relative* L^**-growth with respect to at least any one of* f_1 *or* f_2*. The sign of equality holds when* $\rho_{f_1}^{L^*}(g_1) \neq \rho_{f_2}^{L^*}(g_1)$ *and*

(iii)

$$\rho_{f_1 \pm f_2}^{L^*}(g_1 \pm g_2) \leqslant \max\left[\min\left\{\rho_{f_1}^{L^*}(g_1), \rho_{f_2}^{L^*}(g_1)\right\}, \min\left\{\rho_{f_1}^{L^*}(g_2), \rho_{f_2}^{L^*}(g_2)\right\}\right]$$

when $\rho_{f_1}^{L^*}(g_1) \neq \rho_{f_2}^{L^*}(g_1)$, $\rho_{f_1}^{L^*}(g_2) \neq \rho_{f_2}^{L^*}(g_2)$ *and* g_1 *and* g_1 *are both of regular relative* L^**-growth with respect to at least any one of* f_1 *or* f_2*. The sign of equality holds when* $\min\left\{\rho_{f_1}^{L^*}(g_1), \rho_{f_2}^{L^*}(g_1)\right\} \neq \min\left\{\rho_{f_1}^{L^*}(g_2), \rho_{f_2}^{L^*}(g_2)\right\}$.

THEOREM B. [4] *Let* f_1, f_2, g_1 *and* g_2 *are any four entire functions. Then*

(i)

$$\lambda_{f_1 \pm f_2}^{L^*}(g_1) \geqslant \lambda_{f_i}^{L^*}(g_1)$$

where $\lambda_{f_i}^{L^*}(g_1) = \min\left\{\lambda_{f_k}^{L^*}(g_1) \mid k = i = 1,2\right\}$. *The sign of equality holds when* $\lambda_{f_1}^{L^*}(g_1) \neq \lambda_{f_2}^{L^*}(g_1)$

(ii)

$$\lambda_{f_1}^{L^*}(g_1 \pm g_2) \leqslant \lambda_{f_1}^{L^*}(g_i)$$

where $\lambda_{f_1}^{L^*}(g_i) = \max\left\{\lambda_{f_1}^{L^*}(g_k) \mid k = i = 1,2\right\}$ *and at least* g_1 *or* g_2 *is of regular relative* L^**-growth with respect to* f_1*. The sign of equality holds when* $\lambda_{f_1}^{L^*}(g_1) \neq \lambda_{f_1}^{L^*}(g_2)$ *and*

(iii)

$$\lambda_{f_1 \pm f_2}^{L^*}(g_1 \pm g_2) \geqslant \min\left[\max\left\{\lambda_{f_1}^{L^*}(g_1), \lambda_{f_2}^{L^*}(g_1)\right\}, \max\left\{\lambda_{f_1}^{L^*}(g_2), \lambda_{f_2}^{L^*}(g_2)\right\}\right]$$

when $\lambda_{f_1}^{L^*}(g_1) \neq \lambda_{f_2}^{L^*}(g_1)$, $\lambda_{f_1}^{L^*}(g_2) \neq \lambda_{f_2}^{L^*}(g_2)$ *and at least* g_1 *or* g_2 *is of regular relative* L^**-growth with respect to* f_1 *and* f_2 *respectively. The sign of equality holds when* $\max\left\{\lambda_{f_1}^{L^*}(g_1), \lambda_{f_2}^{L^*}(g_1)\right\} \neq \max\left\{\lambda_{f_1}^{L^*}(g_2), \lambda_{f_2}^{L^*}(g_2)\right\}$.

THEOREM C. [4] *Let* f_1, f_2, g_1 *and* g_2 *are any four entire functions. Then*

(i)

$$\rho_{f_1}^{L^*}(g_1 \cdot g_2) \leqslant \rho_{f_1}^{L^*}(g_i)$$

where $\rho_{f_1}^{L^*}(g_i) = \max\left\{\rho_{f_1}^{L^*}(g_k) \mid k = i = 1,2\right\}$ *and* f_1 *has the Property (A). The sign of equality holds when* $\rho_{f_1}^{L^*}(g_1) \neq \rho_{f_1}^{L^*}(g_2)$*. Similar results hold for the quotient* $\frac{g_1}{g_2}$ *provided* $\frac{g_1}{g_2}$ *is entire;*

(ii)

$$\rho_{f_1 \cdot f_2}^{L^*}(g_1) \geqslant \rho_{f_i}^{L^*}(g_1)$$

where $\rho_{f_i}^{L^*}(g_1) = \min\left\{\rho_{f_k}^{L^*}(g_1) \mid k = i = 1,2\right\}$, $f_1 \cdot f_2$ *has the Property (A) and* g_1 *is of regular relative* L^**-growth with respect to at least any one of* f_1 *or* f_2*. The sign of equality holds when* $\rho_{f_1}^{L^*}(g_1) \neq \rho_{f_2}^{L^*}(g_1)$*. Similar results hold for the quotient* $\frac{f_1}{f_2}$ *provided* $\frac{f_1}{f_2}$ *is entire having the Property (A) and*

(iii)

$$(a)\ \rho_{f_1 \cdot f_2}^{L^*}(g_1 \cdot g_2) \leqslant \max\left[\min\left\{\rho_{f_1}^{L^*}(g_1), \rho_{f_2}^{L^*}(g_1)\right\}, \min\left\{\rho_{f_1}^{L^*}(g_2), \rho_{f_2}^{L^*}(g_2)\right\}\right],$$

$$(b)\ \rho_{\frac{f_1}{f_2}}^{L^*}\left(\frac{g_1}{g_2}\right) \leqslant \max\left[\min\left\{\rho_{f_1}^{L^*}(g_1), \rho_{f_2}^{L^*}(g_1)\right\}, \min\left\{\rho_{f_1}^{L^*}(g_2), \rho_{f_2}^{L^*}(g_2)\right\}\right]$$

when (i) $\rho_{f_1}^{L^*}(g_1) \neq \rho_{f_2}^{L^*}(g_1)$, (ii) $\rho_{f_1}^{L^*}(g_2) \neq \rho_{f_2}^{L^*}(g_2)$ (iii) $f_1 \cdot f_2$, g_1 *and* g_2 *has the Property (A) and* (iv) g_1 *and* g_2 *are both of regular relative* L^**-growth with respect to at least any one of* f_1 *or* f_2*. The sign of equality holds when* $\min\left\{\rho_{f_1}^{L^*}(g_1), \rho_{f_2}^{L^*}(g_1)\right\} \neq \min\left\{\rho_{f_1}^{L^*}(g_2), \rho_{f_2}^{L^*}(g_2)\right\}$.

THEOREM D. [4] *Let* f_1, f_2, g_1 *and* g_2 *be any four entire functions. Then*

(i)

$$\lambda_{f_1 \cdot f_2}^{L^*}(g_1) \geqslant \lambda_{f_i}^{L^*}(g_1)$$

where $\lambda_{f_i}^{L^*}(g_1) = \min\left\{\lambda_{f_k}^{L^*}(g_1) \mid k = i = 1,2\right\}$*. The sign of equality holds when* $\lambda_{f_1}^{L^*}(g_1) \neq \lambda_{f_2}^{L^*}(g_1)$*. Similar results hold for the quotient* $\frac{f_1}{f_2}$ *provided* $\frac{f_1}{f_2}$ *is entire;*

(ii)

$$\lambda_{f_1}^{L^*}(g_1 \cdot g_2) \leqslant \lambda_{f_1}^{L^*}(g_i)$$

where $\lambda_{f_1}^{L^*}(g_i) = \max\left\{\lambda_{f_1}^{L^*}(g_k) \mid k = i = 1,2\right\}$, f_1 *has the Property (A) and at least* g_1 *or* g_2 *is of regular relative* L^**-growth with respect to* f_1*. The sign of equality holds when* $\lambda_{f_1}^{L^*}(g_1) \neq \lambda_{f_1}^{L^*}(g_2)$*. Similar results hold for the quotient* $\frac{g_1}{g_2}$ *provided* $\frac{g_1}{g_2}$ *is entire and*

(iii)

$$(a)\ \lambda_{f_1 \cdot f_2}^{L^*}(g_1 \cdot g_2) \geqslant \min\left[\max\left\{\lambda_{f_1}^{L^*}(g_1), \lambda_{f_2}^{L^*}(g_1)\right\}, \max\left\{\lambda_{f_1}^{L^*}(g_2), \lambda_{f_2}^{L^*}(g_2)\right\}\right],$$

$$(b)\ \lambda_{\frac{f_1}{f_2}}^{L^*}\left(\frac{g_1}{g_2}\right) \geqslant \min\left[\max\left\{\lambda_{f_1}^{L^*}(g_1), \lambda_{f_2}^{L^*}(g_1)\right\}, \max\left\{\lambda_{f_1}^{L^*}(g_2), \lambda_{f_2}^{L^*}(g_2)\right\}\right]$$

when (i) $\lambda_{f_1}^{L^*}(g_1) \neq \lambda_{f_2}^{L^*}(g_1)$, (ii) $\lambda_{f_1}^{L^*}(g_2) \neq \lambda_{f_2}^{L^*}(g_2)$, (iii) $g \cdot g_2$, f_1 *and* f_2 *have the Property (A) and* (iv) *at least* g_1 *or* g_2 *is of regular relative* L^**-growth with respect to* f_1 *and* f_2 *respectively. The sign of equality holds when* $\max\left\{\lambda_{f_1}^{L^*}(g_1), \lambda_{f_2}^{L^*}(g_1)\right\} \neq \max\left\{\lambda_{f_1}^{L^*}(g_2), \lambda_{f_2}^{L^*}(g_2)\right\}$.:

Now in the case of *relative* L^**-type* and *relative* L^**-weak type*, it therefore seems reasonable to study a parallel investigations of its basic properties, which is the prime concern of the paper. In fact, in this paper, under somewhat different conditions we obtain the following theorem related to *relative* L^**-type* (*relative* L^**-lower type*) and *relative weak* L^**-type*:

THEOREM 1. *Let* f_1, f_2, g_1 *and* g_2 *are any four entire functions such that* $\rho_{f_k}^{L^*}(g_k) \mid k = 1,2$ *are non-zero finite.*

(I) If (A) $\rho_{f_1}^{L^*}(g_i) = \max\left\{\rho_{f_1}^{L^*}(g_k) \mid k = i = 1,2\right\}$ *and* (B) $\rho_{f_1}^{L^*}(g_1) \neq \rho_{f_1}^{L^*}(g_2)$, *then*

$$(i)\ \sigma_{f_1}^{L^*}(g_1 \pm g_2) = \sigma_{f_1}^{L^*}(g_i) \mid i = 1,2 \text{ and } (ii)\ \overline{\sigma}_{f_1}^{L^*}(g_1 \pm g_2) = \overline{\sigma}_{f_1}^{L^*}(g_i) \mid i = 1,2.$$

(II) If (A) $\rho_{f_i}^{L^*}(g_1) = \min\left\{\rho_{f_k}^{L^*}(g_1) \mid k = 1,2\right\}$, (B) $\rho_{f_1}^{L^*}(g_1) \neq \rho_{f_2}^{L^*}(g_1)$ *and* (C) g_1 *is of regular relative* L^**-growth with respect to at least any one of* f_1 *or* f_2, *then*

$$(i)\ \sigma_{f_1 \pm f_2}^{L^*}(g_1) = \sigma_{f_i}^{L^*}(g_1) \mid i = 1,2 \text{ and } (ii)\ \overline{\sigma}_{f_1 \pm f_2}^{L^*}(g_1) = \overline{\sigma}_{f_i}^{L^*}(g_1) \mid i = 1,2.$$

(III) If (A) $\rho_{f_i}^{L^*}(g_k) = \max\left[\min\left\{\rho_{f_1}^{L^*}(g_1), \rho_{f_2}^{L^*}(g_1)\right\}, \min\left\{\rho_{f_1}^{L^*}(g_2), \rho_{f_2}^{L^*}(g_2)\right\}\right]$, (B) $\rho_{f_1}^{L^*}(g_1) \neq \rho_{f_2}^{L^*}(g_1)$, (C) $\rho_{f_1}^{L^*}(g_2) \neq \rho_{f_2}^{L^*}(g_2)$, (D) $\min\left\{\rho_{f_1}^{L^*}(g_1), \rho_{f_2}^{L^*}(g_1)\right\} \neq \min\left\{\rho_{f_1}^{L^*}(g_2), \rho_{f_2}^{L^*}(g_2)\right\}$ *and* (E) g_1 *and* g_2 *are both of regular relative* L^**-growth with respect to at least any one of* f_1 *or* f_2, *then*

$$(i)\ \sigma_{f_1 \pm f_2}^{L^*}(g_1 \pm g_2) = \sigma_{f_i}^{L^*}(g_k) \mid i = k = 1,2 \text{ and}$$
$$(ii)\ \overline{\sigma}_{f_1 \pm f_2}^{L^*}(g_1 \pm g_2) = \overline{\sigma}_{f_i}^{L^*}(g_k) \mid i = k = 1,2.$$

THEOREM 2. *Let* f_1, f_2, g_1 *and* g_2 *are any four entire functions such that* $\lambda_{f_k}^{L^*}(g_k) \mid k = 1,2$ *are non-zero finite.*

(I) If (A) $\lambda_{f_1}^{L^*}(g_i) = \max\left\{\lambda_{f_1}^{L^*}(g_k) \mid k = 1,2\right\}$, (B) $\lambda_{f_1}^{L^*}(g_1) \neq \lambda_{f_1}^{L^*}(g_2)$ *and* (C) *at least* g_1 *or* g_2 *is of regular relative* L^**-growth with respect to* f_1, *then*

$$(i)\ \tau_{f_1}^{L^*}(g_1 \pm g_2) = \tau_{f_1}^{L^*}(g_i) \mid i = 1,2 \text{ and } (ii)\ \overline{\tau}_{f_1}^{L^*}(g_1 \pm g_2) = \overline{\tau}_{f_1}^{L^*}(g_i) \mid i = 1,2.$$

(II) If (A) $\lambda_{f_i}^{L^*}(g_1) = \min\left\{\lambda_{f_k}^{L^*}(g_1) \mid k = 1,2\right\}$ *and* (B) $\lambda_{f_1}^{L^*}(g_1) \neq \lambda_{f_2}^{L^*}(g_1)$, *then*

$$(i)\ \tau_{f_1 \pm f_2}^{L^*}(g_1) = \tau_{f_i}^{L^*}(g_1) \mid i = 1,2 \text{ and } (ii)\ \overline{\tau}_{f_1 \pm f_2}^{L^*}(g_1) = \overline{\tau}_{f_i}^{L^*}(g_1) \mid i = 1,2.$$

(III) If (A) $\lambda_{f_i}^{L^*}(g_k) = \min\left[\max\left\{\lambda_{f_1}^{L^*}(g_1), \lambda_{f_2}^{L^*}(g_1)\right\}, \max\left\{\lambda_{f_1}^{L^*}(g_2), \lambda_{f_2}^{L^*}(g_2)\right\}\right]$, (B) $\lambda_{f_1}^{L^*}(g_1) \neq \lambda_{f_2}^{L^*}(g_1)$, (C) $\lambda_{f_1}^{L^*}(g_2) \neq \lambda_{f_2}^{L^*}(g_2)$, (D) $\max\left\{\lambda_{f_1}^{L^*}(g_1), \lambda_{f_2}^{L^*}(g_1)\right\} \neq \max\left\{\lambda_{f_1}^{L^*}(g_2), \lambda_{f_2}^{L^*}(g_2)\right\}$ *and* (E) *at least* g_1 *or* g_2 *is of regular relative* L^**-growth with respect to* f_1 *and* f_2 *respectively, then*

$$(i)\ \tau_{f_1 \pm f_2}^{L^*}(g_1 \pm g_2) = \tau_{f_i}^{L^*}(g_k) \mid i = k = 1,2 \quad \text{and}$$
$$(ii)\ \overline{\tau}_{f_1 \pm f_2}^{L^*}(g_1 \pm g_2) = \overline{\tau}_{f_i}^{L^*}(g_k) \mid i = k = 1,2.$$

THEOREM 3. *Let f_1, f_2, g_1 and g_2 are any four entire functions such that $\rho_{f_k}^{L^*}(g_k) \mid k = 1,2$ are non-zero finite.*

(I) If (A) $\rho_{f_1}^{L^*}(g_i) = \max\left\{\rho_{f_1}^{L^*}(g_k) \mid k = i = 1,2\right\}$, (B) $\rho_{f_1}^{L^*}(g_1) \neq \rho_{f_1}^{L^*}(g_2)$ and (C) f_1 has the Property (A), then

$$(i)\ \sigma_{f_1}^{L^*}(g_1 \cdot g_2) \leqslant \sigma_{f_1}^{L^*}(g_i) \mid i = 1,2 \ \text{and}\ (ii)\ \overline{\sigma}_{f_1}^{L^*}(g_1 \cdot g_2) \leqslant \overline{\sigma}_{f_1}^{L^*}(g_i) \mid i = 1,2,$$

For both the cases the equality holds only when $2^{\rho_{f_1}^{L^*}(g_i)} < 1$.

(II) If (A) $\rho_{f_i}^{L^*}(g_1) = \min\left\{\rho_{f_k}^{L^*}(g_1) \mid k = 1,2\right\}$, (B) $\rho_{f_1}^{L^*}(g_1) \neq \rho_{f_2}^{L^*}(g_1)$ and (C) g_1 has the Property (A) and also g_1 is of regular relative L^*-growth with respect to at least any one of f_1 or f_2, then

$$(i)\ \sigma_{f_1 \cdot f_2}^{L^*}(g_1) \geqslant \sigma_{f_i}^{L^*}(g_1) \mid i = 1,2 \ \text{and}\ (ii)\ \overline{\sigma}_{f_1 \cdot f_2}^{L^*}(g_1) \geqslant \overline{\sigma}_{f_i}^{L^*}(g_1) \mid i = 1,2,$$

For both the cases the equality holds only when $2^{\rho_{f_i}^{L^*}(g_1)} > 1$.

(III) If (A) $\rho_{f_i}^{L^*}(g_k) = \max\ [\min\ \{\rho_{f_1}^{L^*}(g_1),\ \rho_{f_2}^{L^*}(g_1)\},\ \min\ \{\rho_{f_1}^{L^*}(g_2),\ \rho_{f_2}^{L^*}(g_2)\}]$, (B) $\rho_{f_1}^{L^*}(g_1) \neq \rho_{f_2}^{L^*}(g_1)$, (C) $\rho_{f_1}^{L^*}(g_2) \neq \rho_{f_2}^{L^*}(g_2)$, (D) $\min\ \{\rho_{f_1}^{L^*}(g_1),\ \rho_{f_2}^{L^*}(g_1)\} \neq \min\ \{\rho_{f_1}^{L^*}(g_2),\ \rho_{f_2}^{L^*}(g_2)\}$, (E) $f_1 \cdot f_2$, g_1 and g_2 have the Property (A), (F) $2^{\rho_{f_1 \cdot f_2}^{L^*}(g_k)} < 1$, (G) $2^{\rho_{f_k}^{L^*}(g_k)} > 1$ and (H) g_1 and g_2 are both of regular relative L^*-growth with respect to at least any one of f_1 or f_2, then

$$(i)\ \sigma_{f_1 \cdot f_2}^{L^*}(g_1 \cdot g_2) = \sigma_{f_i}^{L^*}(g_k) \mid i = k = 1,2 \quad \text{and}$$
$$(ii)\ \overline{\sigma}_{f_1 \cdot f_2}^{L^*}(g_1 \cdot g_2) = \overline{\sigma}_{f_i}^{L^*}(g_k) \mid i = k = 1,2.$$

Similar results for equality of the above three cases are hold for the quotient $\frac{f_1}{f_2}$ provided $\frac{f_1}{f_2}$ is entire.

THEOREM 4. *Let f_1, f_2, g_1 and g_2 are any four entire functions such that $\rho_{f_k}^{L^*}(g_k) \mid k = 1,2$ are non-zero finite.*

(I) If (A) $\lambda_{f_1}^{L^*}(g_i) = \max\left\{\lambda_{f_1}^{L^*}(g_k) \mid k = i = 1,2\right\}$, (B) $\lambda_{f_1}^{L^*}(g_1) \neq \lambda_{f_1}^{L^*}(g_2)$ and (C) f_1 has the Property (A) and and at least g_1 or g_2 is of regular relative L^*-growth with respect to f_1, then

$$(i)\ \tau_{f_1}^{L^*}(g_1 \cdot g_2) \leqslant \tau_{f_1}^{L^*}(g_i) \mid i = 1,2 \ \text{and}\ (ii)\ \overline{\tau}_{f_1}^{L^*}(g_1 \cdot g_2) \leqslant \overline{\tau}_{f_1}^{L^*}(g_i) \mid i = 1,2,$$

For both the cases the equality holds only when $2^{\lambda_{f_1}^{L^*}(g_i)} < 1$.

(II) If (A) $\lambda_{f_i}^{L^*}(g_1) = \min\left\{\lambda_{f_k}^{L^*}(g_1) \mid k = 1,2\right\}$, (B) $\lambda_{f_1}^{L^*}(g_1) \neq \lambda_{f_2}^{L^*}(g_1)$ and (C) g_1 has the Property (A), then

$$(i)\ \tau_{f_1 \cdot f_2}^{L^*}(g_1) \geqslant \tau_{f_i}^{L^*}(g_1) \mid i = 1,2 \ \text{and}\ (ii)\ \overline{\tau}_{f_1 \cdot f_2}^{L^*}(g_1) \geqslant \overline{\tau}_{f_i}^{L^*}(g_1) \mid i = 1,2,$$

For both the cases the equality holds only when $2^{\lambda_{f_i}^{L^*}(g_1)} > 1$.

(III) If (A) $\lambda_{f_i}^{L^*}(g_k) = \min\ [\max\ \{\lambda_{f_1}^{L^*}(g_1),\ \lambda_{f_2}^{L^*}(g_1)\},\ \max\ \{\lambda_{f_1}^{L^*}(g_2),\ \lambda_{f_2}^{L^*}(g_2)\}]$, (B) $\lambda_{f_1}^{L^*}(g_1) \neq \lambda_{f_2}^{L^*}(g_1)$, (C) $\lambda_{f_1}^{L^*}(g_2) \neq \lambda_{f_2}^{L^*}(g_2)$, (D) $\max\ \{\lambda_{f_1}^{L^*}(g_1),\ \lambda_{f_2}^{L^*}(g_1)\}$ $\neq$ $\max\ \{\lambda_{f_1}^{L^*}(g_2),\ \lambda_{f_2}^{L^*}(g_2)\}$, (E) $g_1 \cdot g_2$, f_1 *and* f_2 *have the Property (A),* (F) $2^{\lambda_{f_1 \cdot f_2}^{L^*}(g_k)} < 1$, (G) $2^{\lambda_{f_k}^{L^*}(g_k)} > 1$ *and* (H) *at least* g_1 *or* g_2 *is of regular relative* L^**-growth with respect to* f_1 *and* f_2 *respectively, then*

$$(i)\ \tau_{f_1 \cdot f_2}^{L^*}(g_1 \cdot g_2) = \tau_{f_i}^{L^*}(g_k) \mid i = k = 1,2 \quad \text{and}$$
$$(ii)\ \overline{\tau}_{f_1 \cdot f_2}^{L^*}(g_1 \cdot g_2) = \overline{\tau}_{f_i}^{L^*}(g_k) \mid i = k = 1,2.$$

Similar results for equality of the above three cases are hold for the quotient $\frac{f_1}{f_2}$ *provided* $\frac{f_1}{f_2}$ *is entire.*

The other aim of this paper is to revisit ideas of equality as mentioned in the first and second part of Theorem A, Theorem B, Theorem C and Theorem D under some different conditions and we prove the following four theorems:

THEOREM 5. *Let* f_1, f_2, g_1 *and* g_2 *are any four entire functions.*
(I) If either $\sigma_{f_1}^{L^*}(g_1) \neq \sigma_{f_1}^{L^*}(g_2)$ *or* $\overline{\sigma}_{f_1}^{L^*}(g_1) \neq \overline{\sigma}_{f_1}^{L^*}(g_2)$ *hold, then*

$$\rho_{f_1}^{L^*}(g_1 \pm g_2) = \rho_{f_1}^{L^*}(g_1) = \rho_{f_1}^{L^*}(g_2).$$

(II) If (A) *either* $\sigma_{f_1}^{L^*}(g_1) \neq \sigma_{f_2}^{L^*}(g_1)$ *or* $\overline{\sigma}_{f_1}^{L^*}(g_1) \neq \overline{\sigma}_{f_2}^{L^*}(g_1)$ *hold and* (B) g_1 *is of regular relative* L^**-growth with respect to at least any one of* f_1 *or* f_2, *then*

$$\rho_{f_1 \pm f_2}^{L^*}(g_1) = \rho_{f_1}^{L^*}(g_1) = \rho_{f_2}^{L^*}(g_1).$$

THEOREM 6. *Let* f_1, f_2, g_1 *and* g_2 *are any four entire functions.*
(I) If (A) *either* $\tau_{f_1}^{L^*}(g_1) \neq \tau_{f_1}^{L^*}(g_2)$ *or* $\overline{\tau}_{f_1}^{L^*}(g_1) \neq \overline{\tau}_{f_1}^{L^*}(g_2)$ *hold and* (B) *at least* g_1 *or* g_2 *is of regular relative* L^**-growth with respect to* f_1, *then*

$$\lambda_{f_1}^{L^*}(g_1 \pm g_2) = \lambda_{f_1}^{L^*}(g_1) = \lambda_{f_1}^{L^*}(g_2).$$

(II) If either $\tau_{f_1}^{L^*}(g_1) \neq \tau_{f_2}^{L^*}(g_1)$ *or* $\overline{\tau}_{f_1}^{L^*}(g_1) \neq \overline{\tau}_{f_2}^{L^*}(g_1)$ *hold, then*

$$\lambda_{f_1 \pm f_2}^{L^*}(g_1) = \lambda_{f_1}^{L^*}(g_1) = \lambda_{f_2}^{L^*}(g_1).$$

THEOREM 7. *Let* f_1, f_2, g_1 *and* g_2 *are any four entire functions.*
(I) If (A) *either* $\sigma_{f_1}^{L^*}(g_1) \neq \sigma_{f_1}^{L^*}(g_2)$ *or* $\overline{\sigma}_{f_1}^{L^*}(g_1) \neq \overline{\sigma}_{f_1}^{L^*}(g_2)$ *hold* (B) $2^{\rho_{f_1}^{L^*}(g_i)} < 1$ *and* (C) f_1 *has the Property (A), then*

$$\rho_{f_1}^{L^*}(g_1 \cdot g_2) = \rho_{f_1}^{L^*}(g_1) = \rho_{f_1}^{L^*}(g_2).$$

(II) If (A) *either* $\sigma_{f_1}^{L^*}(g_1) \neq \sigma_{f_2}^{L^*}(g_1)$ *or* $\overline{\sigma}_{f_1}^{L^*}(g_1) \neq \overline{\sigma}_{f_2}^{L^*}(g_1)$ *hold* (B) $2^{\rho_{f_i}^{L^*}(g_1)} > 1$ *and* (C) g_1 *has the Property (A) and also* g_1 *is of regular relative* L^**-growth with respect to at least any one of* f_1 *or* f_2, *then*

$$\rho_{f_1 \cdot f_2}^{L^*}(g_1) = \rho_{f_1}^{L^*}(g_1) = \rho_{f_2}^{L^*}(g_1).$$

Similar results of above two cases are hold for the quotient $\frac{f_1}{f_2}$ *provided* $\frac{f_1}{f_2}$ *is entire.*

THEOREM 8. *Let* f_1, f_2, g_1 *and* g_2 *are any four entire functions.*

(I) If (A) *either* $\tau_{f_1}^{L^*}(g_1) \neq \tau_{f_1}^{L^*}(g_2)$ *or* $\overline{\tau}_{f_1}^{L^*}(g_1) \neq \overline{\tau}_{f_1}^{L^*}(g_2)$ *hold* (B) $2^{\lambda_{f_1}^{L^*}(g_i)} < 1$ *and* (C) f_1 *has the Property (A) and at least* g_1 *or* g_2 *is of regular relative* L^**-growth with respect to* f_1*, then*

$$\lambda_{f_1}^{L^*}(g_1 \cdot g_2) = \lambda_{f_1}^{L^*}(g_1) = \lambda_{f_1}^{L^*}(g_2).$$

(II) If (A) *either* $\tau_{f_1}^{L^*}(g_1) \neq \tau_{f_1}^{L^*}(g_2)$ *or* $\overline{\tau}_{f_1}^{L^*}(g_1) \neq \overline{\tau}_{f_1}^{L^*}(g_2)$ *hold* (B) $2^{\lambda_{f_i}^{L^*}(g_1)} > 1$ *and* (C) g_1 *has the Property (A), then*

$$\lambda_{f_1 \cdot f_2}^{L^*}(g_1) = \lambda_{f_1}^{L^*}(g_1) = \lambda_{f_2}^{L^*}(g_1).$$

Similar results of above three cases are hold for the quotient $\frac{f_1}{f_2}$ *provided* $\frac{f_1}{f_2}$ *is entire.*

4. Lemmas

In this section we present some lemmas which will be needed in the sequel.

LEMMA 1. [1] *Suppose* f *be an entire function and* α, β *be such that* $\alpha > 1$ *and* $0 < \beta < \alpha$*. Then*

$$M_f(\alpha r) > \beta M_f(r).$$

LEMMA 2. [1] *Let* f *be an entire function satisfying the Property (A). Then for all sufficiently large* r*,*

$$\left[M_f(r)\right]^2 \leqslant M_f\left(r^\delta\right)$$

holds for $\delta > 1$*.*

LEMMA 3. [7] *Every entire function* f *satisfying the Property (A) is transcendental.*

LEMMA 4. *Let* f *be an entire function. Then for all sufficiently large values of* r*,*

$$T_f(r) \leqslant \log M_f(r) \leqslant 3T_f(2r) \ \{cf.\ [5], p.\ 18\}.$$

5. Proofs of the Theorems

In this section we present the proofs of the main results.

Proof of Theorem 1. From the definition of relative L^*-type and relative L^*-lower type of entire function, we have for all sufficiently large values of r that

$$M_{g_k}(r) \leqslant M_{f_k}\left[\left(\sigma_{f_k}^{L^*}(g_k) + \varepsilon\right)\left[re^{L(r)}\right]^{\rho_{f_k}^{L^*}(g_k)}\right], \tag{1}$$

$$M_{g_k}(r) \geqslant M_{f_k}\left[\left(\overline{\sigma}_{f_k}^{L^*}(g_k)-\varepsilon\right)\left[re^{L(r)}\right]^{\rho_{f_k}^{L^*}(g_k)}\right], \tag{2}$$

and also for a sequence values of r tending to infinity we get that

$$M_{g_k}(r) \geqslant M_{f_k}\left[\left(\sigma_{f_k}^{L^*}(g_k)-\varepsilon\right)\left[re^{L(r)}\right]^{\rho_{f_k}^{L^*}(g_k)}\right], \tag{3}$$

$$M_{g_k}(r) \leqslant M_{f_k}\left[\left(\overline{\sigma}_{f_k}^{L^*}(g_k)+\varepsilon\right)\left[re^{L(r)}\right]^{\rho_{f_k}^{L^*}(g_k)}\right] \tag{4}$$

where $\varepsilon\,(>0)$ is any arbitrary positive number and $k=1,2$.

Case I. Let $\rho_{f_1}^{L^*}(g_k) < \rho_{f_1}^{L^*}(g_i)$ where $k=i=1,2$ with $g_k \neq g_i$.

Now from (1) we get for all sufficiently large values of r that

$$M_{g_1\pm g_2}(r) < M_{g_1}(r) + M_{g_2}(r) \tag{5}$$

i.e.,

$$M_{g_1\pm g_2}(r) < M_{f_1}\left[\left(\sigma_{f_1}^{L^*}(g_k)+\varepsilon\right)\left[re^{L(r)}\right]^{\rho_{f_1}^{L^*}(g_k)}\right] + M_{f_1}\left[\left(\sigma_{f_1}^{L^*}(g_i)+\varepsilon\right)\left[re^{L(r)}\right]^{\rho_{f_1}^{L^*}(g_i)}\right]$$

i.e.,

$$\begin{aligned} M_{g_1\pm g_2}(r) < {} & M_{f_1}\left[\left(\sigma_{f_1}^{L^*}(g_i)+\varepsilon\right)\left[re^{L(r)}\right]^{\rho_{f_1}^{L^*}(g_i)}\right] \\ & \times\left[1+\frac{M_{f_1}\left[\left(\sigma_{f_1}^{L^*}(g_k)+\varepsilon\right)\left[re^{L(r)}\right]^{\rho_{f_1}^{L^*}(g_k)}\right]}{M_{f_1}\left[\left(\sigma_{f_1}^{L^*}(g_i)+\varepsilon\right)\left[re^{L(r)}\right]^{\rho_{f_1}^{L^*}(g_i)}\right]}\right]. \end{aligned}$$

Since $\rho_{f_1}^{L^*}(g_k) < \rho_{f_1}^{L^*}(g_i)$, one can make the term $\frac{M_{f_1}\left[\left(\sigma_{f_1}^{L^*}(g_k)+\varepsilon\right)\left[re^{L(r)}\right]^{\rho_{f_1}^{L^*}(g_k)}\right]}{M_{f_1}\left[\left(\sigma_{f_1}^{L^*}(g_i)+\varepsilon\right)\left[re^{L(r)}\right]^{\rho_{f_1}^{L^*}(g_i)}\right]}$ sufficiently small by taking r sufficiently large. Therefore in view of Lemma 1 and the above inequality we get for all sufficiently large values of r that

$$M_{g_1\pm g_2}(r) < M_{f_1}\left[\left(\sigma_{f_1}^{L^*}(g_i)+\varepsilon\right)\left[re^{L(r)}\right]^{\rho_{f_1}^{L^*}(g_i)}\right](1+\varepsilon_1)$$

i.e.,

$$M_{g_1\pm g_2}(r) < M_{f_1}\left[\alpha\left(\sigma_{f_1}^{L^*}(g_i)+\varepsilon\right)\left[re^{L(r)}\right]^{\rho_{f_1}^{L^*}(g_i)}\right]$$

where $\alpha > (1+\varepsilon_1)$.

Now making $\alpha \to 1+$ we obtain from above and Theorem A (i) for all sufficiently large values of r that

$$M_{f_1}^{-1} M_{g_1 \pm g_2}(r) < \left(\sigma_{f_1}^{L^*}(g_i) + \varepsilon\right) \left[re^{L(r)}\right]^{\rho_{f_1}^{L^*}(g_i)}$$

i.e.,

$$\frac{M_{f_1}^{-1} M_{g_1 \pm g_2}(r_n)}{\left[re^{L(r)}\right]^{\rho_{f_1}^{L^*}(g_1 \pm g_2)}} < \left(\sigma_{f_1}^{L^*}(g_i) + \varepsilon\right).$$

Since $\varepsilon > 0$ is arbitrary, we get from above that

$$\sigma_{f_1}^{L^*}(g_1 \pm g_2) \leqslant \sigma_{f_1}^{L^*}(g_i).$$

Further without loss of generality, let $\rho_{f_1}^{L^*}(g_1) < \rho_{f_1}^{L^*}(g_2)$ and $g = g_1 \pm g_2$. Then $\sigma_{f_1}^{L^*}(g) \leqslant \sigma_{f_1}^{L^*}(g_2)$. Further let $g_2 = \pm(g - g_1)$ and in this case we obtain from Theorem A(i) that $\rho_{f_1}^{L^*}(g_1) < \rho_{f_1}^{L^*}(g)$. So $\sigma_{f_1}^{L^*}(g_2) \leqslant \sigma_{f1}^{L^*}(g)$. Hence $\sigma_{f_1}^{L^*}(g) = \sigma_{f_1}^{L^*}(g_2) \Rightarrow \sigma_{f_1}^{L^*}(g_1 \pm g_2) = \sigma_{f_1}^{L^*}(g_2)$. Thus, $\sigma_{f_1}^{L^*}(g_1 \pm g_2) = \sigma_{f_1}^{L^*}(g_i) \mid i = 1,2$ where $\rho_{f_1}^{L^*}(g_i) = \max\left\{\rho_{f_1}^{L^*}(g_k) \mid k = i = 1,2\right\}$ and $\rho_{f_1}^{L^*}(g_1) \neq \rho_{f_1}^{L^*}(g_2)$.

Case II. Further let $\rho_{f_1}^{L^*}(g_k) < \rho_{f_1}^{L^*}(g_i)$ where $k = i = 1,2$ with $g_k \neq g_i$.

Now from (1) and (4) and in view of (5) we get for a sequence of values of r tending to infinity that

$$M_{g_1 \pm g_2}(r_n) < M_{g_1}(r_n) + M_{g_2}(r_n) \tag{6}$$

$$\begin{aligned} M_{g_1 \pm g_2}(r_n) < M_{f_1} &\left[\left(\sigma_{f_1}^{L^*}(g_k) + \varepsilon\right) \left[r_n e^{L(r_n)}\right]^{\rho_{f_1}^{L^*}(g_k)}\right] \\ &+ M_{f_1}\left[\left(\overline{\sigma}_{f_1}^{L^*}(g_i) + \varepsilon\right) \left[r_n e^{L(r_n)}\right]^{\rho_{f_1}^{L^*}(g_i)}\right] \end{aligned}$$

i.e.,

$$\begin{aligned} M_{g_1 \pm g_2}(r_n) < M_{f_1} &\left[\left(\overline{\sigma}_{f_1}^{L^*}(g_i) + \varepsilon\right) \left[r_n e^{L(r_n)}\right]^{\rho_{f_1}^{L^*}(g_i)}\right] \\ &\times \left[1 + \frac{M_{f_1}\left[\left(\sigma_{f_1}^{L^*}(g_k) + \varepsilon\right) \left[r_n e^{L(r_n)}\right]^{\rho_{f_1}^{L^*}(g_k)}\right]}{M_{f_1}\left[\left(\overline{\sigma}_{f_1}^{L^*}(g_i) + \varepsilon\right) \left[r_n e^{L(r_n)}\right]^{\rho_{f_1}^{L^*}(g_i)}\right]}\right]. \end{aligned} \tag{7}$$

Since $\rho_{f_1}^{L^*}(g_k) < \rho_{f_1}^{L^*}(g_i)$, one can make the term $\frac{M_{f_1}\left[\left(\sigma_{f_1}^{L^*}(g_k)+\varepsilon\right)\left[r_n e^{L(r_n)}\right]^{\rho_{f_1}^{L^*}(g_k)}\right]}{M_{f_1}\left[\left(\overline{\sigma}_{f_1}^{L^*}(g_i)+\varepsilon\right)\left[r_n e^{L(r_n)}\right]^{\rho_{f_1}^{L^*}(g_i)}\right]}$ sufficiently small by taking n sufficiently large. Therefore in view of Lemma 1 and Theorem A (i) and using the similar technique of case I we obtain from (7) that

$$\overline{\sigma}_{f_1}^{L^*}(g_1 \pm g_2) \leqslant \overline{\sigma}_{f_1}^{L^*}(g_i).$$

Further without loss of generality, let $\rho_{f_1}^{L^*}(g_1) < \rho_{f_1}^{L^*}(g_2)$ and $g = g_1 \pm g_2$. Then $\overline{\sigma}_{f_1}^{L^*}(g) \leqslant \overline{\sigma}_{f_1}^{L^*}(g_2)$. Further let $g_2 = \pm(g - g_1)$ and in this case we obtain from Theorem C that $\rho_{f_1}^{L^*}(g_1) < \rho_{f_1}^{L^*}(g)$. So $\overline{\sigma}_{f_1}^{L^*}(g_2) \leqslant \overline{\sigma}_{f1}^{L^*}(g)$. Hence $\overline{\sigma}_{f_1}^{L^*}(g) = \overline{\sigma}_{f_1}^{L^*}(g_2) \Rightarrow \overline{\sigma}_{f_1}^{L^*}(g_1 \pm g_2) = \overline{\sigma}_{f_1}^{L^*}(g_2)$. Thus, $\overline{\sigma}_{f_1}^{L^*}(g_1 \pm g_2) = \overline{\sigma}_{f_1}^{L^*}(g_i) \mid i = 1,2$ where $\rho_{f_1}^{L^*}(g_i) = \max\left\{\rho_{f_1}^{L^*}(g_k) \mid k = i = 1,2\right\}$ and $\rho_{f_1}^{L^*}(g_1) \neq \rho_{f_1}^{L^*}(g_2)$.

Thus the first part of the theorem follows from Case I and Case II.

Case III. Now suppose that $\rho_{f_i}^{L^*}(g_1) < \rho_{f_k}^{L^*}(g_1)$ where $k = i = 1,2$ with $f_i \neq f_k$ and g_1 is of regular relative L^*-growth with respect to at least any one of f_1 or f_2.

We can make the term $\frac{M_{f_k}\left(\left(\sigma_{f_i}^{L^*}(g_1)-\varepsilon\right)\left[r_n e^{L(r_n)}\right]^{\rho_{f_i}^{L^*}(g_1)}\right)}{M_{f_k}\left(\left(\overline{\sigma}_{f_k}^{L^*}(g_1)-\varepsilon\right)\left[r_n e^{L(r_n)}\right]^{\rho_{f_k}^{L^*}(g_1)}\right)}$ sufficiently small by taking n sufficiently large since $\rho_{f_i}^{L^*}(g_1) < \rho_{f_k}^{L^*}(g_1)$. Therefore

$$\frac{M_{f_k}\left(\left(\sigma_{f_i}^{L^*}(g_1)-\varepsilon\right)\left[r_n e^{L(r_n)}\right]^{\rho_{f_i}^{L^*}(g_1)}\right)}{M_{f_k}\left(\left(\overline{\sigma}_{f_k}^{L^*}(g_1)-\varepsilon\right)\left[r_n e^{L(r_n)}\right]^{\rho_{f_k}^{L^*}(g_1)}\right)} < \varepsilon_1 \tag{8}$$

for sufficiently large n.

Now in view of (2), (3) and (8) we obtain for a sequence of values of r tending to infinity that

$$M_{f_1 \pm f_2}\left[\left(\sigma_{f_i}^{L^*}(g_1)-\varepsilon\right)\left[r_n e^{L(r_n)}\right]^{\rho_{f_i}^{L^*}(g_1)}\right]$$
$$< M_{f_1}\left(\left(\sigma_{f_i}^{L^*}(g_1)-\varepsilon\right)\left[r_n e^{L(r_n)}\right]^{\rho_{f_i}^{L^*}(g_1)}\right) + M_{f_2}\left(\left(\sigma_{f_i}^{L^*}(g_1)-\varepsilon\right)\left[r_n e^{L(r_n)}\right]^{\rho_{f_i}^{L^*}(g_1)}\right)$$

i.e.,

$$M_{f_1 \pm f_2}\left[\left(\sigma_{f_i}^{L^*}(g_1)-\varepsilon\right)\left[r_n e^{L(r_n)}\right]^{\rho_{f_i}^{L^*}(g_1)}\right]$$
$$< M_{f_i}\left(\left(\sigma_{f_i}^{L^*}(g_1)-\varepsilon\right)\left[r_n e^{L(r_n)}\right]^{\rho_{f_i}^{L^*}(g_1)}\right) + M_{f_k}\left(\left(\sigma_{f_i}^{L^*}(g_1)-\varepsilon\right)\left[r_n e^{L(r_n)}\right]^{\rho_{f_i}^{L^*}(g_1)}\right)$$

i.e.,

$$M_{f_1 \pm f_2}\left[\left(\sigma_{f_i}^{L^*}(g_1)-\varepsilon\right)\left[r_n e^{L(r_n)}\right]^{\rho_{f_i}^{L^*}(g_1)}\right]$$
$$< M_{f_i}\left(\left(\sigma_{f_i}^{L^*}(g_1)-\varepsilon\right)\left[r_n e^{L(r_n)}\right]^{\rho_{f_i}^{L^*}(g_1)}\right)+\varepsilon_1 M_{f_k}\left(\left(\overline{\sigma}_{f_k}^{L^*}(g_1)-\varepsilon\right)\left[r_n e^{L(r_n)}\right]^{\rho_{f_k}^{L^*}(g_1)}\right)$$

i.e.,

$$M_{f_1 \pm f_2}\left[\left(\sigma_{f_i}^{L^*}(g_1)-\varepsilon\right)\left[r_n e^{L(r_n)}\right]^{\rho_{f_i}^{L^*}(g_1)}\right] < M_{g_1}(r_n)(1+\varepsilon_1)$$

i.e.,

$$M_{f_1 \pm f_2}\left[\left(\sigma_{f_i}^{L^*}(g_1)-\varepsilon\right)\left[r_n e^{L(r_n)}\right]^{\rho_{f_i}^{L^*}(g_1)}\right] < M_{g_1}(\alpha r_n)$$

where $\alpha > (1+\varepsilon_1)$.

Hence making $\alpha \to 1+$ we obtain from above and Theorem A(ii) for a sequence of values of r tending to infinity that

$$\left(\sigma_{f_i}^{L^*}(g_1)-\varepsilon\right)\left[r_n e^{L(r_n)}\right]^{\rho_{f_i}^{L^*}(g_1)} < M_{f_1 \pm f_2}^{-1} M_{g_1}(r_n)$$

i.e.,

$$\left(\sigma_{f_i}^{L^*}(g_1)-\varepsilon\right) < \frac{M_{f_1 \pm f_2}^{-1} M_{g_1}(r_n)}{\left[r_n e^{L(r_n)}\right]^{\rho_{f_1 \pm f_2}^{L^*}(g_1)}}.$$

Since $\varepsilon > 0$ is arbitrary, it follows from above that

$$\sigma_{f_1 \pm f_2}^{L^*}(g_1) \geqslant \sigma_{f_i}^{L^*}(g_1).$$

Now without loss of generality, we may consider that $\rho_{f_1}^{L^*}(g_1) < \rho_{f_2}^{L^*}(g_1)$ and $f = f_1 \pm f_2$. Then $\sigma_f^{L^*}(g_1) \geqslant \sigma_{f_1}^{L^*}(g_1)$. Further let $f_1 = (f \pm f_2)$. Therefore in view of Theorem A (ii), $\rho_f^{L^*}(g_1) < \rho_{f_2}^{L^*}(g_1)$ and accordingly $\sigma_{f_1}^{L^*}(g_1) \geqslant \sigma_f^{L^*}(g_1)$. Hence $\sigma_f^{L^*}(g_1) = \sigma_{f_1}^{L^*}(g_1) \Rightarrow \sigma_{f_1 \pm f_2}^{L^*}(g_1) = \sigma_{f_1}^{L^*}(g_1)$. So, $\sigma_{f_1 \pm f_2}^{L^*}(g_1) = \sigma_{f_i}^{L^*}(g_1) \mid i = 1,2$ where $\rho_{f_i}^{L^*}(g_1) = \min\{\rho_{f_k}^{L^*}(g_1) \mid k = i = 1,2\}$ provided $\rho_{f_1}^{L^*}(g_1) \neq \rho_{f_2}^{L^*}(g_1)$ and g_1 is of regular relative L^*-growth with respect to at least any one of f_1 or f_2.

Case IV. In this case also one can clearly assume that $\rho_{f_i}^{L^*}(g_1) < \rho_{f_k}^{L^*}(g_1)$ where $k = i = 1,2$ with $f_i \neq f_k$ and g_1 is of regular relative L^*-growth with respect to at least any one of f_1 or f_2.

We can also make the term $\frac{M_{f_k}\left(\left(\overline{\sigma}_{f_i}^{L^*}(g_1)-\varepsilon\right)\left[re^{L(r)}\right]^{\rho_{f_i}^{L^*}(g_1)}\right)}{M_{f_k}\left(\left(\overline{\sigma}_{f_k}^{L^*}(g_1)-\varepsilon\right)\left[re^{L(r)}\right]^{\rho_{f_k}^{L^*}(g_1)}\right)}$ sufficiently small by taking r sufficiently large as $\rho_{f_i}^{L^*}(g_1) < \rho_{f_k}^{L^*}(g_1)$. So

$$\frac{M_{f_k}\left(\left(\overline{\sigma}_{f_i}^{L^*}(g_1)-\varepsilon\right)\left[re^{L(r)}\right]^{\rho_{f_i}^{L^*}(g_1)}\right)}{M_{f_k}\left(\left(\overline{\sigma}_{f_k}^{L^*}(g_1)-\varepsilon\right)\left[re^{L(r)}\right]^{\rho_{f_k}^{L^*}(g_1)}\right)} < \varepsilon_1 \tag{9}$$

for sufficiently large r.

Then in view of (2) and (9) we obtain for all sufficiently large values of r that

$$M_{f_1\pm f_2}\left[\left(\overline{\sigma}_{f_i}^{L^*}(g_1)-\varepsilon\right)\left[re^{L(r)}\right]^{\rho_{f_i}^{L^*}(g_1)}\right] < M_{f_1}\left(\left(\overline{\sigma}_{f_i}^{L^*}(g_1)-\varepsilon\right)\left[re^{L(r)}\right]^{\rho_{f_i}^{L^*}(g_1)}\right)$$
$$+M_{f_2}\left(\left(\overline{\sigma}_{f_i}^{L^*}(g_1)-\varepsilon\right)\left[re^{L(r)}\right]^{\rho_{f_i}^{L^*}(g_1)}\right)$$

i.e.,

$$M_{f_1\pm f_2}\left[\left(\overline{\sigma}_{f_i}^{L^*}(g_1)-\varepsilon\right)\left[re^{L(r)}\right]^{\rho_{f_i}^{L^*}(g_1)}\right] < M_{f_i}\left(\left(\overline{\sigma}_{f_i}^{L^*}(g_1)-\varepsilon\right)\left[re^{L(r)}\right]^{\rho_{f_i}^{L^*}(g_1)}\right)$$
$$+\varepsilon_1 M_{f_k}\left(\left(\overline{\sigma}_{f_k}^{L^*}(g_1)-\varepsilon\right)\left[re^{L(r)}\right]^{\rho_{f_k}^{L^*}(g_1)}\right). \tag{10}$$

Therefore using the similar technique for all sufficiently large values of r as executed in the proof of case III we get from (10) that $\overline{\sigma}_{f_1\pm f_2}^{L^*}(g_1) = \overline{\sigma}_{f_i}^{L^*}(g_1) \mid i = 1,2$ where $\rho_{f_i}^{L^*}(g_1) = \min\{\rho_{f_k}^{L^*}(g_1) \mid k = i = 1,2\}$ provided $\rho_{f_1}^{L^*}(g_1) \neq \rho_{f_2}^{L^*}(g_1)$ and g_1 is of regular relative L^*-growth with respect to at least any one of f_1 or f_2.

Thus combining Case III and Case IV we obtain the second part of the theorem.

The third part of the theorem is a natural consequence of Theorem A (iii) and the first part and second part of the theorem. Hence its proof is omitted. □

Proof of Theorem 2. For any arbitrary positive number $\varepsilon(>0)$, we have from definition 6 for all sufficiently large values of r that

$$M_{g_k}(r) \leqslant M_{f_k}\left[\left(\overline{\tau}_{f_k}^{L^*}(g_k)+\varepsilon\right)\left[re^{L(r)}\right]^{\lambda_{f_k}^{L^*}(g_k)}\right], \tag{11}$$

$$M_{g_k}(r) \geqslant M_{f_k}\left[\left(\tau_{f_k}^{L^*}(g_k)-\varepsilon\right)\left[re^{L(r)}\right]^{\lambda_{f_k}^{L^*}(g_k)}\right], \tag{12}$$

and for a sequence $\{r_n\} \to \infty$, we have

$$M_{g_k}(r) \geqslant M_{f_k}\left[\left(\overline{\tau}_{f_k}^{L^*}(g_k)-\varepsilon\right)\left[r_n e^{L(r_n)}\right]^{\lambda_{f_k}^{L^*}(g_k)}\right], \tag{13}$$

$$M_{g_k}(r) \leqslant M_{f_k}\left[\left(\tau_{f_k}^{L^*}(g_k)+\varepsilon\right)\left[r_n e^{L(r_n)}\right]^{\lambda_{f_k}^{L^*}(g_k)}\right] \tag{14}$$

where $k=1,2$.

Case I. Let us consider $\lambda_{f_1}^{L^*}(g_k) < \lambda_{f_1}^{L^*}(g_i)$ where $k=i=1,2$ with $g_k \neq g_i$ and at least g_1 or g_2 is of regular relative L^*-growth with respect to f_1.

Therefore from (6), (11) and (14) we get for a sequence $\{r_n\} \to \infty$ that

$$M_{g_1 \pm g_2}(r_n) < M_{f_1}\left[\left(\overline{\tau}_{f_1}^{L^*}(g_k)+\varepsilon\right)\left[r_n e^{L(r_n)}\right]^{\lambda_{f_1}^{L^*}(g_k)}\right] + M_{f_1}\left[\left(\tau_{f_1}^{L^*}(g_i)+\varepsilon\right)\left[r_n e^{L(r_n)}\right]^{\lambda_{f_1}^{L^*}(g_i)}\right]$$

i.e.,

$$M_{g_1 \pm g_2}(r_n) < M_{f_1}\left[\left(\tau_{f_1}^{L^*}(g_i)+\varepsilon\right)\left[r_n e^{L(r_n)}\right]^{\lambda_{f_1}^{L^*}(g_i)}\right] \times \left[1+\frac{M_{f_1}\left[\left(\overline{\tau}_{f_1}^{L^*}(g_k)+\varepsilon\right)\left[r_n e^{L(r_n)}\right]^{\lambda_{f_1}^{L^*}(g_k)}\right]}{M_{f_1}\left[\left(\tau_{f_1}^{L^*}(g_i)+\varepsilon\right)\left[r_n e^{L(r_n)}\right]^{\lambda_{f_1}^{L^*}(g_i)}\right]}\right]. \tag{15}$$

Since $\lambda_{f_1}^{L^*}(g_k) < \lambda_{f_1}^{L^*}(g_i)$, we can make the term $\frac{M_{f_1}\left[\left(\overline{\tau}_{f_1}^{L^*}(g_k)+\varepsilon\right)\left[r_n e^{L(r_n)}\right]^{\lambda_{f_1}^{L^*}(g_k)}\right]}{M_{f_1}\left[\left(\tau_{f_1}^{L^*}(g_i)+\varepsilon\right)\left[r_n e^{L(r_n)}\right]^{\lambda_{f_1}^{L^*}(g_i)}\right]}$ sufficiently small by taking n sufficiently large. So with the help of Lemma 1 and the second part of Theorem B and using the similar technique of case I of Theorem 1, we get from (15) that

$$\tau_{f_1}^{L^*}(g_1 \pm g_2) \leqslant \tau_{f_1}^{L^*}(g_i).$$

Now without loss of generality, suppose that $\lambda_{f_1}^{L^*}(g_1) < \lambda_{f_1}^{L^*}(g_2)$ and $g = g_1 \pm g_2$. So $\tau_{f_1}^{L^*}(g) \leqslant \tau^{L^*}(g_2)$. Also let $g_2 = \pm(g - g_1)$ and in this case we have from the first part of Theorem B that $\lambda_{f_1}^{L^*}(g_1) < \lambda_{f_1}^{L^*}(g)$. Therefore $\tau_{f_1}^{L^*}(g_2) \leqslant \tau_{f_1}^{L^*}(g)$. Hence $\tau_{f_1}^{L^*}(g) = \tau_{f_1}^{L^*}(g_2) \Rightarrow \tau_{f_1}^{L^*}(g_1 \pm g_2) = \tau_{f_1}^{L^*}(g_2)$. Thus, $\tau_{f_1}^{L^*}(g_1 \pm g_2) = \tau_{f_1}^{L^*}(g_i) \mid i = 1,2$ where $\lambda_{f_1}^{L^*}(g_i) = \max\left\{\lambda_{f_1}^{L^*}(g_k) \mid k=i=1,2\right\}$ and $\lambda_{f_1}^{L^*}(g_1) \neq \lambda_{f_1}^{L^*}(g_2)$.

Case II. Let us consider that $\lambda_{f_1}^{L^*}(g_k) < \lambda_{f_1}^{L^*}(g_i)$ where $k=i=1,2$ with $g_k \neq g_i$. Now in view of (11) we get for all sufficiently large values of r that

$$M_{g_1 \pm g_2}(r) < M_{g_1}(r) + M_{g_2}(r)$$

i.e.,

$$M_{g_1 \pm g_2}(r) < M_{f_1}\left[\left(\overline{\tau}_{f_1}^{L^*}(g_k)+\varepsilon\right)\left[r e^{L(r)}\right]^{\lambda_{f_1}^{L^*}(g_k)}\right] + M_{f_1}\left[\left(\overline{\tau}_{f_1}^{L^*}(g_i)+\varepsilon\right)\left[r e^{L(r)}\right]^{\lambda_{f_1}^{L^*}(g_i)}\right]$$

i.e.,

$$M_{g_1 \pm g_2}(r) < M_{f_1}\left[\left(\overline{\tau}_{f_1}^{L^*}(g_i)+\varepsilon\right)\left[re^{L(r)}\right]^{\lambda_{f_1}^{L^*}(g_i)}\right] \times \left[1+\frac{M_{f_1}\left[\left(\overline{\tau}_{f_1}^{L^*}(g_k)+\varepsilon\right)\left[re^{L(r)}\right]^{\lambda_{f_1}^{L^*}(g_k)}\right]}{M_{f_1}\left[\left(\overline{\tau}_{f_1}^{L^*}(g_i)+\varepsilon\right)\left[re^{L(r)}\right]^{\lambda_{f_1}^{L^*}(g_i)}\right]}\right].$$

As $\lambda_{f_1}^{L^*}(g_k) < \lambda_{f_1}^{L^*}(g_i)$, by taking r sufficiently large one can make the term $\frac{M_{f_1}\left[\left(\overline{\tau}_{f_1}^{L^*}(g_k)+\varepsilon\right)\left[re^{L(r)}\right]^{\lambda_{f_1}^{L^*}(g_k)}\right]}{M_{f_1}\left[\left(\overline{\tau}_{f_1}^{L^*}(g_i)+\varepsilon\right)\left[re^{L(r)}\right]^{\lambda_{f_1}^{L^*}(g_i)}\right]}$ sufficiently small and therefore for similar reasoning of Case-I we get from above that $\overline{\tau}_{f_1}^{L^*}(g_1 \pm g_2) = \overline{\tau}_{f_1}^{L^*}(g_i) \mid i = 1,2$ where $\lambda_{f_1}^{L^*}(g_i) = \max\left\{\lambda_{f_1}^{L^*}(g_k) \mid k = i = 1,2\right\}$ and $\lambda_{f_1}^{L^*}(g_1) \neq \lambda_{f_1}^{L^*}(g_2)$ and hence its detail proof is omitted.

Thus the first part of the theorem follows from Case I and Case II.

Case III. Now suppose that $\lambda_{f_i}^{L^*}(g_1) < \lambda_{f_k}^{L^*}(g_1)$ where $k = i = 1,2$ with $f_i \neq f_k$.

Therefore we can make the term $\frac{M_{f_k}\left(\left(\tau_{f_i}^{L^*}(g_1)-\varepsilon\right)\left[re^{L(r)}\right]^{\lambda_{f_i}^{L^*}(g_1)}\right)}{M_{f_k}\left(\left(\tau_{f_k}^{L^*}(g_1)-\varepsilon\right)\left[re^{L(r)}\right]^{\lambda_{f_k}^{L^*}(g_1)}\right)}$ sufficiently small by taking r sufficiently large since $\rho_{f_i}^{L^*}(g_1) < \rho_{f_k}^{L^*}(g_1)$. So

$$\frac{M_{f_k}\left(\left(\tau_{f_i}^{L^*}(g_1)-\varepsilon\right)\left[re^{L(r)}\right]^{\lambda_{f_i}^{L^*}(g_1)}\right)}{M_{f_k}\left(\left(\tau_{f_k}^{L^*}(g_1)-\varepsilon\right)\left[re^{L(r)}\right]^{\lambda_{f_k}^{L^*}(g_1)}\right)} < \varepsilon_1 \tag{16}$$

for sufficiently large r.

Then in view of (2) and (16) we obtain for all sufficiently large values of r that

$$M_{f_1 \pm f_2}\left[\left(\tau_{f_i}^{L^*}(g_1)-\varepsilon\right)\left[re^{L(r)}\right]^{\lambda_{f_i}^{L^*}(g_1)}\right] < M_{f_1}\left(\left(\tau_{f_i}^{L^*}(g_1)-\varepsilon\right)\left[re^{L(r)}\right]^{\lambda_{f_i}^{L^*}(g_1)}\right) + M_{f_2}\left(\left(\tau_{f_i}^{L^*}(g_1)-\varepsilon\right)\left[re^{L(r)}\right]^{\lambda_{f_i}^{L^*}(g_1)}\right)$$

i.e.,

$$M_{f_1 \pm f_2}\left[\left(\tau_{f_i}^{L^*}(g_1)-\varepsilon\right)\left[re^{L(r)}\right]^{\lambda_{f_i}^{L^*}(g_1)}\right] < M_{f_i}\left(\left(\tau_{f_i}^{L^*}(g_1)-\varepsilon\right)\left[re^{L(r)}\right]^{\lambda_{f_i}^{L^*}(g_1)}\right)$$
$$+\varepsilon_1 M_{f_k}\left(\left(\tau_{f_k}^{L^*}(g_1)-\varepsilon\right)\left[re^{L(r)}\right]^{\lambda_{f_k}^{L^*}(g_1)}\right).$$

Therefore using the similar technique as executed in the proof of case IV of Theorem 1, it follows from above and the first part of Theorem B that

$$\tau_{f_1 \pm f_2}^{L^*}(g_1) \geqslant \tau_{f_i}^{L^*}(g_1).$$

At this time without loss of generality, we may consider that $\lambda_{f_1}^{L^*}(g_1) < \lambda_{f_2}^{L^*}(g_1)$ and $f = f_1 \pm f_2$. Then $\tau_f^{L^*}(g_1) \geqslant \tau_{f_1}^{L^*}(g_1)$. Further let $f_1 = (f \pm f_2)$. Therefore in view of Theorem B (i), $\lambda_f^{L^*}(g_1) < \lambda_{f_2}^{L^*}(g_1)$ and accordingly $\tau_{f_1}^{L^*}(g_1) \geqslant \tau_f^{L^*}(g_1)$. Hence $\tau_f^{L^*}(g_1) = \tau_{f_1}^{L^*}(g_1) \Rightarrow \tau_{f_1 \pm f_2}^{L^*}(g_1) = \tau_{f_1}^{L^*}(g_1)$. So, $\tau_{f_1 \pm f_2}^{L^*}(g_1) = \tau_{f_i}^{L^*}(g_1) \mid i = 1,2$ where $\lambda_{f_i}^{L^*}(g_1) = \min\left\{\lambda_{f_k}^{L^*}(g_1) \mid k = i = 1,2\right\}$ provided $\lambda_{f_1}^{L^*}(g_1) \neq \lambda_{f_2}^{L^*}(g_1)$.

Case IV. Now let us consider $\lambda_{f_i}^{L^*}(g_1) < \lambda_{f_k}^{L^*}(g_1)$ where $k = i = 1,2$ with $f_i \neq f_k$.

Further we can make the term $\dfrac{M_{f_k}\left(\left(\overline{\tau}_{f_i}^{L^*}(g_1)-\varepsilon\right)\left[r_n e^{L(r_n)}\right]^{\lambda_{f_i}^{L^*}(g_1)}\right)}{M_{f_k}\left(\left(\tau_{f_k}^{L^*}(g_1)-\varepsilon\right)\left[r_n e^{L(r_n)}\right]^{\lambda_{f_k}^{L^*}(g_1)}\right)}$ sufficiently small by taking n sufficiently large since $\rho_{f_i}^{L^*}(g_1) < \rho_{f_k}^{L^*}(g_1)$. Therefore

$$\frac{M_{f_k}\left(\left(\overline{\tau}_{f_i}^{L^*}(g_1)-\varepsilon\right)\left[r_n e^{L(r_n)}\right]^{\lambda_{f_i}^{L^*}(g_1)}\right)}{M_{f_k}\left(\left(\tau_{f_k}^{L^*}(g_1)-\varepsilon\right)\left[r_n e^{L(r_n)}\right]^{\lambda_{f_k}^{L^*}(g_1)}\right)} < \varepsilon_1 \tag{17}$$

for sufficiently large n.

Now in view of (2), (3) and (16) we obtain for a sequence of values of r tending to infinity that

$$M_{f_1 \pm f_2}\left[\left(\overline{\tau}_{f_i}^{L^*}(g_1)-\varepsilon\right)\left[r_n e^{L(r_n)}\right]^{\lambda_{f_i}^{L^*}(g_1)}\right] < M_{f_1}\left(\left(\overline{\tau}_{f_i}^{L^*}(g_1)-\varepsilon\right)\left[r_n e^{L(r_n)}\right]^{\lambda_{f_i}^{L^*}(g_1)}\right)$$
$$+M_{f_2}\left(\left(\overline{\tau}_{f_i}^{L^*}(g_1)-\varepsilon\right)\left[r_n e^{L(r_n)}\right]^{\lambda_{f_i}^{L^*}(g_1)}\right)$$

i.e.,

$$M_{f_1 \pm f_2}\left[\left(\overline{\tau}_{f_i}^{L^*}(g_1)-\varepsilon\right)\left[r_n e^{L(r_n)}\right]^{\lambda_{f_i}^{L^*}(g_1)}\right] < M_{f_i}\left(\left(\overline{\tau}_{f_i}^{L^*}(g_1)-\varepsilon\right)\left[r_n e^{L(r_n)}\right]^{\lambda_{f_i}^{L^*}(g_1)}\right)$$
$$+\varepsilon_1 M_{f_k}\left(\left(\tau_{f_k}^{L^*}(g_1)-\varepsilon\right)\left[r_n e^{L(r_n)}\right]^{\lambda_{f_k}^{L^*}(g_1)}\right).$$

Hence using the similar technique of case III of Theorem 1, we obtain conclusion from above that $\overline{\tau}^{L^*}_{f_1 \pm f_2}(g_1) = \overline{\tau}^{L^*}_{f_i}(g_1) \mid i = 1,2$ where $\lambda^{L^*}_{f_i}(g_1) = \min\{\lambda^{L^*}_{f_k}(g_1) \mid k = i = 1,2\}$ provided $\lambda^{L^*}_{f_1}(g_1) \neq \lambda^{L^*}_{f_2}(g_1)$.

So the second part of the theorem follows from Case III and Case IV.

The proof of the third part of the Theorem is omitted as it can be carried in view of the third part of Theorem B and the above cases. □

Proof of Theorem 3. *Case I.* By Lemma 3, f_1 is transcendental. Suppose that $\rho^{L^*}_{f_1}(g_k) < \rho^{L^*}_{f_1}(g_i)$ where $k = i = 1,2$ with $g_k \neq g_i$. Now for any arbitrary $\varepsilon > 0$, we have from (1) for all sufficiently large values of r that

$$M_{g_1 \cdot g_2}(r) \leqslant M_{g_1}(r) \cdot M_{g_2}(r). \tag{18}$$

$$M_{g_1 \cdot g_2}(r) \leqslant M_{f_1}\left[\left(\sigma^{L^*}_{f_1}(g_k) + \frac{\varepsilon}{2}\right)\left[re^{L(r)}\right]^{\rho^{L^*}_{f_1}(g_k)}\right] M_{f_1}\left[\left(\sigma^{L^*}_{f_1}(g_i) + \frac{\varepsilon}{2}\right)\left[re^{L(r)}\right]^{\rho^{L^*}_{f_1}(g_i)}\right].$$

Since $\rho^{L^*}_{f_1}(g_k) < \rho^{L^*}_{f_1}(g_i)$, we get for all sufficiently large values of r that $(\sigma^{L^*}_{f_1}(g_i) + \varepsilon)\left[re^{L(r)}\right]^{\rho^{L^*}_{f_1}(g_i)} > (\sigma^{L^*}_{f_1}(g_k) + \varepsilon)\left[re^{L(r)}\right]^{\rho^{L^*}_{f_1}(g_k)}$.

Therefore $M_{f_1}\left[(\sigma^{L^*}_{f_1}(g_i) + \varepsilon)\left[re^{L(r)}\right]^{\rho^{L^*}_{f_1}(g_i)}\right] > M_{f_1}\left[(\sigma^{L^*}_{f_1}(g_k) + \varepsilon)\left[re^{L(r)}\right]^{\rho^{L^*}_{f_1}(g_k)}\right]$ and from the above arguments it follows for all sufficiently large values of r that

$$M_{g_1 \cdot g_2}(r) < M_{f_1}\left[\left(\sigma^{L^*}_{f_1}(g_i) + \frac{\varepsilon}{2}\right)\left[re^{L(r)}\right]^{\rho^{L^*}_{f_1}(g_i)}\right]^2. \tag{19}$$

Let us observe that

$$\delta_1 := \frac{\sigma^{L^*}_{f_1}(g_i) + \varepsilon}{\sigma^{L^*}_{f_1}(g_i) + \frac{\varepsilon}{2}} > 1$$

$$\Rightarrow \log\left(\sigma^{L^*}_{f_1}(g_i) + \varepsilon\right)\left[re^{L(r)}\right]^{\rho^{L^*}_{f_1}(g_i)} > \log\left(\sigma^{L^*}_{f_1}(g_i) + \frac{\varepsilon}{2}\right)\left[re^{L(r)}\right]^{\rho^{L^*}_{f_1}(g_i)}$$

$$\Rightarrow \frac{\log\left(\sigma^{L^*}_{f_1}(g_i) + \varepsilon\right)\left[re^{L(r)}\right]^{\rho^{L^*}_{f_1}(g_i)}}{\log\left(\sigma^{L^*}_{f_1}(g_i) + \frac{\varepsilon}{2}\right)\left[re^{L(r)}\right]^{\rho^{L^*}_{f_1}(g_i)}} = \delta\,(\text{say}) > 1$$

$$\Rightarrow \log\left(\sigma^{L^*}_{f_1}(g_i) + \varepsilon\right)\left[re^{L(r)}\right]^{\rho^{L^*}_{f_1}(g_i)} = \delta \log\left(\sigma^{L^*}_{f_1}(g_i) + \frac{\varepsilon}{2}\right)\left[re^{L(r)}\right]^{\rho^{L^*}_{f_1}(g_i)}. \tag{20}$$

Since f_1 has the Property (A), in view of Lemma 2, Theorem C (i) and (20) we obtain from (19) for all sufficiently large values of r that

$$M_{g_1 \cdot g_2}(r) < M_{f_1}\left[\left(\left(\sigma^{L^*}_{f_1}(g_i) + \frac{\varepsilon}{2}\right)\left[re^{L(r)}\right]^{\rho^{L^*}_{f_1}(g_i)}\right)^{\delta}\right]$$

i.e.,

$$M_{g_1 \cdot g_2}(r) < M_{f_1}\left[\left(\sigma_{f_1}^{L^*}(g_i)+\varepsilon\right)\left[re^{L(r)}\right]^{\rho_{f_1}^{L^*}(g_i)}\right].$$

That is, we have

$$\frac{M_{f_1}^{-1} M_{g_1 \cdot g_2}(r)}{\left[re^{L(r)}\right]^{\rho_{f_1}^{L^*}(g_i)}} < \left(\sigma_{f_1}^{L^*}(g_i)+\varepsilon\right)$$

i.e.,

$$\frac{M_{f_1}^{-1} M_{g_1 \cdot g_2}(r)}{\left[re^{L(r)}\right]^{\rho_{f_1}^{L^*}(g_1 \cdot g_2)}} < \left(\sigma_{f_1}^{L^*}(g_i)+\varepsilon\right).$$

i.e.,

$$\sigma_{f_1}^{L^*}(g_1 \cdot g_2) \leqslant \sigma_{f_1}^{L^*}(g_i). \tag{21}$$

In order to establish the equality of (21), let us restrict ourselves on the functions f_1 and g_i with the property $2^{\rho_{f_1}^{L^*}(g_i)} \mid i = 1, 2 < 1$. Now let h, h_1, h_2 and k be any four entire functions such that $h = \frac{h_2}{h_1}$ and $\rho_k^{L^*}(h_1) < \rho_k^{L^*}(h_2)$. So $T_h(r) = T_{\frac{h_2}{h_1}}(r) \leqslant T_{h_2}(r) + T_{h_1}(r) + O(1)$. Now in view of Lemma 4 and in the line of the construction of the proof as above it follows for all sufficiently large values of r that

$$M_k^{-1} M_h\left(\frac{r}{2}\right) < \left(\sigma_k^{L^*}(h_2)+\varepsilon\right)\left[re^{L(r)}\right]^{\rho_k^{L^*}(h_2)} + O(1).$$

Since $L(ar) \sim L(r)$ as $r \to \infty$ for every positive constant a, we get from above for all sufficiently large values of r that

$$\frac{M_k^{-1} M_{\frac{h_2}{h_1}}(r)}{\left[\left(\frac{r}{2}\right) e^{L\left(\frac{r}{2}\right)}\right]^{\rho_k^{L^*}\left(\frac{h_2}{h_1}\right)}} < 2^{\rho_k^{L^*}(h_2)} \times \left(\sigma_k^{L^*}(h_2)+\varepsilon\right) + \frac{O(1)}{\left[re^{L(r)}\right]^{\rho_k^{L^*}(h_2)}}.$$

Now if we consider $2^{\rho_k^{L^*}(h_2)} < 1$ then it follows from above for all sufficiently large values of r that

$$\frac{M_k^{-1} M_{\frac{h_2}{h_1}}(r)}{\left[\left(\frac{r}{2}\right) e^{L\left(\frac{r}{2}\right)}\right]^{\rho_k^{L^*}\left(\frac{h_2}{h_1}\right)}} < \left(\sigma_k^{L^*}(h_2)+\varepsilon\right) + \frac{O(1)}{\left[re^{L(r)}\right]^{\rho_k^{L^*}(h_2)}}$$

i.e.,

$$\sigma_k(h) = \sigma_k\left(\frac{h_2}{h_1}\right) \leqslant \sigma_k(h_2).$$

Further without loss of any generality, let $g = g_1 \cdot g_2$ and $\rho_{f_1}^{L^*}(g_1) < \rho_{f_1}^{L^*}(g_2) = \rho_{f_1}^{L^*}(g)$. Then $\sigma_{f_1}^{L^*}(g) \leqslant \sigma_{f_1}^{L^*}(g_2)$. Also let $g_2 = \frac{g}{g_1}$ and $2^{\rho_{f_1}^{L^*}(g_2)} < 1$. So in this case we obtain from above arguments that $\sigma_{f_1}^{L^*}(g_2) \leqslant \sigma_{f_1}^{L^*}(g)$. Hence $\sigma_{f_1}^{L^*}(g) = \sigma_{f_1}^{L^*}(g_2) \Rightarrow \sigma_{f_1}^{L^*}(g_1 \cdot g_2) = \sigma_{f_1}^{L^*}(g_2)$. Thus, $\sigma_{f_1}^{L^*}(g_1 \cdot g_2) = \sigma_{f_1}^{L^*}(g_i) \mid i = 1,2$ where $\rho_{f_1}^{L^*}(g_i) = \max\left\{\rho_{f_1}^{L^*}(g_k) \mid k = i = 1,2\right\}$, $\rho_{f_1}^{L^*}(g_1) \neq \rho_{f_1}^{L^*}(g_2)$ and $2^{\rho_{f_1}^{L^*}(g_i)} \mid i = 1,2 < 1$.

Next we may suppose that $g = \frac{g_1}{g_2}$ with $g_1,\ g_2,\ g$ are all entire functions and also suppose that $\rho_{f_1}^{L^*}(g_2) < \rho_{f_1}^{L^*}(g_1)$. We have $g_1 = g \cdot g_2$. Therefore $\sigma_{f_1}^{L^*}(g_1) = \sigma_{f_1}^{L^*}(g)$ as $\rho_{f_1}^{L^*}(g) > \rho_{f_1}^{L^*}(g_2)$ and $2^{\rho_{f_1}^{L^*}(g_1)} < 1$.

Case II. In view of Lemma 3, f_1 is transcendental. Now let $\rho_{f_1}^{L^*}(g_k) < \rho_{f_1}^{L^*}(g_i)$ where $k = i = 1,2$ with $g_k \neq g_i$. Therefore from (1) and (4) it follows for a sequence $\{r_n\}$ of values of r tending to infinity that

$$M_{g_1 \cdot g_2}(r_n) \leqslant M_{g_1}(r_n) \cdot M_{g_2}(r_n). \tag{22}$$

That is, we have

$$\begin{aligned} M_{g_1 \cdot g_2}(r_n) \leqslant\ & M_{f_1}\left[\left(\sigma_{f_1}^{L^*}(g_k) + \frac{\varepsilon}{2}\right)\left[r_n e^{L(r_n)}\right]^{\rho_{f_1}^{L^*}(g_k)}\right] \\ & \times M_{f_1}\left[\left(\overline{\sigma}_{f_1}^{L^*}(g_i) + \frac{\varepsilon}{2}\right)\left[r_n e^{L(r_n)}\right]^{\rho_{f_1}^{L^*}(g_i)}\right]. \end{aligned} \tag{23}$$

Since $\rho_{f_1}^{L^*}(g_k) < \rho_{f_1}^{L^*}(g_i)$, so for a sequence of values of r tending to infinity $M_{f_1}\left[\left(\overline{\sigma}_{f_1}^{L^*}(g_i) + \frac{\varepsilon}{2}\right)\left[r_n e^{L(r_n)}\right]^{\rho_{f_1}^{L^*}(g_i)}\right] > M_{f_1}\left[\left(\sigma_{f_1}^{L^*}(g_k) + \frac{\varepsilon}{2}\right)\left[r_n e^{L(r_n)}\right]^{\rho_{f_1}^{L^*}(g_k)}\right]$ holds and therefore from (23) we obtain for a sequence $\{r_n\}$ of values of r tending to infinity that

$$M_{g_1 \cdot g_2}(r_n) < M_{f_1}\left[\left(\overline{\sigma}_{f_1}^{L^*}(g_i) + \frac{\varepsilon}{2}\right)\left[r_n e^{L(r_n)}\right]^{\rho_{f_1}^{L^*}(g_i)}\right]^2. \tag{24}$$

Now using the similar technique for a sequence of values of r tending to infinity as explored in the proof of Case I, the second part of Theorem 3 I (ii) follows from (24).

Therefore the first part of theorem follows Case I and Case II.

Case III. By Lemma 3, g_1 is transcendental. Suppose that $\rho_{f_i}^{L^*}(g_1) < \rho_{f_k}^{L^*}(g_1)$ where $k = i = 1,2$ with $f_i \neq f_k$ and g_1 is of regular relative L^*-growth with respect to at least any one of f_1 or f_2.

Now for all sufficiently large values of n and $\rho_{f_i}^{L^*}(g_1) < \rho_{f_k}^{L^*}(g_1)$

$$\left(\overline{\sigma}_{f_k}^{L^*}(g_1) - \varepsilon\right)\left[r_n e^{L(r_n)}\right]^{\rho_{f_k}^{L^*}(g_1)} > \left(\sigma_{f_i}^{L^*}(g_1) - \varepsilon\right)\left[r_n e^{L(r_n)}\right]^{\rho_{f_i}^{L^*}(g_1)}$$

holds. Consequently

$$M_{f_k}\left(\left(\overline{\sigma}_{f_k}^{L^*}(g_1)-\varepsilon\right)\left[r_n e^{L(r_n)}\right]^{\rho_{f_k}^{L^*}(g_1)}\right) > M_{f_k}\left(\left(\sigma_{f_i}^{L^*}(g_1)-\varepsilon\right)\left[r_n e^{L(r_n)}\right]^{\rho_{f_i}^{L^*}(g_1)}\right)$$

also holds.

Therefore in view of (2), (3) and above we obtain for a sequence of values of r tending to infinity that

$$M_{f_1\cdot f_2}\left(\left(\sigma_{f_i}^{L^*}(g_1)-\varepsilon\right)\left[r_n e^{L(r_n)}\right]^{\rho_{f_i}^{L^*}(g_1)}\right)$$
$$< M_{f_1}\left(\left(\sigma_{f_i}^{L^*}(g_1)-\varepsilon\right)\left[r_n e^{L(r_n)}\right]^{\rho_{f_i}^{L^*}(g_1)}\right)\times M_{f_2}\left(\left(\sigma_{f_i}^{L^*}(g_1)-\varepsilon\right)\left[r_n e^{L(r_n)}\right]^{\rho_{f_i}^{L^*}(g_1)}\right)$$

i.e.,

$$M_{f_1\cdot f_2}\left(\left(\sigma_{f_i}^{L^*}(g_1)-\varepsilon\right)\left[r_n e^{L(r_n)}\right]^{\rho_{f_i}^{L^*}(g_1)}\right)$$
$$< M_{f_i}\left(\left(\sigma_{f_i}^{L^*}(g_1)-\varepsilon\right)\left[r_n e^{L(r_n)}\right]^{\rho_{f_i}^{L^*}(g_1)}\right)\times M_{f_k}\left(\left(\sigma_{f_i}^{L^*}(g_1)-\varepsilon\right)\left[r_n e^{L(r_n)}\right]^{\rho_{f_i}^{L^*}(g_1)}\right)$$

i.e.,

$$M_{f_1\cdot f_2}\left(\left(\sigma_{f_i}^{L^*}(g_1)-\varepsilon\right)\left[r_n e^{L(r_n)}\right]^{\rho_{f_i}^{L^*}(g_1)}\right)$$
$$< M_{f_i}\left(\left(\sigma_{f_i}^{L^*}(g_1)-\varepsilon\right)\left[r_n e^{L(r_n)}\right]^{\rho_{f_i}^{L^*}(g_1)}\right)\times M_{f_k}\left(\left(\overline{\sigma}_{f_k}^{L^*}(g_1)-\varepsilon\right)\left[r_n e^{L(r_n)}\right]^{\rho_{f_k}^{L^*}(g_1)}\right)$$

i.e.,

$$M_{f_1\cdot f_2}\left(\left(\sigma_{f_i}^{L^*}(g_1)-\varepsilon\right)\left[r_n e^{L(r_n)}\right]^{\rho_{f_i}^{L^*}(g_1)}\right) < \left[M_{g_1}(r_n)\right]^2.$$

Since g_1 has the Property (A), in view of Lemma 2 we obtain from above for a sequence of values of r tending to infinity that

$$i.e.,\quad M_{f_1\cdot f_2}\left(\left(\sigma_{f_i}^{L^*}(g_1)-\varepsilon\right)\left[r_n e^{L(r_n)}\right]^{\rho_{f_i}^{L^*}(g_1)}\right) < M_{g_1}\left(r_n^{\delta}\right). \tag{25}$$

Now making $\delta \to 1+$ we obtain from (25) and the second part of Theorem C for a sequence $\{r_n\}$ of values of r tending to infinity that

$$\left(\sigma_{f_i}^{L^*}(g_1)-\varepsilon\right)\left[r_n e^{L(r_n)}\right]^{\rho_{f_i}^{L^*}(g_1)} < M_{f_1\cdot f_2}^{-1} M_{g_1}(r_n)$$

i.e.,

$$\left(\sigma_{f_i}^{L^*}(g_1)-\varepsilon\right) < \frac{M_{f_1\cdot f_2}^{-1}M_{g_1}(r_n)}{\left[r_n e^{L(r_n)}\right]^{\rho_{f_1\cdot f_2}^{L^*}(g_1)}}.$$

Since $\varepsilon>0$ is arbitrary, it follows from above arguments that

$$\sigma_{f_1\cdot f_2}^{L^*}(g_1)\geqslant\sigma_{f_i}^{L^*}(g_1). \tag{26}$$

In order to establish the equality of (26), let us restrict ourselves on the functions f_i and g_1 with the property $2^{\rho_{f_i}^{L^*}(g_1)} \mid i=1,2>1$. Now let $h,\ h_1,\ h_2$ and k be any four entire functions such that $h=\frac{h_1}{h_2}$ and $\rho_{h_1}^{L^*}(k)<\rho_{h_2}^{L^*}(k)$. So $T_h(r)=T_{\frac{h_1}{h_2}}(r)\leqslant T_{h_1}(r)+T_{h_2}(r)+O(1)$. Now if we consider $2^{\rho_{h_1}^{L^*}(k)}>1$ then in view of Lemma 4 and in the line of the construction of the proof as above and case I of Theorem 3 it follows that $\sigma_{h_1}^{L^*}(k)\leqslant\sigma_h^{L^*}(k)=\sigma_{\frac{h_1}{h_2}}^{L^*}(k)$ as $L(ar)\sim L(r)$ as $r\to\infty$ for every positive constant a.

Further without loss of any generality, let $f=f_1\cdot f_2$ and $\rho_{f_1}^{L^*}(g_1)=\rho_f^{L^*}(g_1)<\rho_{f_2}^{L^*}(g_1)$. Then $\sigma_f^{L^*}(g_1)\geqslant\sigma_{f_1}^{L^*}(g_1)$. Also let $f_1=\frac{f}{f_2}$ and in this case we obtain from above that $\sigma_{f_1}^{L^*}(g_1)\geqslant\sigma_f^{L^*}(g_1)$ if $2^{\rho_{f_1}^{L^*}(g_1)}>1$. Hence $\sigma_f^{L^*}(g_1)=\sigma_{f_1}^{L^*}(g_1)$ implies that $\sigma_{f_1\cdot f_2}^{L^*}(g_1)=\sigma_{f_1}^{L^*}(g_1)$. Thus, $\sigma_{f_1\cdot f_2}^{L^*}(g_1)=\sigma_{f_i}^{L^*}(g_1)\mid i=1,2$ where $\rho_{f_i}^{L^*}(g_1)=\min\left\{\rho_{f_k}^{L^*}(g_1)\mid k=1,2\right\}$, $\rho_{f_1}^{L^*}(g_1)\neq\rho_{f_2}^{L^*}(g_1)$ and $2^{\rho_{f_i}^{L^*}(g_1)}\mid i=1,2>1$.

Next one may suppose that $f=\frac{f_2}{f_1}$ with $f_1,\ f_2,\ f$ are all entire and $\rho_{f_2}^{L^*}(g_1)<\rho_{f_1}^{L^*}(g_1)$. We have $f_2=f\cdot f_1$. Therefore $\sigma_{f_2}^{L^*}(g_1)=\sigma_f^{L^*}(g_1)$ as $\rho_{f_1}^{L^*}(g_1)>\rho_f^{L^*}(g_1)$ and $2^{\rho_{f_1}^{L^*}(g_1)}\mid i=1,2>1$.

Case IV. By Lemma 3, g_1 is transcendental. Suppose $\rho_{f_i}^{L^*}(g_1)<\rho_{f_k}^{L^*}(g_1)$ where $k=i=1,2$ with $f_i\neq f_k$ $(i\neq k)$ and g_1 is of regular relative L^*-growth with respect to at least any one of f_1 or f_2.

Therefore for all sufficiently large values of r and $\rho_{f_i}^{L^*}(g_1)<\rho_{f_k}^{L^*}(g_1)$

$$\left(\overline{\sigma}_{f_k}^{L^*}(g_1)-\varepsilon\right)\left[re^{L(r)}\right]^{\rho_{f_k}^{L^*}(g_1)} > \left(\overline{\sigma}_{f_i}^{L^*}(g_1)-\varepsilon\right)\left[re^{L(r)}\right]^{\rho_{f_i}^{L^*}(g_1)}$$

holds. As a result

$$M_{f_k}\left(\left(\overline{\sigma}_{f_k}^{L^*}(g_1)-\varepsilon\right)\left[re^{L(r)}\right]^{\rho_{f_k}^{L^*}(g_1)}\right) > M_{f_k}\left(\left(\overline{\sigma}_{f_i}^{L^*}(g_1)-\varepsilon\right)\left[re^{L(r)}\right]^{\rho_{f_i}^{L^*}(g_1)}\right)$$

also holds.

Therefore in view of (2) and from above arguments we obtain for all sufficiently large values of r that

$$M_{f_1\cdot f_2}\left(\left(\overline{\sigma}_{f_i}^{L^*}(g_1)-\varepsilon\right)\left[re^{L(r)}\right]^{\rho_{f_i}^{L^*}(g_1)}\right)$$
$$< M_{f_1}\left(\left(\overline{\sigma}_{f_i}^{L^*}(g_1)-\varepsilon\right)\left[re^{L(r)}\right]^{\rho_{f_i}^{L^*}(g_1)}\right)\times M_{f_2}\left(\left(\overline{\sigma}_{f_i}^{L^*}(g_1)-\varepsilon\right)\left[re^{L(r)}\right]^{\rho_{f_i}^{L^*}(g_1)}\right)$$

i.e.,

$$M_{f_1\cdot f_2}\left(\left(\overline{\sigma}_{f_i}^{L^*}(g_1)-\varepsilon\right)\left[re^{L(r)}\right]^{\rho_{f_i}^{L^*}(g_1)}\right)<[M_{g_1}(r)]^2. \tag{27}$$

Thus Theorem 3 II (ii) follows from (27) by using the similar technique for all sufficiently large values of r of Case III.

Therefore the second part of the theorem follows from Case III and Case IV.

Proof of the third part of the Theorem is omitted as it can be carried out in view of Theorem C (iii) and the above cases. □

Proof of Theorem 4. Case I. By Lemma 3, f_1 is transcendental. Suppose that $\lambda_{f_1}^{L^*}(g_k)<\lambda_{f_1}^{L^*}(g_i)$ where $k=i=1,2$ with $g_k\neq g_i$ and at least g_1 or g_2 is of regular relative L^*-growth with respect to f_1. Now for any arbitrary $\varepsilon>0$, from (11), (14) and (22), we obtain for a sequence $\{r_n\}$ of values of r tending to infinity that

$$M_{g_1\cdot g_2}(r_n)\leqslant M_{f_1}\left[\left(\overline{\tau}_{f_1}^{L^*}(g_k)+\frac{\varepsilon}{2}\right)\left[r_ne^{L(r_n)}\right]^{\lambda_{f_1}^{L^*}(g_k)}\right]$$
$$\times M_{f_1}\left[\left(\tau_{f_1}^{L^*}(g_i)+\frac{\varepsilon}{2}\right)\left[r_ne^{L(r_n)}\right]^{\lambda_{f_1}^{L^*}(g_i)}\right].$$

As $\lambda_{f_1}^{L^*}(g_k)<\lambda_{f_1}^{L^*}(g_i)$, we get from above arguments for a sequence $\{r_n\}$ of values of r tending to infinity that

$$M_{g_1\cdot g_2}(r_n)<M_{f_1}\left[\left(\tau_{f_1}^{L^*}(g_i)+\frac{\varepsilon}{2}\right)\left[r_ne^{L(r_n)}\right]^{\lambda_{f_1}^{L^*}(g_i)}\right]^2. \tag{28}$$

Now using the similar technique as explored in the proof of Case II of Theorem 3, we have from (28) and the second part of Theorem D that

$$\tau_{f_1}^{L^*}(g_1\cdot g_2)\leqslant\tau_{f_1}^{L^*}(g_i). \tag{29}$$

In order to establish the equality of (29), let us restrict ourselves on the functions f_1 and g_i with the property $2^{\lambda_{f_1}^{L^*}(g_i)}\mid i=1,2<1$. Now let h, h_1, h_2 and k be any four entire functions such that $h=\frac{h_2}{h_1}$ and $\lambda_k^{L^*}(h_1)<\lambda_k^{L^*}(h_2)$. So $T_h(r)=T_{\frac{h_2}{h_1}}(r)\leqslant T_{h_2}(r)+T_{h_1}(r)+O(1)$. Now if we take $\lambda_k^{L^*}(h_2)<1$ then in view of Lemma 4 and in the line of the construction of the proof as above and case I of Theorem 3 it follows that $\tau_k^{L^*}(h) = \tau_k^{L^*}\left(\frac{h_2}{h_1}\right) \leqslant \tau_k^{L^*}(h_2)$.

Further without loss of any generality, let $g = g_1 \cdot g_2$ and $\lambda_{f_1}^{L^*}(g_1) < \lambda_{f_1}^{L^*}(g_2) = \lambda_{f_1}^{L^*}(g)$. Then $\tau_{f_1}^{L^*}(g) \leqslant \tau_{f_1}^{L^*}(g_2)$. Also let $g_2 = \frac{g}{g_1}$ and in this case we obtain from above arguments that $\tau_{f_1}^{L^*}(g_2) \leqslant \tau_{f_1}^{L^*}(g)$ when $2^{\lambda_{f_1}^{L^*}(g_2)} < 1$. Hence $\tau_{f_1}^{L^*}(g) = \tau_{f_1}^{L^*}(g_2) \Rightarrow \tau_{f_1}^{L^*}(g_1 \cdot g_2) = \tau_{f_1}^{L^*}(g_2)$. Thus, $\tau_{f_1}^{L^*}(g_1 \cdot g_2) = \tau_{f_1}^{L^*}(g_i) \mid i = 1,2$ where $\lambda_{f_1}^{L^*}(g_i) = \max\left\{\lambda_{f_1}^{L^*}(g_k) \mid k = i = 1,2\right\}$, $\lambda_{f_1}^{L^*}(g_1) \neq \lambda_{f_1}^{L^*}(g_2)$ and $2^{\lambda_{f_1}^{L^*}(g_i)} \mid i = 1,2 < 1$.

Next we may suppose that $g = \frac{g_1}{g_2}$ with g_1, g_2, g are all entire functions and also suppose that $\lambda_{f_1}^{L^*}(g_2) < \lambda_{f_1}^{L^*}(g_1)$. We have $g_1 = g \cdot g_2$. Therefore $\lambda_{f_1}^{L^*}(g_1) = \lambda_{f_1}^{L^*}(g)$ as $\lambda_{f_1}^{L^*}(g) > \lambda_{f_1}^{L^*}(g_2)$ and $2^{\lambda_{f_1}^{L^*}(g_1)} < 1$.

Case II. In view of Lemma 3, f_1 is transcendental. Now let $\lambda_{f_1}^{L^*}(g_k) < \lambda_{f_1}^{L^*}(g_i)$ where $k = i = 1,2$ with $g_k \neq g_i$ and at least g_1 or g_2 is of regular relative L^*-growth with respect to f_1. Therefore from (18) and (11) it follows for all sufficiently large values of r that

$$M_{g_1 \cdot g_2}(r) \leqslant M_{f_1}\left[\left(\overline{\tau}_{f_1}^{L^*}(g_k) + \frac{\varepsilon}{2}\right)\left[re^{L(r)}\right]^{\lambda_{f_1}^{L^*}(g_k)}\right]$$
$$\times M_{f_1}\left[\left(\overline{\tau}_{f_1}^{L^*}(g_i) + \frac{\varepsilon}{2}\right)\left[re^{L(r)}\right]^{\lambda_{f_1}^{L^*}(g_i)}\right]. \tag{30}$$

Since $\lambda_{f_1}^{L^*}(g_k) < \lambda_{f_1}^{L^*}(g_i)$, so for all sufficiently large values of r,

$$M_{f_1}\left[\left(\overline{\tau}_{f_1}^{L^*}(g_i) + \frac{\varepsilon}{2}\right)\left[re^{L(r)}\right]^{\lambda_{f_1}^{L^*}(g_i)}\right] > M_{f_1}\left[\left(\overline{\tau}_{f_1}^{L^*}(g_k) + \frac{\varepsilon}{2}\right)\left[re^{L(r)}\right]^{\lambda_{f_1}^{L^*}(g_k)}\right]$$

holds and therefore from (30) we get for all sufficiently large values of r that

$$M_{g_1 \cdot g_2}(r) < M_{f_1}\left[\left(\overline{\tau}_{f_1}^{L^*}(g_i) + \frac{\varepsilon}{2}\right)\left[re^{L(r)}\right]^{\lambda_{f_1}^{L^*}(g_i)}\right]^2. \tag{31}$$

Now using the similar technique of Case I of Theorem 3, Theorem 4 I (i) follows from (31).

Therefore combining Case I and Case II, the first part of the theorem follows.

Case III. By Lemma 3, g_1 is transcendental. Suppose that $\lambda_{f_i}^{L^*}(g_1) < \lambda_{f_k}^{L^*}(g_1)$ where $k = i = 1, 2$ with $f_i \neq f_k$.

Since $\lambda_{f_i}^{L^*}(g_1) < \lambda_{f_k}^{L^*}(g_1)$, then for all sufficiently large values of r

$$\left(\tau_{f_k}^{L^*}(g_1) - \varepsilon\right)\left[re^{L(r)}\right]^{\lambda_{f_k}^{L^*}(g_1)} > \left(\tau_{f_i}^{L^*}(g_1) - \varepsilon\right)\left[re^{L(r)}\right]^{\lambda_{f_i}^{L^*}(g_1)}$$

holds. Therefore

$$M_{f_k}\left(\left(\tau_{f_k}^{L^*}(g_1)-\varepsilon\right)\left[re^{L(r)}\right]^{\lambda_{f_k}^{L^*}(g_1)}\right) > M_{f_k}\left(\left(\tau_{f_i}^{L^*}(g_1)-\varepsilon\right)\left[re^{L(r)}\right]^{\lambda_{f_i}^{L^*}(g_1)}\right)$$

also holds.

Now in view of (12) and from above arguments we obtain for all sufficiently large values of r that

$$M_{f_1\cdot f_2}\left(\left(\tau_{f_i}^{L^*}(g_1)-\varepsilon\right)\left[re^{L(r)}\right]^{\lambda_{f_i}^{L^*}(g_1)}\right)$$
$$< M_{f_1}\left(\left(\tau_{f_i}^{L^*}(g_1)-\varepsilon\right)\left[re^{L(r)}\right]^{\lambda_{f_i}^{L^*}(g_1)}\right)\times M_{f_2}\left(\left(\tau_{f_i}^{L^*}(g_1)-\varepsilon\right)\left[re^{L(r)}\right]^{\lambda_{f_i}^{L^*}(g_1)}\right)$$

i.e.,

$$M_{f_1\cdot f_2}\left(\left(\tau_{f_i}^{L^*}(g_1)-\varepsilon\right)\left[re^{L(r)}\right]^{\lambda_{f_i}^{L^*}(g_1)}\right) < \left[M_{g_1}(r)\right]^2. \tag{32}$$

Further using the similar technique as explored in the proof of Case II of Theorem 3, we have from (32) and the first part of Theorem D that

$$\tau_{f_1\cdot f_2}^{L^*}(g_1) \geqslant \tau_{f_i}^{L^*}(g_1). \tag{33}$$

In order to establish the equality of (33), let us restrict ourselves on the functions f_i and g_1 with the property $2^{\lambda_{f_i}^{L^*}(g_1)} \mid i=1,2>1$. Now let $h,\ h_1,\ h_2$ and k be any four entire functions such that $h=\frac{h_1}{h_2}$ and $\lambda_{h_1}^{L^*}(k)<\lambda_{h_2}^{L^*}(k)$. So $T_h(r)=T_{\frac{h_1}{h_2}}(r)\leqslant T_{h_1}(r)+T_{h_2}(r)+O(1)$. Now if we consider $2^{\lambda_{h_1}^{L^*}(k)}>1$ then in view of Lemma 4 and in the line of the construction of the proof as above and case I of Theorem 3 it follows that $\tau_{h_1}^{L^*}(k)\leqslant\tau_h^{L^*}(k)=\tau_{\frac{h_1}{h_2}}^{L^*}(k)$ since $L(ar)\sim L(r)$ as $r\to\infty$ for every positive constant a.

Further without loss of any generality, let $f=f_1\cdot f_2$ and $\lambda_{f_1}^{L^*}(g_1)=\lambda_f^{L^*}(g_1)<\lambda_{f_2}^{L^*}(g_1)$. Then $\tau_f^{L^*}(g_1)\geqslant\tau_{f_1}^{L^*}(g_1)$. Also let $f_1=\frac{f}{f_2}$ and $2^{\lambda_{f_1}^{L^*}(g_1)}>1$. Therefore in this case we obtain from above that $\tau_{f_1}^{L^*}(g_1)\geqslant\tau_f^{L^*}(g_1)$. Hence $\tau_f^{L^*}(g_1)=\tau_{f_1}^{L^*}(g_1)$ implies that $\tau_{f_1\cdot f_2}^{L^*}(g_1)=\tau_{f_1}^{L^*}(g_1)$. Thus, $\tau_{f_1\cdot f_2}^{L^*}(g_1)=\tau_{f_i}^{L^*}(g_1)\mid i=1,2$ where $\lambda_{f_i}^{L^*}(g_1)=\min\left\{\lambda_{f_k}^{L^*}(g_1)\mid k=1,2\right\}$, $\lambda_{f_1}^{L^*}(g_1)\neq\lambda_{f_2}^{L^*}(g_1)$ and $2^{\lambda_{f_i}^{L^*}(g_1)}\mid i=1,2>1$.

Next one may suppose that $f=\frac{f_2}{f_1}$ with $f_1,\ f_2,\ f$ are all entire and $\lambda_{f_2}^{L^*}(g_1)<\lambda_{f_1}^{L^*}(g_1)$. We have $f_2=f\cdot f_1$. Therefore $\tau_{f_2}^{L^*}(g_1)=\tau_f^{L^*}(g_1)$ as $\lambda_{f_1}^{L^*}(g_1)>\lambda_f^{L^*}(g_1)$ and $2^{\lambda_{f_1}^{L^*}(g_1)}\mid i=1,2>1$.

Case IV. By Lemma 3, g_1 is transcendental. Suppose $\lambda_{f_i}^{L^*}(g_1)<\lambda_{f_k}^{L^*}(g_1)$ where $k=i=1,2$ with $f_i\neq f_k$. Therefore for all sufficiently large values of r we obtain that

$$\left(\tau_{f_k}^{L^*}(g_1)-\varepsilon\right)\left[re^{L(r)}\right]^{\lambda_{f_k}^{L^*}(g_1)} > \left(\overline{\tau}_{f_i}^{L^*}(g_1)-\varepsilon\right)\left[re^{L(r)}\right]^{\lambda_{f_i}^{L^*}(g_1)}.$$

Naturally,

$$M_{f_k}\left(\left(\tau_{f_k}^{L^*}(g_1)-\varepsilon\right)\left[re^{L(r)}\right]^{\lambda_{f_k}^{L^*}(g_1)}\right) > M_{f_k}\left(\left(\overline{\tau}_{f_i}^{L^*}(g_1)-\varepsilon\right)\left[re^{L(r)}\right]^{\lambda_{f_i}^{L^*}(g_1)}\right)$$

holds.

Therefore in view of (12), (13) and from above arguments we obtain for a sequence $\{r_n\}$ of values of r tending to infinity that

$$\begin{aligned} &M_{f_1\cdot f_2}\left(\left(\overline{\tau}_{f_i}^{L^*}(g_1)-\varepsilon\right)\left[re^{L(r)}\right]^{\lambda_{f_i}^{L^*}(g_1)}\right) \\ &< M_{f_1}\left(\left(\overline{\tau}_{f_i}^{L^*}(g_1)-\varepsilon\right)\left[re^{L(r)}\right]^{\lambda_{f_i}^{L^*}(g_1)}\right)\times M_{f_2}\left(\left(\overline{\tau}_{f_i}^{L^*}(g_1)-\varepsilon\right)\left[re^{L(r)}\right]^{\lambda_{f_i}^{L^*}(g_1)}\right) \end{aligned}$$

i.e.,

$$M_{f_1\cdot f_2}\left(\left(\overline{\tau}_{f_i}^{L^*}(g_1)-\varepsilon\right)\left[re^{L(r)}\right]^{\lambda_{f_i}^{L^*}(g_1)}\right) < [M_{g_1}(r)]^2. \tag{34}$$

Therefore using the similar technique for a sequence of values of r tending to infinity of Case III , the second part of Theorem 4 II (ii) follows from (34).

Thus the second part of the theorem follows from Case III and Case IV.

Proof of the third part of the theorem is omitted as it can be carried out in view of Theorem D (iii) and the above cases. □

Proof of Theorem 5. Case I. Suppose that $\rho_{f_1}^{L^*}(g_1)=\rho_{f_1}^{L^*}(g_2)$ $(0<\rho_{f_1}^{L^*}(g_1),$ $\rho_{f_1}^{L^*}(g_2)<\infty)$. Now in view of Theorem A (i) it is easy to see that $\rho_{f_1}^{L^*}(g_1\pm g_2)\leqslant \rho_{f_1}^{L^*}(g_1)=\rho_{f_1}^{L^*}(g_2)$. If possible let

$$\rho_{f_1}^{L^*}(g_1\pm g_2)<\rho_{f_1}^{L^*}(g_1)=\rho_{f_1}^{L^*}(g_2). \tag{35}$$

Let $\sigma_{f_1}^{L^*}(g_1)\neq\sigma_{f_1}(g_2)$. Then in view of Theorem 1 I (i) and (35) we obtain that $\sigma_{f_1}^{L^*}(g_1)=\sigma_{f_1}^{L^*}(g_1\pm g_2\mp g_2)=\sigma_{f_1}^{L^*}(g_2)$ which is a contradiction. Hence $\rho_{f_1}^{L^*}(g_1\pm g_2)$ $=\rho_{f_1}^{L^*}(g_1)=\rho_{f_1}^{L^*}(g_2)$. Similarly with the help of Theorem 1 I (ii), one can obtain the same conclusion under the hypothesis $\overline{\sigma}_{f_1}^{L^*}(g_1)\neq\overline{\sigma}_{f_1}^{L^*}(g_2)$. This proves the first part of the theorem.

Case II. Let us consider that $\rho_{f_1}^{L^*}(g_1)=\rho_{f_2}^{L^*}(g_1)$ $(0<\rho_{f_1}^{L^*}(g_1),\rho_{f_2}^{L^*}(g_1)<\infty)$ and g_1 is of regular relative L^*-growth with respect to at least any one of f_1 or f_2. Therefore in view of the second part of Theorem A, it follows that $\rho_{f_1\pm f_2}^{L^*}(g_1)\geqslant\rho_{f_1}^{L^*}(g_1)=\rho_{f_2}^{L^*}(g_1)$ and if possible let

$$\rho_{f_1\pm f_2}^{L^*}(g_1)>\rho_{f_1}^{L^*}(g_1)=\rho_{f_2}^{L^*}(g_1). \tag{36}$$

Let us consider that $\sigma_{f_1}^{L^*}(g_1)\neq\sigma_{f_2}^{L^*}(g_1)$. Then. in view of the Theorem 1 II (i) and (36) we obtain that $\sigma_{f_1}^{L^*}(g_1)=\sigma_{f_1\pm f_2\mp f_2}^{L^*}(g_1)=\sigma_{f_2}^{L^*}(g_1)$ which is a contradiction.

Hence $\rho_{f_1\pm f_2}^{L^*}(g_1)=\rho_{f_1}^{L^*}(g_1)=\rho_{f_2}^{L^*}(g_1)$. Also in view of Theorem 1 II (ii) one can derive the same conclusion for the condition $\overline{\sigma}_{f_1}^{L^*}(g_1)\neq\overline{\sigma}_{f_2}^{L^*}(g_1)$ and therefore the second part of the theorem is established. □

Proof of Theorem 6. Case I. Let $\lambda_{f_1}^{L^*}(g_1)=\lambda_{f_1}^{L^*}(g_2)$ $(0<\lambda_{f_1}^{L^*}(g_1),\lambda_{f_1}^{L^*}(g_2)<\infty)$ and at least g_1 or g_2 is of regular relative L^*-growth with respect to f_1. Now, in view of Theorem B (ii), it is easy to see that $\lambda_{f_1}^{L^*}(g_1\pm g_2)\leqslant\lambda_{f_1}^{L^*}(g_1)=\lambda_{f_1}^{L^*}(g_2)$. If possible let

$$\lambda_{f_1}^{L^*}(g_1\pm g_2)<\lambda_{f_1}^{L^*}(g_1)=\lambda_{f_1}^{L^*}(g_2). \tag{37}$$

Let $\tau_{f_1}^{L^*}(g_1)\neq\tau_{f_1}^{L^*}(g_2)$. Then in view of Theorem 2 I (i) and (37) we obtain that $\tau_{f_1}^{L^*}(g_1)=\tau_{f_1}^{L^*}(g_1\pm g_2\mp g_2)=\tau_{f_1}^{L^*}(g_2)$ which is a contradiction. Hence $\lambda_{f_1}^{L^*}(g_1\pm g_2)=\lambda_{f_1}^{L^*}(g_1)=\lambda_{f_1}^{L^*}(g_2)$. Similarly with the help of Theorem 2 I (ii) , one can establish the same conclusion under the hypothesis $\overline{\tau}_{f_1}^{L^*}(g_1)\neq\overline{\tau}_{f_1}^{L^*}(g_2)$. This prove the first part of the theorem.

Case II. Let us consider that $\lambda_{f_1}^{L^*}(g_1)=\lambda_{f_2}^{L^*}(g_1)$ $(0<\lambda_{f_1}^{L^*}(g_1),\lambda_{f_2}^{L^*}(g_1)<\infty$. Therefore in view of Theorem B (i) it follows that $\lambda_{f_1\pm f_2}^{L^*}(g_1)\geqslant\lambda_{f_1}^{L^*}(g_1)=\lambda_{f_2}^{L^*}(g_1)$ and if if possible let

$$\lambda_{f_1\pm f_2}^{L^*}(g_1)>\lambda_{f_1}^{L^*}(g_1)=\lambda_{f_2}^{L^*}(g_1). \tag{38}$$

Suppose $\tau_{f_1}^{L^*}(g_1)\neq\tau_{f_2}^{L^*}(g_1)$. Then in view of Theorem 2 II (i) and (38) we obtain that $\tau_{f_1}^{L^*}(g_1)=\tau_{f_1\pm f_2\mp f_2}^{L^*}(g_1)=\tau_{f_2}^{L^*}(g_1)$ which is a contradiction. Hence $\lambda_{f_1\pm f_2}^{L^*}(g_1)=\lambda_{f_1}^{L^*}(g_1)=\lambda_{f_2}^{L^*}(g_1)$. Analogously with the help of Theorem 2 II (ii), the same conclusion can also be derived under the condition $\overline{\tau}_{f_1}^{L^*}(g_1)\neq\overline{\tau}_{f_2}^{L^*}(g_1)$ and therefore the second part of the theorem is established. □

Proof of Theorem 7. Case I. Suppose that $\rho_{f_1}^{L^*}(g_1)=\rho_{f_1}^{L^*}(g_2)$ $(0<\rho_{f_1}^{L^*}(g_1),\rho_{f_1}^{L^*}(g_2)<\infty)$. Now in view of Theorem C (i) it is easy to see that $\rho_{f_1}^{L^*}(g_1\cdot g_2)\leqslant\rho_{f_1}^{L^*}(g_1)=\rho_{f_1}^{L^*}(g_2)$. If possible let

$$\rho_{f_1}^{L^*}(g_1\cdot g_2)<\rho_{f_1}^{L^*}(g_1)=\rho_{f_1}^{L^*}(g_2). \tag{39}$$

Let $\sigma_{f_1}^{L^*}(g_1)\neq\sigma_{f_1}^{L^*}(g_2)$. Now in view of Theorem 3 I (i) and (39) we obtain that $\sigma_{f_1}^{L^*}(g_1)=\sigma_{f_1}^{L^*}\left(\frac{g_1\cdot g_2}{g_2}\right)=\sigma_{f_1}^{L^*}(g_2)$ which is a contradiction. Hence $\rho_{f_1}^{L^*}(g_1\cdot g_2)=\rho_{f_1}^{L^*}(g_1)=\rho_{f_1}^{L^*}(g_2)$. Similarly with the help of Theorem 3 I (ii), one can obtain the same conclusion under the hypothesis $\overline{\sigma}_{f_1}^{L^*}(g_1)\neq\overline{\sigma}_{f_1}^{L^*}(g_2)$. This prove the first part of the theorem.

Case II. Let us consider that $\rho_{f_1}^{L^*}(g_1)=\rho_{f_2}^{L^*}(g_1)$ $(0<\rho_{f_1}^{L^*}(g_1),\rho_{f_2}^{L^*}(g_1)<\infty)$ and g_1 is of regular relative L^*-growth with respect to at least any one of f_1 or f_2. Therefore in view of the second part of Theorem C, it follows that $\rho_{f_1\cdot f_2}^{L^*}(g_1)\geqslant\rho_{f_1}^{L^*}(g_1)=\rho_{f_2}(g_1)$ and if possible let

$$\rho_{f_1\cdot f_2}^{L^*}(g_1)>\rho_{f_1}^{L^*}(g_1)=\rho_{f_2}^{L^*}(g_1). \tag{40}$$

Further suppose that $\sigma_{f_1}^{L^*}(g_1) \neq \sigma_{f_2}^{L^*}(g_1)$. Therefore in view of the first part of Theorem 3 II (i) and (40), we obtain that $\sigma_{f_1}^{L^*}(g_1) = \sigma_{\frac{f_1 \cdot f_2}{f_2}}^{L^*}(g_1) = \sigma_{f_2}^{L^*}(g_1)$ which is a contradiction. Hence $\rho_{f_1 \cdot f_2}^{L^*}(g_1) = \rho_{f_1}^{L^*}(g_1) = \rho_{f_2}^{L^*}(g_1)$. Likewise with the help of Theorem 3 II (ii), one can obtain the same conclusion under the hypothesis $\overline{\sigma}_{f_1}^{L^*}(g_1) \neq \overline{\sigma}_{f_2}^{L^*}(g_1)$. This prove the second part of the theorem.

We omit the proof for quotient as it is an easy consequence of the above two cases. □

Proof of Theorem 8. Case I. Let $\lambda_{f_1}^{L^*}(g_1) = \lambda_{f_1}^{L^*}(g_2)$ $(0 < \lambda_{f_1}^{L^*}(g_1), \lambda_{f_1}^{L^*}(g_2) < \infty)$ and at least g_1 or g_2 is of regular relative L^*-growth with respect to f_1. Now in view of Theorem B (ii) it is easy to see that $\lambda_{f_1}^{L^*}(g_1 \cdot g_2) \leqslant \lambda_{f_1}^{L^*}(g_1) = \lambda_{f_1}^{L^*}(g_2)$. If possible let

$$\lambda_{f_1}^{L^*}(g_1 \cdot g_2) < \lambda_{f_1}^{L^*}(g_1) = \lambda_{f_1}^{L^*}(g_2). \tag{41}$$

Also let $\tau_{f_1}^{L^*}(g_1) \neq \tau_{f_1}^{L^*}(g_2)$. Then in view of Theorem 4 I (i) and (41), we obtain that $\tau_{f_1}^{L^*}(g_1) = \tau_{f_1}^{L^*}\left(\frac{g_1 \cdot g_2}{g_2}\right) = \tau_{f_1}^{L^*}(g_2)$ which is a contradiction. Hence $\lambda_{f_1}^{L^*}(g_1 \cdot g_2) = \lambda_{f_1}^{L^*}(g_1) = \lambda_{f_1}^{L^*}(g_2)$. Analogously with the help of Theorem 4 I (ii), the same conclusion can also be derived under the condition $\overline{\tau}_{f_1}^{L^*}(g_1) \neq \overline{\tau}_{f_1}^{L^*}(g_2)$. Hence the first part of the theorem is established.

Case II. Let us consider that $\lambda_{f_1}^{L^*}(g_1) = \lambda_{f_2}^{L^*}(g_1)$ $(0 < \lambda_{f_1}^{L^*}(g_1), \lambda_{f_2}^{L^*}(g_1) < \infty$. Therefore in view of Theorem B (i) it follows that $\lambda_{f_1 \cdot f_2}^{L^*}(g_1) \geqslant \lambda_{f_1}^{L^*}(g_1) = \lambda_{f_2}^{L^*}(g_1)$ and if possible let

$$\lambda_{f_1 \cdot f_2}^{L^*}(g_1) > \lambda_{f_1}^{L^*}(g_1) = \lambda_{f_2}^{L^*}(g_1). \tag{42}$$

Further let $\tau_{f_1}^{L^*}(g_1) \neq \tau_{f_2}^{L^*}(g_1)$. Then in view of the second part of Theorem 4 II (i) and (42) we obtain that $\tau_{f_1}^{L^*}(g_1) = \tau_{\frac{f_1 \cdot f_2}{f_2}}^{L^*}(g_1) = \tau_{f_2}^{L^*}(g_1)$ which is a contradiction. Hence $\lambda_{f_1 \cdot f_2}^{L^*}(g_1) = \lambda_{f_1}^{L^*}(g_1) = \lambda_{f_2}^{L^*}(g_1)$. Similarly by Theorem 4 II (ii), we get the same conclusion when $\overline{\tau}_{f_1}^{L^*}(g_1) \neq \overline{\tau}_{f_2}^{L^*}(g_1)$ and therefore the second part of the theorem follows.

We omit the proof for quotient as it is an easy consequence of the above two cases. □

6. Concluding Remarks

In this paper, we have investigated some properties of *relative L^*-type* (*relative L^*-lower type*) and *relative L^*-weak type* of entire functions. Here we actually prove Theorem 1 to Theorem 4 under some different conditions stated in Theorem A to Theorem D, respectively. Further some natural questions may arise about the sum and product properties for *relative L^*-type* (*relative L^*-lower type*) and *relative L^*-weak type* of entire functions when the conditions of Theorem 5 to Theorem 8 are respectively provided. Answers of these last questions are left to the interested researchers in this area.

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