ON THE LIMIT INFERIOR AND LIMIT SUPERIOR FOR DOUBLE SEQUENCES

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Abstract. Let \((u_{mn})\) be a double sequence of real numbers such that \(\limsup \sigma_{mn}(u) = \beta\) and \(\liminf \sigma_{mn}(u) = \alpha\), where \(\sigma_{mn}(u) = \frac{1}{(m+1)(n+1)} \sum_{j=0}^{m} \sum_{k=0}^{n} u_{jk}\), and \(\alpha \neq \beta\). In this paper, it is presented that \(\limsup u_{mn} = \beta\) and \(\liminf u_{mn} = \alpha\) if the following conditions hold: For \(\lambda > 1\)

\[
\liminf \frac{1}{(\lceil \lambda m \rceil - m)(\lceil \lambda n \rceil - n)} \sum_{j=m+1}^{\lceil \lambda m \rceil} \sum_{k=n+1}^{\lceil \lambda n \rceil} (u_{jk} - u_{mn}) \geq 2(\beta - \alpha) \frac{\lambda(2\lambda - 1)}{(\lambda - 1)^2},
\]

for \(0 < \lambda < 1\)

\[
\liminf \frac{1}{(m - \lceil \lambda m \rceil)(n - \lceil \lambda n \rceil)} \sum_{j=\lceil \lambda m \rceil + 1}^{m} \sum_{k=\lceil \lambda n \rceil + 1}^{n} (u_{mn} - u_{jk}) \geq 2(\beta - \alpha) \frac{\lambda}{(\lambda - 1)^2},
\]

where \(\lceil \lambda n \rceil\) denotes the integer part of \(\lambda n\).

1. Introduction

Let \(u = (u_{mn})\) be a double sequence of real numbers. A double sequence \((u_{mn})\) is said to be Pringsheim convergent (or \(P\)-convergent) \([10]\) to \(s\) if for a given \(\varepsilon > 0\) there exists a positive integer \(N\) such that \(|u_{mn} - s| < \varepsilon\) for all nonnegative integer \(m, n \geq N\), and we write \(\lim u_{mn} = s\). A double sequence \((u_{mn})\) is bounded if there exists a real number \(C > 0\) such that \(|u_{mn}| \leq C\) for all nonnegative \(m, n\). Notice that some \(P\)-convergent double sequence can be unbounded. For example, the double sequence

\[
u_{mn} = \begin{cases} n^2, & \text{if } m = 0; \\ m, & \text{if } n = 0; \\ 0, & \text{otherwise} \end{cases}
\]

is \(P\)-convergent to 0, but it is not bounded.

Definition 1. \([9]\) Let \((u_{mn})\) be a double sequence and let \(\alpha_j = \sup \{u_{mn} : m, n \geq j\}\) for each \(j\). The Pringsheim limit superior of \((u_{mn})\) is defined as follows:

i) if \(\alpha = +\infty\) for each \(j\), then \(\limsup u_{mn} := +\infty\)

ii) if \(\alpha < +\infty\) for some \(j\), then \(\limsup u_{mn} := \inf_j \{\alpha_j\}\).

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Similarly, the Pringsheim limit inferior of \((u_{mn})\) is defined.

The propositions of the limit superior and limit inferior like single dimensional sequences are held [9]: If \(u = (u_{mn})\) and \(v = (v_{mn})\) are double sequences, then

i) \(\liminf u_{mn} \leq \limsup u_{mn}\),

ii) \(\lim u_{mn} = s\) if and only if \(\liminf u_{mn} = \limsup u_{mn} = s\),

iii) \(\limsup(-u_{mn}) = -\liminf u_{mn}\),

iv) \(\limsup(u_{mn} + v_{mn}) \leq \limsup u_{mn} + \limsup v_{mn}\),

v) \(\liminf(u_{mn} + v_{mn}) \geq \liminf u_{mn} + \liminf v_{mn}\).

In the example below, the double sequence \((u_{mn})\) is not bounded above and is not bounded below either; but, both of the Pringsheim limit superior and inferior are finite numbers. The sequence

\[
u_{mn} = \begin{cases}n^2, & \text{if } m = 0; \\ -m, & \text{if } n = 0; \\ (-1)^m, & \text{if } m = n > 0; \\ 0, & \text{otherwise}
\end{cases}
\]

is not \(P\)-convergence but \(\liminf u_{mn} = -1\) and \(\limsup u_{mn} = 1\).

The \((C, 1, 1)\) means of \((u_{mn})\) are defined by

\[
\sigma_{mn}(u) = \frac{1}{(m+1)(n+1)} \sum_{j=0}^{m} \sum_{k=0}^{n} u_{jk}
\]

for nonnegative integers \(m, n\) ([6]). The sequence \((u_{mn})\) is said to be \((C, 1, 1)\) summable to a finite number \(s\) if \(\lim \sigma_{mn}(u) = s\).

If a double sequence is \(P\)-convergent to \(s\), then it is \((C, 1, 1)\) summable to \(s\) on the condition that it is bounded. Nevertheless, the converse case is not necessarily true. That is to say, the bounded and \((C, 1, 1)\) summable double sequence might not be \(P\)-convergent.

**Example 1.** [11] The sequence \((u_{mn}) = \left(\sum_{i=0}^{m} \sum_{j=0}^{n} (-1)^{i+j}\right)\) is not \(P\)-convergent. But it is \((C, 1, 1)\) summable to \(\frac{1}{4}\). In fact, from the definition of \((C, 1, 1)\) means, we get

\[
\sigma_{mn}(u) = \begin{cases}
\frac{1}{4}, & \text{if } m, n \text{ are odd}; \\
\frac{m + 2}{4m + 4}, & \text{if } m \text{ is even, } n \text{ is odd}; \\
\frac{n + 2}{4n + 4}, & \text{if } m \text{ is odd, } n \text{ is even}; \\
\frac{(m + 2)(n + 2)}{(2m + 2)(2n + 2)}, & \text{if } m, n \text{ are even}.
\end{cases}
\]
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Hence, we have \( \lim \sigma_{mn}(u) = \frac{1}{4} \).

Note that the \( P \)-convergence of a double sequence \( (u_{mn}) \) can be recovered out of the \( P \)-convergence of its \((C,1,1)\) means under some suitable conditions.

The relation between limit inferior and limit superior of \( (u_{mn}) \) and the sequence of its arithmetic means \( (\sigma_{mn}(u)) \) for bounded sequences as follows [9]:

\[
\liminf u_{mn} \leq \liminf \sigma_{mn}(u) \leq \limsup \sigma_{mn}(u) \leq \limsup u_{mn}. \tag{1}
\]

In the following example, \( \liminf \sigma_{mn}(u) \) and \( \limsup \sigma_{mn}(u) \) exist and are finite, but \( \liminf u_{mn} \) and \( \limsup u_{mn} \) are not finite.

**Example 2.** The double sequence is defined by

\[
u_{mn} = \begin{cases} n, & \text{if } m = 0; \\ -m^2, & \text{if } n = 0; \\ (-1)^mm, & \text{if } m = n > 0; \\ 0, & \text{otherwise.} \end{cases}
\]

It is clear that \( \liminf \sigma_{mn}(u) = -\frac{1}{2} \) and \( \limsup \sigma_{mn}(u) = \frac{1}{2} \), but \( \liminf u_{mn} = -\infty \) and \( \limsup u_{mn} = \infty \).

The problem of getting ordinary convergence from the convergence of arithmetic mean of a sequence has been a issue for over a century about which researches are carried out. In recent years, there have been studies on both the \((C,1)\) summability of single dimensional sequences [1, 3, 4, 5] and the \((C,1,1)\) summability of two dimensional sequences [11, 7, 8]. Çanak [2] has investigated under which conditions a sequence has \( \liminf \) and \( \limsup \) if the sequence of its arithmetic means has \( \liminf \) and \( \limsup \).

In this study, we extend the theorem given by Çanak to double sequences. More clearly, we will focus on the suitable conditions in order to show that the existence of \( \limsup \) and \( \liminf \) of the sequence \( (u_{mn}) \) out of the existence of \( \limsup \) and \( \liminf \) of the sequence \( (\sigma_{mn}(u)) \). Our main theorem is generalized the theorem given by Móricz [7].

**2. Main results**

**Theorem 2.** For a double bounded sequence \( (u_{mn}) \) of real numbers, let \( \limsup \sigma_{mn}(u) = \beta \) and \( \liminf \sigma_{mn}(u) = \alpha \), where \( \alpha \neq \beta \). If for \( \lambda > 1 \)

\[
\liminf \frac{1}{(\lfloor \lambda m \rfloor - m)(\lfloor \lambda n \rfloor - n)} \sum_{j=m+1}^{\lfloor \lambda m \rfloor} \sum_{k=n+1}^{\lfloor \lambda n \rfloor} (u_{jk} - u_{mn}) \geq 2(\beta - \alpha) \frac{\lambda(2\lambda - 1)}{(\lambda - 1)^2}, \tag{2}
\]

and for \( 0 < \lambda < 1 \)

\[
\liminf \frac{1}{(m - \lfloor \lambda m \rfloor)(n - \lfloor \lambda n \rfloor)} \sum_{j=\lfloor \lambda m \rfloor + 1}^{m} \sum_{k=\lfloor \lambda n \rfloor + 1}^{n} (u_{mn} - u_{jk}) \geq 2(\beta - \alpha) \frac{\lambda}{(\lambda - 1)^2}, \tag{3}
\]
where \([\lambda n]\) denotes the integer part of \(\lambda n\), then \(\limsup u_{mn} = \beta\) and \(\liminf u_{mn} = \alpha\).

**Corollary 3.** Let \((u_{mn})\) be \((C,1,1)\) summable to \(s\). If for \(\lambda > 1\)

\[
\liminf_{\lambda \to 1^+} \frac{1}{([\lambda m] - m)([\lambda n] - n)} \sum_{j=m+1}^{[\lambda m]} \sum_{k=n+1}^{[\lambda n]} (u_{jk} - u_{mn}) \geq 0, \tag{4}
\]

and for \(0 < \lambda < 1\)

\[
\liminf_{\lambda \to 1^-} \frac{1}{(m - [\lambda m])(n - [\lambda n])} \sum_{j=[\lambda m]+1}^{m} \sum_{k=[\lambda n]+1}^{n} (u_{mn} - u_{jk}) \geq 0, \tag{5}
\]

then \((u_{mn})\) is \(P\)-convergent to \(s\).

We can replace the conditions (4) and (5) by the conditions which are weaker

\[
\limsup_{\lambda \to 1^+} \liminf_{\lambda \to 1^-} \frac{1}{([\lambda m] - m)([\lambda n] - n)} \sum_{j=m+1}^{[\lambda m]} \sum_{k=n+1}^{[\lambda n]} (u_{jk} - u_{mn}) \geq 0, \tag{6}
\]

and

\[
\limsup_{\lambda \to 1^-} \liminf_{\lambda \to 1^+} \frac{1}{(m - [\lambda m])(n - [\lambda n])} \sum_{j=[\lambda m]+1}^{m} \sum_{k=[\lambda n]+1}^{n} (u_{mn} - u_{jk}) \geq 0, \tag{7}
\]

since the conditions hold for all \(\lambda > 1\) and all \(0 < \lambda < 1\), respectively.

It is showed by Moricz [7] that the conditions (6) and (7) are sufficient conditions for convergence from the \((C,1,1)\) summability.

In order to prove our main results, the following lemma is necessary.

**Lemma 4.** [11] Let \((u_{mn})\) be a double sequence. For sufficiently large \(m,n\):

(i) If \(\lambda > 1\)

\[
|u_{mn} - \sigma_{mn}(u)| = \frac{([\lambda m]+1)([\lambda n]+1)}{([\lambda m] - m)([\lambda n] - n)} (\sigma_{[\lambda m],[\lambda n]}(u) - \sigma_{[\lambda m],[\lambda n]}(u) - \sigma_{m,[\lambda n]}(u) + \sigma_{mn}(u))
\]

\[
+ \frac{[\lambda m]+1}{[\lambda m] - m} (\sigma_{[\lambda m],[\lambda n]}(u) - \sigma_{m,[\lambda n]}(u)) + \frac{[\lambda n]+1}{[\lambda n] - n} (\sigma_{m,[\lambda n]}(u) - \sigma_{m,n}(u))
\]

\[
- \frac{1}{([\lambda m] - m)([\lambda n] - n)} \sum_{j=m+1}^{[\lambda m]} \sum_{k=n+1}^{[\lambda n]} (u_{jk} - u_{mn}).
\]

(ii) If \(0 < \lambda < 1\)

\[
|u_{mn} - \sigma_{mn}(u)| = \frac{([\lambda m]+1)([\lambda n]+1)}{(m - [\lambda m])(n - [\lambda n])} (\sigma_{mn}(u) - \sigma_{[\lambda m],[\lambda n]}(u) - \sigma_{m,[\lambda n]}(u) + \sigma_{[\lambda m],[\lambda n]}(u))
\]

\[
+ \frac{[\lambda m]+1}{m - [\lambda m]} (\sigma_{mn}(u) - \sigma_{[\lambda m],[\lambda n]}(u)) + \frac{[\lambda n]+1}{n - [\lambda n]} (\sigma_{mn}(u) - \sigma_{m,[\lambda n]}(u))
\]

\[
+ \frac{1}{(m - [\lambda m])(n - [\lambda n])} \sum_{j=[\lambda m]+1}^{m} \sum_{k=[\lambda n]+1}^{n} (u_{mn} - u_{jk}).
\]
3. Proof of Theorem 2

Suppose that \( \limsup \sigma_{mn}(u) = \beta \) and \( \liminf \sigma_{mn}(u) = \alpha \), where \( \alpha \neq \beta \). Since \( \lim_{x \to \infty} \frac{[\lambda x] + 1}{[\lambda x]} = \frac{\lambda}{\lambda - 1} \) for \( x = m, n \), we have

\[
\limsup_{x \to \infty} \frac{[\lambda m] + 1}{[\lambda m] - m} \frac{[\lambda n] + 1}{[\lambda n] - n} \left( \sigma_{[\lambda m],[\lambda n]}^{(11)}(u) - \sigma_{[\lambda m],[\lambda n]}^{(11)}(u) - \sigma_{m,[\lambda n]}^{(11)}(u) + \sigma_{mn}^{(11)}(u) \right) \leq 2(\beta - \alpha) \frac{\lambda^2}{(\lambda - 1)^2},
\]
and

\[
\limsup_{x \to \infty} \frac{\lambda m}{\lambda m - m} \left( \sigma_{[\lambda m],[\lambda n]}^{(11)}(u) - \sigma_{m,[\lambda n]}^{(11)}(u) \right) \leq (\beta - \alpha) \frac{\lambda}{\lambda - 1}
\]

and

\[
\limsup_{x \to \infty} \frac{\lambda n}{\lambda n - n} \left( \sigma_{m,[\lambda n]}^{(11)}(u) - \sigma_{m,[\lambda n]}^{(11)}(u) \right) \leq (\beta - \alpha) \frac{\lambda}{\lambda - 1}
\]
for \( \lambda > 1 \). We get

\[
\limsup_{x \to \infty} u_{mn} \leq \beta + 2(\beta - \alpha) \frac{\lambda^2}{(\lambda - 1)^2} + 2(\beta - \alpha) \frac{\lambda}{\lambda - 1} - \liminf \frac{1}{(\lambda m - m)(\lambda n - n)} \sum_{j=m+1}^{[\lambda m]} \sum_{k=n+1}^{[\lambda n]} (u_{jk} - u_{mn})
\]

by taking the limsup of both sides of the identity in Lemma 4 (i) and using (8), (9), (10). When we take (2) into account, we obtain

\[
\limsup_{x \to \infty} u_{mn} \leq \beta.
\]

By (1),

\[
\beta \leq \limsup_{x \to \infty} u_{mn}.
\]

Combining (11) and (12), we have \( \lim_{x \to \infty} u_{mn} = \beta \).

On the other hand, since \( \lim_{x \to \infty} \frac{[\lambda x] + 1}{[\lambda x]} = \frac{\lambda}{1 - \lambda} \) for \( x = m, n \), we have

\[
\liminf_{x \to \infty} \frac{[\lambda m] + 1}{m - [\lambda m]} \frac{[\lambda n] + 1}{n - [\lambda n]} \left( \sigma_{mn}^{(11)}(u) - \sigma_{m,[\lambda n]}^{(11)}(u) + \sigma_{[\lambda m],[\lambda n]}^{(11)}(u) \right) \geq 2(\beta - \alpha) \frac{\lambda^2}{(1 - \lambda)^2},
\]
and

\[
\liminf_{x \to \infty} \frac{[\lambda m] + 1}{m - [\lambda m]} \left( \sigma_{mn}^{(11)}(u) - \sigma_{m,[\lambda n]}^{(11)}(u) \right) \geq (\beta - \alpha) \frac{\lambda}{1 - \lambda}
\]

and

\[
\liminf_{x \to \infty} \frac{[\lambda n] + 1}{n - [\lambda n]} \left( \sigma_{mn}^{(11)}(u) - \sigma_{m,[\lambda n]}^{(11)}(u) \right) \geq (\beta - \alpha) \frac{\lambda}{1 - \lambda}
\]
for $0 < \lambda < 1$. Taking the liminf of both sides of the identity in Lemma 4 (ii) and using (13), (14), (15), we obtain

$$
\liminf u_{mn} \geq \beta + 2(\beta - \alpha) \frac{\lambda^2}{(1 - \lambda)^2} + 2(\beta - \alpha) \frac{\lambda}{1 - \lambda} + \liminf \frac{1}{(\lfloor \lambda m \rfloor - m)(\lfloor \lambda n \rfloor - n)} \sum_{j=m+1}^{\lfloor \lambda m \rfloor} \sum_{k=n+1}^{\lfloor \lambda n \rfloor} (u_{jk} - u_{mn}).
$$

Taking (3) into account, we get

$$
\liminf u_{mn} \geq \alpha. \quad (16)
$$

By (1),

$$
\alpha \geq \liminf u_{mn}. \quad (17)
$$

Combining (16) and (17), we have $\liminf u_{mn} = \alpha$.

REFERENCES


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