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SOME APPROXIMATION PROPERTIES OF HEXAGONAL FOURIER SERIES

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Dedicated to Professor Daniyal M. Israfilov on the occasion of his 60th birthday

Abstract. L. Leindler, A. Meir and V. Totik considered the φ-norm on $C_{2\pi}$ (the space 2π-periodic continuous functions) and estimated the deviation $\|A_n(f)-f\|_{\varphi}$ in terms of the modulus of continuity of $f \in C_{2\pi}$, where (A_n) is a sequence of convolution operators from $C_{2\pi}$ into itself and φ is an increasing function on $(0,\infty)$ (Acta Math. Hung. 45 (1985), 441-443). In the present paper, an analogue of the theorem of Leindler, Meir and Totik is proved for functions periodic with respect to the hexagon lattice. Also, this theorem is applied to obtain estimates for approximation by partial sums of hexagonal Fourier series in Hölder and generalized Hölder norms.

1. Introduction

Approximation properties of Fourier series of 2π-periodic of functions are studied by several mathematicians. In particular, the degree of approximation by partial sums and some linear means of Fourier series was investigated in different metrics. Results of these investigations and more information on approximation properties of Fourier series can be found in the excellent monographs [2], [15] and [17]. Also, there are many theorems about approximation of functions of several real variables by multiple Fourier series. Approximation properties of functions on cubes of the d-dimensional Euclidean space $\mathbb{R}^d$ are studied by assuming that the functions are periodic in each of their variables (see, for example [15, Sections 5.3 and 6.3] and [17, Vol. II, Chapter XVII]). But, in the non-tensor product domains another periodicity is needed. The most notable periodicity is the periodicity defined by lattices. The discrete Fourier analysis on lattices was developed in [11].

A lattice is the discrete subgroup $A\mathbb{Z}^d = \{Ak : k \in \mathbb{Z}^d\}$ of the Euclidean space $\mathbb{R}^d$, where A is a non-singular $d \times d$ matrix – the generator matrix of the lattice. The lattice $A^{-tr}\mathbb{Z}^d$, where A^{-tr} is the transpose of the inverse matrix A^{-1}, is called the dual lattice of $A\mathbb{Z}^d$. A bounded open set $\Omega \subset \mathbb{R}^d$ is said to tile $\mathbb{R}^d$ with the lattice $A\mathbb{Z}^d$ if

$$\sum_{\alpha \in A\mathbb{Z}^d} \chi_{\Omega}(x+\alpha) = 1$$

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for almost all $x \in \mathbb{R}^d$. In this case the set Ω is called a spectral set for the lattice $A\mathbb{Z}^d$ and write $\Omega + A\mathbb{Z}^d = \mathbb{R}^d$. The spectral set Ω is not unique. It is specified that it contains 0 as an interior point and tiles $\mathbb{R}^d$ with the lattice $A\mathbb{Z}^d$ without overlapping and without gap, i. e.

$$\sum_{k \in \mathbb{Z}^d} \chi_\Omega (x + Ak) = 1$$

for all $x \in \mathbb{R}^d$ and $\Omega + Ak$ and $\Omega + Aj$ are disjoint if $k \neq j$. For example we can take $\Omega = \left[-\frac{1}{2}, \frac{1}{2}\right)^d$ for the standard lattice $\mathbb{Z}^d$ (the lattice generated by the identity matrix).

Let Ω be the spectral set of the lattice $A\mathbb{Z}^d$. $L^2(\Omega)$ becomes a Hilbert space with respect to the inner product

$$\langle f, g \rangle_\Omega := \frac{1}{|\Omega|} \int_\Omega f(x) \overline{g(x)} dx,$$

where $|\Omega|$ is the d-dimensional Lebesgue measure of Ω. A theorem of Fuglede states that the set $\left\{ e^{2\pi i \langle \alpha, x \rangle} : \alpha \in A^{-tr}\mathbb{Z}^d \right\}$ is an orthonormal basis of the Hilbert space $L^2(\Omega)$, where $\langle \alpha, x \rangle$ is the usual Euclidean inner product of α and x ([3]). This theorem suggests that, by using the exponentials $e^{2\pi i \langle \alpha, x \rangle}$ $\left(\alpha \in A^{-tr}\mathbb{Z}^d\right)$ one can study Fourier series and approximation on the spectral set of the lattice $A\mathbb{Z}^d$.

A function f is said to be periodic with respect to the lattice $A\mathbb{Z}^d$ if

$$f(x + Ak) = f(x)$$

for all $k \in \mathbb{Z}^d$.

If we consider the standard lattice $\mathbb{Z}^d$ and its spectral set $\left[-\frac{1}{2}, \frac{1}{2}\right)^d$, Fourier series with respect to this lattice coincide with usual multiple Fourier series of functions of d-variables.

2. Hexagon lattice and hexagonal Fourier series

In the Euclidean plane $\mathbb{R}^2$, besides the standard lattice $\mathbb{Z}^2$ and the rectangular domain $\left[-\frac{1}{2}, \frac{1}{2}\right)^2$, the simplest lattice is the hexagon lattice and the simplest spectral set is the regular hexagon. Also, it is well known that the hexagon lattice offers the densest packing of $\mathbb{R}^2$ with unit balls. Thus, the hexagon lattice and hexagonal Fourier series have great importance in Fourier analysis.

The generator matrix and the spectral set of the hexagonal lattice $H\mathbb{Z}^2$ are given by

$$H = \begin{bmatrix} \sqrt{3} & 0 \\ -1 & 2 \end{bmatrix}$$

and

$$\Omega_H = \left\{ (x_1, x_2) \in \mathbb{R}^2 : -1 \leqslant x_2, \frac{\sqrt{3}}{2} x_1 \pm \frac{1}{2} x_2 < 1 \right\}.$$

It is more convenient to use the homogeneous coordinates (t_1,t_2,t_3) that satisfies $t_1+t_2+t_3=0$. If we define

$$t_1 := -\frac{x_2}{2}+\frac{\sqrt{3}x_1}{2},\ t_2 := x_2,\ t_3 := -\frac{x_2}{2}-\frac{\sqrt{3}x_1}{2}, \tag{1}$$

the hexagon Ω_H becomes

$$\Omega = \left\{(t_1,t_2,t_3)\in\mathbb{R}^3 : -1\leqslant t_1,t_2,-t_3<1,\ t_1+t_2+t_3=0\right\},$$

which is the intersection of the plane $t_1+t_2+t_3=0$ with the cube $[-1,1]^3$.

We use bold letters $\mathbf{t}$ for homogeneous coordinates and we denote by $\mathbb{R}^3_H$ the plane $t_1+t_2+t_3=0$, that is

$$\mathbb{R}^3_H = \left\{\mathbf{t}=(t_1,t_2,t_3)\in\mathbb{R}^3 : t_1+t_2+t_3=0\right\}.$$

Also we use the notation $\mathbb{Z}^3_H$ for the set of points in $\mathbb{R}^3_H$ with integer components, that is $\mathbb{Z}^3_H=\mathbb{Z}^3\cap\mathbb{R}^3_H$.

In the homogeneous coordinates, the inner product on $L^2(\Omega)$ becomes

$$\langle f,g\rangle_H = \frac{1}{|\Omega|}\int_\Omega f(\mathbf{t})\overline{g(\mathbf{t})}d\mathbf{t},$$

where $|\Omega|$ denotes the area of Ω, and the orthonormal basis of $L^2(\Omega)$ becomes

$$\left\{\phi_{\mathbf{j}}(\mathbf{t}) = e^{\frac{2\pi i}{3}\langle \mathbf{j},\mathbf{t}\rangle} : \mathbf{j}\in\mathbb{Z}^3_H,\ \mathbf{t}\in\mathbb{R}^3_H\right\}.$$

Also, a function f is periodic with respect to the hexagonal lattice (or H-periodic) if and only if $f(\mathbf{t})=f(\mathbf{t}+\mathbf{s})$ whenever $\mathbf{s}\equiv\mathbf{0} \pmod 3$, where $\mathbf{t}\equiv\mathbf{s} \pmod 3$ defined as

$$t_1-s_1\equiv t_2-s_2\equiv t_3-s_3 \pmod 3.$$

It is clear that the functions $\phi_{\mathbf{j}}(\mathbf{t})$ are H-periodic and if the function f is H-periodic then

$$\int_\Omega f(\mathbf{t}+\mathbf{s})\,d\mathbf{t} = \int_\Omega f(\mathbf{t})\,d\mathbf{t},\ \left(\mathbf{s}\in\mathbb{R}^3_H\right).$$

For every natural number n, we define a subset of $\mathbb{Z}^3_H$ by

$$\mathbb{H}_n := \left\{\mathbf{j}=(j_1,j_2,j_3)\in\mathbb{Z}^3_H : -n\leqslant j_1,j_2,j_3\leqslant n\right\}.$$

Note that, $\mathbb{H}_n$ consists of all points with integer components inside the hexagon $n\overline{\Omega}$. Members of the set

$$\mathscr{H}_n := \operatorname{span}\left\{\phi_{\mathbf{j}} : \mathbf{j}\in\mathbb{H}_n\right\},\ (n\in\mathbb{N})$$

are called hexagonal trigonometric polynomials. It is clear that the dimension of $\mathscr{H}_n$ is $\#\mathbb{H}_n = 3n^2+3n+1$.

The hexagonal Fourier series of an H-periodic function $f \in L^1(\Omega)$ is

$$f(\mathbf{t}) \sim \sum_{\mathbf{j} \in \mathbb{Z}_H^3} \widehat{f}_{\mathbf{j}} \phi_{\mathbf{j}}(\mathbf{t}), \tag{2}$$

where

$$\widehat{f}_{\mathbf{j}} = \frac{1}{|\Omega|} \int_{\Omega} f(\mathbf{t}) e^{-\frac{2\pi i}{3}\langle \mathbf{j}, \mathbf{t} \rangle} d\mathbf{t}, \ \left(\mathbf{j} \in \mathbb{Z}_H^3\right).$$

The nth partial sum of the series (2) is defined by

$$S_n(f)(\mathbf{t}) := \sum_{\mathbf{j} \in \mathbb{H}_n} \widehat{f}_{\mathbf{j}} \phi_{\mathbf{j}}(\mathbf{t}), \ (n \in \mathbb{N}).$$

We refer to [11] and [16] for more detailed information about Fourier analysis on lattices and hexagonal Fourier series.

Approximation properties of some linear means of hexagonal Fourier series in uniform, Hölder and generalized Hölder norms were investigated in [4]-[8]. But, up to now, there are no results on the degree of approximation of partial sums of hexagonal Fourier series in these norms.

3. Approximation on hexagonal domains

Hereafter, we shall write $A \ll B$ for the quantities A and B, if there exists a constant $K > 0$ such that $A \leqslant KB$ holds.

We denote by $C_{2\pi}$ the space of 2π-periodic continuous functions on the real line, which is a Banach space with respect to the norm

$$\|f\|_{C_{2\pi}} = \sup_{0 \leqslant x \leqslant 2\pi} |f(x)|.$$

The modulus of continuity of the function $f \in C_{2\pi}$ is defined by

$$\omega(f, \delta) := \sup_{0 < |h| \leqslant \delta} \|f - T_h(f)\|_{C_{2\pi}},$$

where $T_h(f)(x) := f(x+h)$.

A function $f \in C_{2\pi}$ belongs to the Hölder class $H_{2\pi}^{\alpha}$ $(0 < \alpha \leqslant 1)$ if

$$\sup_{\delta > 0} \frac{\|f - T_\delta(f)\|_{C_{2\pi}}}{\delta^\alpha} < \infty,$$

or equivalently $\omega(f, \delta) \ll \delta^\alpha$ for every $\delta > 0$. The Hölder norm on $H_{2\pi}^{\alpha}$ is defined by

$$\|f\|_\alpha := \|f\|_{C_{2\pi}} + \sup_{\delta > 0} \frac{\|f - T_\delta(f)\|_{C_{2\pi}}}{\delta^\alpha}.$$

We set $\|f\|_0 := \|f\|_{C_{2\pi}}$ for convenience.

In [10], the authors considered a more general norm, the φ-norm, defined by

$$\|f\|_{\varphi} := \|f\|_{C_{2\pi}} + \sup_{\delta>0} \frac{\|f - T_{\delta}(f)\|_{C_{2\pi}}}{\varphi(\delta)}, \tag{3}$$

where φ is a positive increasing function on $(0,\infty)$, and proved the following theorem.

THEOREM A. *Let (A_n) be a sequence of linear convolution operators from $C_{2\pi}$ into $C_{2\pi}$ with operator norms $\|A_n\|$ and let φ be a positive increasing function on $(0,\infty)$. Then*

$$\begin{aligned} \|A_n(f) - f\|_{\varphi} \leqslant & \left(1 + \frac{2}{\varphi(1/n)}\right) \|A_n(f) - f\|_{C_{2\pi}} \\ & + (1 + \|A_n\|) \sup_{0<\delta\leqslant 1/n} \frac{2\omega(f,\delta)}{\varphi(\delta)} \end{aligned} \tag{4}$$

for each $f \in C_{2\pi}$.

This theorem is useful for obtaining approximation estimates in Hölder norms. For example, if we consider the sequence $(S_n(f))$ of partial sums of the Fourier series of $f \in H_{2\pi}^{\alpha}$ $(0<\alpha\leqslant 1)$, and if we take $\varphi(\delta) = \delta^{\beta}$, where $0 \leqslant \beta < \alpha$, (4) yields the estimate

$$\|f - S_n(f)\|_{\beta} \ll \frac{\log n}{n^{\alpha-\beta}}, \tag{5}$$

which was obtained by Prössdorf in [13].

We denote $C_H(\overline{\Omega})$ the Banach space of H-periodic continuous functions on $\mathbb{R}_H^3$, equipped with the uniform norm

$$\|f\|_{C_H(\overline{\Omega})} = \sup_{\mathbf{t}\in\overline{\Omega}} |f(\mathbf{t})|.$$

The modulus of continuity of the function $f \in C_H(\overline{\Omega})$ is defined by

$$\omega_H(f,\delta) := \sup_{0<\|\mathbf{h}\|\leqslant\delta} \|f - T_{\mathbf{h}}(f)\|_{C_H(\overline{\Omega})},$$

where

$$\|\mathbf{h}\| := \max\{|h_1|, |h_2|, |h_3|\}$$

for $\mathbf{h} = (h_1,h_2,h_3) \in \mathbb{R}_H^3$ and $T_{\mathbf{h}}(f)(\mathbf{t}) = f(\mathbf{t}+\mathbf{h})$. For $0<\alpha\leqslant 1$ the Hölder class $H^{\alpha}(\overline{\Omega})$ consists of functions $f \in C_H(\overline{\Omega})$ such that

$$\sup_{\|\mathbf{h}\|>0} \frac{\|f - T_{\mathbf{h}}(f)\|_{C_H(\overline{\Omega})}}{\|\mathbf{h}\|^{\alpha}} < \infty,$$

or equivalently $\omega_H(f,\delta) \ll \delta^{\alpha}$ for $\delta > 0$. The Hölder norm on $H^{\alpha}(\overline{\Omega})$ is defined by

$$\|f\|_{H^{\alpha}(\overline{\Omega})} := \|f\|_{C_H(\overline{\Omega})} + \sup_{\|\mathbf{h}\|>0} \frac{\|f - T_{\mathbf{h}}(f)\|_{C_H(\overline{\Omega})}}{\|\mathbf{h}\|^{\alpha}}.$$

As in the case of 2π-periodic functions we assume that $\|f\|_{H^0(\overline{\Omega})} = \|f\|_{C_H(\overline{\Omega})}$. To obtain an analogue of estimate (4) for hexagonal Fourier series we define the φ-norm on $C_H\left(\overline{\Omega}\right)$ as

$$\|f\|_{\varphi} := \|f\|_{C_H(\overline{\Omega})} + \sup_{\|\mathbf{h}\|>0} \frac{\|f - T_{\mathbf{h}}(f)\|_{C_H(\overline{\Omega})}}{\varphi(\|\mathbf{h}\|)}.$$

Our main result is the following.

THEOREM 1. *Let (A_n) be a sequence linear convolution operators from $C_H\left(\overline{\Omega}\right)$ into itself and φ be an increasing positive function on $(0,\infty)$. Then, for every $f \in C_H\left(\overline{\Omega}\right)$ we have*

$$\|A_n(f) - f\|_{\varphi} \leqslant \left(1 + \frac{2}{\varphi(1/n)}\right) \|A_n(f) - f\|_{C_H(\overline{\Omega})} + (1 + \|A_n\|) \sup_{0<\|\mathbf{h}\|\leqslant 1/n} \frac{\omega_H(f, \|\mathbf{h}\|)}{\varphi(\|\mathbf{h}\|)}. \tag{6}$$

Proof. We use the method of [10] in proof of this theorem.
Set $R_n := A_n(f) - f$ for $f \in C_H\left(\overline{\Omega}\right)$. Hence,

$$\|A_n(f) - f\|_{\varphi} = \|R_n\|_{\varphi} = \|R_n\|_{C_H(\overline{\Omega})} + \sup_{\|\mathbf{h}\|>0} \frac{\|R_n - T_{\mathbf{h}}(R_n)\|_{C_H(\overline{\Omega})}}{\varphi(\|\mathbf{h}\|)}.$$

For $\|\mathbf{h}\| \geqslant 1/n$, we have

$$\begin{aligned}
\frac{\|R_n - T_{\mathbf{h}}(R_n)\|_{C_H(\overline{\Omega})}}{\varphi(\|\mathbf{h}\|)} &= \frac{\|A_n(f) - f - T_{\mathbf{h}}(A_n(f) - f)\|_{C_H(\overline{\Omega})}}{\varphi(\|\mathbf{h}\|)} \\
&\leqslant \frac{\|A_n(f) - f\|_{C_H(\overline{\Omega})} + \|(A_n(f) - f)(\cdot + \mathbf{h})\|_{C_H(\overline{\Omega})}}{\varphi(\|\mathbf{h}\|)} \\
&= 2\frac{\|A_n(f) - f\|_{C_H(\overline{\Omega})}}{\varphi(\|\mathbf{h}\|)} \leqslant 2\frac{\|A_n(f) - f\|_{C_H(\overline{\Omega})}}{\varphi(1/n)}.
\end{aligned}$$

For $0 < \|\mathbf{h}\| \leqslant 1/n$,

$$\begin{aligned}
\frac{|R_n(\mathbf{t}) - T_{\mathbf{h}}(R_n(\mathbf{t}))|}{\varphi(\|\mathbf{h}\|)} &= \frac{|R_n(\mathbf{t}) - R_n(\mathbf{t} + \mathbf{h})|}{\varphi(\|\mathbf{h}\|)} \\
&= \frac{|A_n(f)(\mathbf{t}) - f(\mathbf{t}) - (A_n(f)(\mathbf{t} + \mathbf{h}) - f(\mathbf{t} + \mathbf{h}))|}{\varphi(\|\mathbf{h}\|)} \\
&\leqslant \frac{|A_n(f)(\mathbf{t}) - A_n(f)(\mathbf{t} + \mathbf{h})|}{\varphi(\|\mathbf{h}\|)} + \frac{|f(\mathbf{t}) - f(\mathbf{t} + \mathbf{h})|}{\varphi(\|\mathbf{h}\|)} \\
&= \frac{|A_n(f)(\mathbf{t}) - T_{\mathbf{h}}(A_n(f))(\mathbf{t})|}{\varphi(\|\mathbf{h}\|)} + \frac{|f(\mathbf{t}) - T_{\mathbf{h}}(f)(\mathbf{t})|}{\varphi(\|\mathbf{h}\|)}
\end{aligned}$$

$$= \frac{|A_n(f)(\mathbf{t}) - A_n(T_{\mathbf{h}}(f))(\mathbf{t})|}{\varphi(\|\mathbf{h}\|)} + \frac{|f(\mathbf{t}) - T_{\mathbf{h}}(f)(\mathbf{t})|}{\varphi(\|\mathbf{h}\|)}$$
$$\leqslant \frac{\|A_n(f - T_{\mathbf{h}}(f))\|_{C_H(\overline{\Omega})}}{\varphi(\|\mathbf{h}\|)} + \frac{\|f - T_{\mathbf{h}}(f)\|_{C_H(\overline{\Omega})}}{\varphi(\|\mathbf{h}\|)}$$
$$\leqslant (1 + \|A_n\|) \frac{\|f - T_{\mathbf{h}}(f)\|_{C_H(\overline{\Omega})}}{\varphi(\|\mathbf{h}\|)}$$
$$\leqslant (1 + \|A_n\|) \frac{\omega_H(f, \|\mathbf{h}\|)}{\varphi(\|\mathbf{h}\|)}.$$

Thus, we get

$$\frac{\|R_n - T_{\mathbf{h}}(R_n)\|_{C_H(\overline{\Omega})}}{\varphi(\|\mathbf{h}\|)} \leqslant 2 \frac{\|A_n(f) - f\|_{C_H(\overline{\Omega})}}{\varphi(1/n)} + (1 + \|A_n\|) \frac{\omega_H(f, \|\mathbf{h}\|)}{\varphi(\|\mathbf{h}\|)}$$

for every $\mathbf{h} \in \mathbb{R}_H^3$ with $\|\mathbf{h}\| > 0$. This inequality and definition of the norm $\|\cdot\|_\varphi$ yield (6). □

Let $A_n = S_n$, where (S_n) is the sequence of partial sums of hexagonal Fourier series. It is known that for the norm of

$$S_n : C_H(\overline{\Omega}) \to C_H(\overline{\Omega})$$

the estimate $\|S_n\| \ll (\log n)^2$ holds ([12], [14]).

Let $E_n(f)$ be the best approximation of $f \in C_H(\overline{\Omega})$ by elements of $\mathscr{H}_n$, and let $L_n^* \in \mathscr{H}_n$ be the hexagonal trigonometric polynomial of best approximation of f in $\mathscr{H}_n$. Hence we have

$$E_n(f) = \inf_{S \in \mathscr{H}_n} \|f - S\|_{C_H(\overline{\Omega})} = \|f - L_n^*\|_{C_H(\overline{\Omega})},$$

and

$$\|f - S_n(f)\|_{C_H(\overline{\Omega})} = \|f - L_n^* + S_n(L_n^* - f)\|_{C_H(\overline{\Omega})}$$
$$\leqslant (1 + \|S\|_n) E_n(f) \ll (\log n)^2 E_n(f).$$

Jackson type theorem ([16, Theorem 4.4]) states that

$$E_n(f) \ll \omega_H\left(f, \frac{1}{n}\right), \ (n = 1, 2, \ldots).$$

Thus, we obtain for each $f \in C_H(\overline{\Omega})$,

$$\|f - S_n(f)\|_{C_H(\overline{\Omega})} \ll (\log n)^2 \omega_H\left(f, \frac{1}{n}\right). \tag{7}$$

Hence we get

$$\|f - S_n(f)\|_{C_H(\overline{\Omega})} \ll \frac{(\log n)^2}{n^\alpha}$$

for $f \in H^{\alpha}\left(\overline{\Omega}\right)$, $0<\alpha\leqslant 1$.

A non-decreasing continuous function $\omega:[0,\infty)\to[0,\infty)$ is called a modulus of continuity if

$$\omega(0)=0,\ \omega(t_1+t_2)\leqslant\omega(t_1)+\omega(t_2).$$

For any modulus of continuity ω, we define the generalized Hölder class $H^{\omega}\left(\overline{\Omega}\right)$ as the set of functions $f\in C_H\left(\overline{\Omega}\right)$ for which

$$\omega_H(f,\delta)\ll\omega(\delta),\ (\delta>0).$$

If $\omega(\delta)=\delta^{\alpha}, 0<\alpha\leqslant 1$, then it is clear that $H^{\omega}\left(\overline{\Omega}\right)$ coincides with $H^{\alpha}\left(\overline{\Omega}\right)$, and $\|f\|_{H^{\omega}(\overline{\Omega})}$ be $\|f\|_{H^{\alpha}(\overline{\Omega})}$.

In [9], L. Leindler introduced a certain class of moduli of continuity:

for $0\leqslant\alpha\leqslant 1$, let $\mathscr{M}_{\alpha}$ denote the class of moduli of continuity ω_{α} having the following properties:

(i) for any $\alpha'>\alpha$ there exists a natural number $\mu=\mu(\alpha')$ such that

$$2^{\mu\alpha'}\omega_{\alpha}\left(2^{-n-\mu}\right)>2\omega_{\alpha}\left(2^{-n}\right),\ (n=1,2,\ldots),$$

(ii) for every natural number ν, there exists a natural number $N(\nu)$ such that

$$2^{\nu\alpha}\omega_{\alpha}\left(2^{-n-\nu}\right)\leqslant 2\omega_{\alpha}\left(2^{-n}\right),\ (n>N(\nu)).$$

It is clear that $\omega(\delta)=\delta^{\alpha}\in\mathscr{M}_{\alpha}$, but $\omega_{\alpha}(\delta)$ is an extension of $\omega(\delta)=\delta^{\alpha}$. Consequently, in general, $H^{\omega_{\alpha}}\left(\overline{\Omega}\right)$ is larger than $H^{\alpha}\left(\overline{\Omega}\right)$.

If $f\in H^{\omega_{\alpha}}\left(\overline{\Omega}\right)$, where $\omega_{\alpha}\in\mathscr{M}_{\alpha}$ $(0\leqslant\alpha\leqslant 1)$, by (7) we get

$$\|f-S_n(f)\|_{C_H(\overline{\Omega})}\ll(\log n)^2\,\omega_{\alpha}(1/n),$$

and (6) gives the following theorem.

THEOREM 2. *Let* $\omega_{\alpha}\in\mathscr{M}_{\alpha}$ $(0\leqslant\alpha\leqslant 1)$, $f\in H^{\omega_{\alpha}}\left(\overline{\Omega}\right)$ *and* φ *be an increasing function such that* $\frac{\omega_{\alpha}(\delta)}{\varphi(\delta)}$ *is non-decreasing. Then*

$$\|f-S_n(f)\|_{\varphi}\ll\left(1+\frac{1}{\varphi(1/n)}\right)\omega_{\alpha}(1/n)(\log n)^2.$$

It is known that if $0\leqslant\beta<\alpha\leqslant 1$, $\omega_{\beta}\in\mathscr{M}_{\beta}$ and $\omega_{\alpha}\in\mathscr{M}_{\alpha}$, then the function $\omega_{\alpha}/\omega_{\beta}$ is non-decreasing. Hence we obtain the following estimate.

COROLLARY 1. *If* $0\leqslant\beta<\alpha\leqslant 1$, $\omega_{\beta}\in\mathscr{M}_{\beta}$, $\omega_{\alpha}\in\mathscr{M}_{\alpha}$ *and* $f\in H^{\omega_{\alpha}}\left(\overline{\Omega}\right)$, *then*

$$\|f-S_n(f)\|_{H^{\omega_{\beta}}(\overline{\Omega})}\ll\frac{\omega_{\alpha}(1/n)}{\omega_{\beta}(1/n)}(\log n)^2.$$

In the case of classical Fourier series, the analogue of Corollary 1 was proved in [9].

In special case we get the following analogue of (5):

COROLLARY 2. *If* $0 \leqslant \beta < \alpha \leqslant 1$ *and* $f \in H^{\alpha}\left(\overline{\Omega}\right)$, *then*

$$\|f - S_n(f)\|_{H^{\beta}(\overline{\Omega})} \ll \frac{(\log n)^2}{n^{\alpha-\beta}}.$$

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