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ON THE LIMIT INFERIOR AND LIMIT SUPERIOR FOR DOUBLE SEQUENCES

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Abstract. Let (u_{mn}) be a double sequence of real numbers such that $\limsup \sigma_{mn}(u) = \beta$ and $\liminf \sigma_{mn}(u) = \alpha$, where $\sigma_{mn}(u) = \frac{1}{(m+1)(n+1)} \sum_{j=0}^{m} \sum_{k=0}^{n} u_{jk}$, and $\alpha \neq \beta$. In this paper, it is presented that $\limsup u_{mn} = \beta$ and $\liminf u_{mn} = \alpha$ if the following conditions hold: For $\lambda > 1$

$$\liminf \frac{1}{([\lambda m]-m)([\lambda n]-n)} \sum_{j=m+1}^{[\lambda m]} \sum_{k=n+1}^{[\lambda n]} (u_{jk}-u_{mn}) \geqslant 2(\beta-\alpha)\frac{\lambda(2\lambda-1)}{(\lambda-1)^2},$$

for $0<\lambda<1$

$$\liminf \frac{1}{(m-[\lambda m])(n-[\lambda n])} \sum_{j=[\lambda m]+1}^{m} \sum_{k=[\lambda n]+1}^{n} (u_{mn}-u_{jk}) \geqslant 2(\beta-\alpha)\frac{\lambda}{(\lambda-1)^2},$$

where $[\lambda n]$ denotes the integer part of λn.

1. Introduction

Let $u=(u_{mn})$ be a double sequence of real numbers. A double sequence (u_{mn}) is said to be Pringsheim convergent (or P- convergent) [10] to s if for a given $\varepsilon > 0$ there exists a positive integer N such that $|u_{mn}-s| < \varepsilon$ for all nonnegative integer $m,n \geqslant N$, and we write $\lim u_{mn} = s$. A double sequence (u_{mn}) is bounded if there exists a real number $C>0$ such that $|u_{mn}| \leqslant C$ for all nonnegative m,n. Notice that some P-convergent double sequence can be unbounded. For example, the double sequence

$$u_{mn} = \begin{cases} n^2, & \text{if } m=0; \\ m, & \text{if } n=0; \\ 0, & \text{otherwise} \end{cases}$$

is P-convergent to 0, but it is not bounded.

DEFINITION 1. [9] Let (u_{mn}) be a double sequence and let $\alpha_j = \sup_j\{u_{mn} : m,n \geqslant j\}$ for each j. The Pringsheim limit superior of (u_{mn}) is defined as follows:

i) if $\alpha = +\infty$ for each j, then $\limsup u_{mn} := +\infty$

ii) if $\alpha < +\infty$ for some j, then $\limsup u_{mn} := \inf_j\{\alpha_j\}$.

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Similarly, the Pringsheim limit inferior of (u_{mn}) is defined.

The propositions of the limit superior and limit inferior like single dimensional sequences are held [9]: If $u=(u_{mn})$ and $v=(v_{mn})$ are double sequences, then

i) $\liminf u_{mn} \leqslant \limsup u_{mn}$,

ii) $\lim u_{mn}=s$ if and only if $\liminf u_{mn}=\limsup u_{mn}=s$,

iii) $\limsup(-u_{mn})=-\liminf u_{mn}$,

iv) $\limsup(u_{mn}+v_{mn}) \leqslant \limsup u_{mn}+\limsup v_{mn}$,

v) $\liminf(u_{mn}+v_{mn}) \geqslant \liminf u_{mn}+\liminf v_{mn}$.

In the example below, the double sequence (u_{mn}) is not bounded above and is not bounded below either; but, both of the Pringsheim limit superior and inferior are finite numbers. The sequence

$$u_{mn}=\begin{cases} n^2, & \text{if } m=0; \\ -m, & \text{if } n=0; \\ (-1)^m, & \text{if } m=n>0; \\ 0, & \text{otherwise} \end{cases}$$

is not P-convergence but $\liminf u_{mn}=-1$ and $\limsup u_{mn}=1$.

The $(C,1,1)$ means of (u_{mn}) are defined by

$$\sigma_{mn}(u)=\frac{1}{(m+1)(n+1)}\sum_{j=0}^{m}\sum_{k=0}^{n}u_{jk}$$

for nonnegative integers m,n ([6]). The sequence (u_{mn}) is said to be $(C,1,1)$ summable to a finite number s if $\lim \sigma_{mn}(u)=s$.

If a double sequence is P-convergent to s, then it is $(C,1,1)$ summable to s on the condition that it is bounded. Nevertheless, the converse case is not necessarily true. That is to say, the bounded and $(C,1,1)$ summable double sequence might not be P-convergent.

EXAMPLE 1. [11] The sequence $(u_{mn})=\left(\sum_{i=0}^{m}\sum_{j=0}^{n}(-1)^{i+j}\right)$ is not P-convergent. But it is $(C,1,1)$ summable to $\frac{1}{4}$. In fact, from the definition of $(C,1,1)$ means, we get

$$\sigma_{mn}(u)=\begin{cases} \dfrac{1}{4} & \text{if } m,n \text{ are odd;} \\ \dfrac{m+2}{4m+4} & \text{if } m \text{ is even, } n \text{ is odd;} \\ \dfrac{n+2}{4n+4} & \text{if } m \text{ is odd, } n \text{ is even;} \\ \dfrac{(m+2)(n+2)}{(2m+2)(2n+2)} & \text{if } m,n \text{ are even.} \end{cases}$$

Hence, we have $\lim \sigma_{mn}(u) = \frac{1}{4}$.

Note that the P-convergence of a double sequence (u_{mn}) can be recovered out of the P-convergence of its $(C,1,1)$ means under some suitable conditions.

The relation between limit inferior and limit superior of (u_{mn}) and the sequence of its arithmetic means $(\sigma_{mn}(u))$ for bounded sequences as follows [9]:

$$\liminf u_{mn} \leqslant \liminf \sigma_{mn}(u) \leqslant \limsup \sigma_{mn}(u) \leqslant \limsup u_{mn}. \tag{1}$$

In the following example, $\liminf \sigma_{mn}(u)$ and $\limsup \sigma_{mn}(u)$ exist and are finite, but $\liminf u_{mn}$ and $\limsup u_{mn}$ are not finite.

EXAMPLE 2. The double sequence is defined by

$$u_{mn} = \begin{cases} n, & \text{if } m = 0; \\ -m^2, & \text{if } n = 0; \\ (-1)^m m, & \text{if } m = n > 0; \\ 0, & \text{otherwise.} \end{cases}$$

It is clear that $\liminf \sigma_{mn}(u) = -\frac{1}{2}$ and $\limsup \sigma_{mn}(u) = \frac{1}{2}$, but $\liminf u_{mn} = -\infty$ and $\limsup u_{mn} = \infty$.

The problem of getting ordinary convergence from the convergence of arithmetic mean of a sequence has been a issue for over a century about which researches are carried out. In recent years, there have been studies on both the $(C,1)$ summability of single dimensional sequences [1, 3, 4, 5] and the $(C,1,1)$ summability of two dimensional sequences [11, 7, 8]. Çanak [2] has investigated under which conditions a sequence has $\liminf$ and $\limsup$ if the sequence of its arithmetic means has $\liminf$ and $\limsup$.

In this study, we extend the theorem given by Çanak to double sequences. More clearly, we will focus on the suitable conditions in order to show that the existence of $\limsup$ and $\liminf$ of the sequence (u_{mn}) out of the existence of $\limsup$ and $\liminf$ of the sequence $(\sigma_{mn}(u))$. Our main theorem is generalized the theorem given by Móricz [7].

2. Main results

THEOREM 2. *For a double bounded sequence (u_{mn}) of real numbers, let $\limsup \sigma_{mn}(u) = \beta$ and $\liminf \sigma_{mn}(u) = \alpha$, where $\alpha \neq \beta$. If for $\lambda > 1$*

$$\liminf \frac{1}{([\lambda m]-m)([\lambda n]-n)} \sum_{j=m+1}^{[\lambda m]} \sum_{k=n+1}^{[\lambda n]} (u_{jk} - u_{mn}) \geqslant 2(\beta-\alpha)\frac{\lambda(2\lambda-1)}{(\lambda-1)^2}, \tag{2}$$

and for $0 < \lambda < 1$

$$\liminf \frac{1}{(m-[\lambda m])(n-[\lambda n])} \sum_{j=[\lambda m]+1}^{m} \sum_{k=[\lambda n]+1}^{n} (u_{mn} - u_{jk}) \geqslant 2(\beta-\alpha)\frac{\lambda}{(\lambda-1)^2}, \tag{3}$$

where $[\lambda n]$ *denotes the integer part of* λn*, then* $\limsup u_{mn} = \beta$ *and* $\liminf u_{mn} = \alpha$*.*

COROLLARY 3. *Let* (u_{mn}) *be* $(C,1,1)$ *summable to* s*. If for* $\lambda > 1$

$$\liminf \frac{1}{(\lambda m] - m)([\lambda n] - n)} \sum_{j=m+1}^{[\lambda m]} \sum_{k=n+1}^{[\lambda n]} (u_{jk} - u_{mn}) \geqslant 0, \tag{4}$$

and for $0 < \lambda < 1$

$$\liminf \frac{1}{(m - [\lambda m])(n - [\lambda n])} \sum_{j=[\lambda m]+1}^{m} \sum_{k=[\lambda n]+1}^{n} (u_{mn} - u_{jk}) \geqslant 0, \tag{5}$$

then (u_{mn}) *is* P*-convergent to* s*.*

We can replace the conditions (4) and (5) by the conditions which are weaker

$$\limsup_{\lambda \to 1^+} \liminf \frac{1}{([\lambda m] - m)([\lambda n] - n)} \sum_{j=m+1}^{[\lambda m]} \sum_{k=n+1}^{[\lambda n]} (u_{jk} - u_{mn}) \geqslant 0, \tag{6}$$

and

$$\limsup_{\lambda \to 1^-} \liminf \frac{1}{(m - [\lambda m])(n - [\lambda n])} \sum_{j=[\lambda m]+1}^{m} \sum_{k=[\lambda n]+1}^{n} (u_{mn} - u_{jk}) \geqslant 0, \tag{7}$$

since the conditions hold for all $\lambda > 1$ and all $0 < \lambda < 1$, respectively.

It is showed by Moricz [7] that the conditions (6) and (7) are sufficient conditions for convergence from the $(C,1,1)$ summability.

In order to prove our main results, the following lemma is necessary.

LEMMA 4. [11] *Let* (u_{mn}) *be a double sequence. For sufficiently large* m,n*:*

(i) If $\lambda > 1$

$$\begin{aligned} u_{mn} - \sigma_{mn}(u) = {} & \frac{([\lambda m]+1)([\lambda n]+1)}{([\lambda m]-m)([\lambda n]-n)} \left(\sigma_{[\lambda m],[\lambda n]}(u) - \sigma_{[\lambda m],n}(u) - \sigma_{m,[\lambda n]}(u) + \sigma_{mn}(u)\right) \\ & + \frac{[\lambda m]+1}{[\lambda m]-m} \left(\sigma_{[\lambda m],n}(u) - \sigma_{m,n}(u)\right) + \frac{[\lambda n]+1}{[\lambda n]-n} \left(\sigma_{m,[\lambda n]}(u) - \sigma_{m,n}(u)\right) \\ & - \frac{1}{([\lambda m]-m)([\lambda n]-n)} \sum_{j=m+1}^{[\lambda m]} \sum_{k=n+1}^{[\lambda n]} (u_{jk} - u_{mn}). \end{aligned}$$

(ii) If $0 < \lambda < 1$

$$\begin{aligned} u_{mn} - \sigma_{mn}(u) = {} & \frac{([\lambda m]+1)([\lambda n]+1)}{(m-[\lambda m])(n-[\lambda n])} \left(\sigma_{mn}(u) - \sigma_{[\lambda m],n}(u) - \sigma_{m,[\lambda n]}(u) + \sigma_{[\lambda m],[\lambda n]}(u)\right) \\ & + \frac{[\lambda m]+1}{m-[\lambda m]} \left(\sigma_{mn}(u) - \sigma_{[\lambda m],n}(u)\right) + \frac{[\lambda n]+1}{n-[\lambda n]} \left(\sigma_{mn}(u) - \sigma_{m,[\lambda n]}(u)\right) \\ & + \frac{1}{(m-[\lambda m])(n-[\lambda n])} \sum_{j=[\lambda m]+1}^{m} \sum_{k=[\lambda n]+1}^{n} (u_{mn} - u_{jk}). \end{aligned}$$

3. Proof of Theorem 2

Suppose that $\limsup \sigma_{mn}(u)=\beta$ and $\liminf \sigma_{mn}(u)=\alpha$, where $\alpha \neq \beta$. Since $\lim_{x\to\infty} \frac{[\lambda x]+1}{[\lambda x]-x}=\frac{\lambda}{\lambda-1}$ for $x=m,n$, we have

$$\limsup \frac{([\lambda m]+1)([\lambda n]+1)}{([\lambda m]-m)([\lambda n]-n)}\left(\sigma^{(11)}_{[\lambda m],[\lambda n]}(u)-\sigma^{(11)}_{[\lambda m],n}(u)-\sigma^{(11)}_{m,[\lambda n]}(u)+\sigma^{(11)}_{mn}(u)\right)$$
$$\leqslant 2(\beta-\alpha)\frac{\lambda^2}{(\lambda-1)^2}, \tag{8}$$

and

$$\limsup \frac{[\lambda m]+1}{[\lambda m]-m}\left(\sigma^{(11)}_{[\lambda m],n}(u)-\sigma^{(11)}_{m,n}(u)\right)\leqslant (\beta-\alpha)\frac{\lambda}{\lambda-1} \tag{9}$$

$$\limsup \frac{[\lambda n]+1}{[\lambda n]-n}\left(\sigma^{(11)}_{m,[\lambda n]}(u)-\sigma^{(11)}_{m,n}(u)\right)\leqslant (\beta-\alpha)\frac{\lambda}{\lambda-1} \tag{10}$$

for $\lambda>1$. We get

$$\limsup u_{mn}\leqslant \beta+2(\beta-\alpha)\frac{\lambda^2}{(\lambda-1)^2}+2(\beta-\alpha)\frac{\lambda}{\lambda-1}$$
$$-\liminf \frac{1}{([\lambda m]-m)([\lambda n]-n)}\sum_{j=m+1}^{[\lambda m]}\sum_{k=n+1}^{[\lambda n]}(u_{jk}-u_{mn})$$

by taking the $\limsup$ of both sides of the identity in Lemma 4 (i) and using (8), (9), (10). When we take (2) into account, we obtain

$$\limsup u_{mn}\leqslant \beta. \tag{11}$$

By (1),

$$\beta \leqslant \limsup u_{mn}. \tag{12}$$

Combining (11) and (12), we have $\limsup u_{mn}=\beta$.

On the other hand, since $\lim_{x\to\infty}\frac{[\lambda x]+1}{x-[\lambda x]}=\frac{\lambda}{1-\lambda}$ for $x=m,n$, we have

$$\liminf \frac{([\lambda m]+1)([\lambda n]+1)}{(m-[\lambda m])(n-[\lambda n])}\left(\sigma^{(11)}_{mn}(u)-\sigma^{(11)}_{[\lambda m],n}(u)-\sigma^{(11)}_{m,[\lambda n]}(u)+\sigma^{(11)}_{[\lambda m],[\lambda n]}(u)\right)$$
$$\geqslant 2(\beta-\alpha)\frac{\lambda^2}{(1-\lambda)^2}, \tag{13}$$

and

$$\liminf \frac{[\lambda m]+1}{m-[\lambda m]}\left(\sigma^{(11)}_{mn}(u)-\sigma^{(11)}_{[\lambda m],n}(u)\right)\geqslant (\beta-\alpha)\frac{\lambda}{1-\lambda} \tag{14}$$

$$\liminf \frac{[\lambda n]+1}{n-[\lambda n]}\left(\sigma^{(11)}_{mn}(u)-\sigma^{(11)}_{m,[\lambda n]}(u)\right)\geqslant (\beta-\alpha)\frac{\lambda}{1-\lambda} \tag{15}$$

for $0 < \lambda < 1$. Taking the $\liminf$ of both sides of the identity in Lemma 4 (ii) and using (13), (14), (15), we obtain

$$\liminf u_{mn} \geqslant \beta + 2(\beta-\alpha)\frac{\lambda^2}{(1-\lambda)^2} + 2(\beta-\alpha)\frac{\lambda}{1-\lambda} + \liminf \frac{1}{([\lambda m]-m)([\lambda n]-n)} \sum_{j=m+1}^{[\lambda m]} \sum_{k=n+1}^{[\lambda n]} (u_{jk}-u_{mn}).$$

Taking (3) into account, we get

$$\liminf u_{mn} \geqslant \alpha. \tag{16}$$

By (1),

$$\alpha \geqslant \liminf u_{mn}. \tag{17}$$

Combining (16) and (17), we have $\liminf u_{mn} = \alpha$.

REFERENCES

[1] İ. ÇANAK, *An extended Tauberian theorem for the $(C,1)$ summability method*, Appl. Math.Lett. **21**, 1 (2008), 74–80.

[2] İ. ÇANAK, *On (C,1) means of sequences*, Comput. Math. Appl. **62**, 9 (2011), 3446–3448.

[3] İ. ÇANAK, *A theorem on the Cesàro summability method*, Comput. Math. Appl. **61**, 4 (2011), 1162–1166.

[4] İ. ÇANAK, Ü. TOTUR, *Some Tauberian conditions for Cesàro summability method*, Math. Slovaca **62**, 2 (2012), 271–280.

[5] İ. ÇANAK, Ü. TOTUR, *A condition under which slow oscillation of a sequence follows from Cesàro summability of its generator sequence*, Appl. Math. Comput. **216**, 5 (2010), 1618–1623.

[6] K. KNOPP, *Limitierungs-Umkehrsätze für Doppelfolgen*, Math. Z. **45**, (1939), 573–589.

[7] F. MÓRICZ, *Tauberian theorems for Cesàro summable double sequences*, Stud. Math. **110**, 1 (1994), 83–96.

[8] M. MURSALEEN, F. BAŞAR, *Domain of Cesàro mean of order one in some spaces of double sequences*, Studia Sci. Math. Hungar. **51**, 3 (2014), 335–356.

[9] R. F. PATTERSON, *Double sequence core theorems*, Int. J. Math. Math. Sci. **22**, 4 (1999), 785–793.

[10] A. PRINGSHEIM, *Zur Theorie der zweifach unendlichen Zahlenfolgen*, Math. Ann. **53**, (1900), 289–321.

[11] Ü. TOTUR, *Classical Tauberian theorems for the $(C,1,1)$ summability method*, An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. **61**, 2 (2015), 401–414.

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