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INEQUALITIES INVOLVING THE INTEGRALS OF POLYNOMIALS AND THEIR POLAR DERIVATIVES

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Abstract. For the class of polynomials $P(z)$ of degree n having all their zeros in $|z| \leqslant k$ where $k \leqslant 1$, Aziz [1] proved that for each $q > 0$,

$$n\left\{\int_0^{2\pi}\left|\frac{P(e^{i\theta})}{P'(e^{i\theta})}\right|^q d\theta\right\}^{1/q} \leqslant \left\{\int_0^{2\pi}\left|k+e^{i\theta}\right|^q d\theta\right\}^{1/q}.$$

In this paper, we extend this inequality to the polar derivative in the sense that we take the polar derivative $D_\alpha P(z)$ in place of ordinary derivative $P'(z)$ of polynomial $P(z)$. We also obtain analogous inequalities for the class of lacunary polynomials $P(z) = a_n z^n + \sum_{j=\mu}^{n} a_{n-j} z^{n-j}$, $1 \leqslant \mu \leqslant n$, of degree n having all their zeros in $|z| \leqslant k$, $k \leqslant 1$.

1. Introduction

Let $P(z)$ be a polynomial of degree n. It was shown by Turán [10] that if $P(z)$ has all its zeros in $|z| \leqslant 1$, then

$$n\max_{|z|=1}|P(z)| \leqslant 2\max_{|z|=1}\left|P'(z)\right|. \tag{1}$$

Inequality (1) is best possible with equality holds for $P(z) = \alpha z^n + \beta$ where $|\alpha| = |\beta| \neq 0$.

As an extension of (1), Malik [5] proved that if $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leqslant k$ where $k \leqslant 1$, then

$$n\max_{|z|=1}|P(z)| \leqslant (1+k)\max_{|z|=1}\left|P'(z)\right|. \tag{2}$$

Equality in (2) holds for $P(z) = (z+k)^n$ where $k \leqslant 1$.

Malik [6] obtained a generalization of (1) in the sense that the left-hand side of (1) is replaced by a factor involving the integral mean of $|P(z)|$ on $|z| = 1$. In fact, he proved that if $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leqslant 1$, then for each $q > 0$,

$$n\left\{\int_0^{2\pi}\left|P\left(e^{i\theta}\right)\right|^q d\theta\right\}^{\frac{1}{q}} \leqslant \left\{\int_0^{2\pi}\left|1+e^{i\theta}\right|^q d\theta\right\}^{\frac{1}{q}}\max_{|z|=1}|P'(z)|. \tag{3}$$

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The corresponding extension of (2), which is a generalization of (3), was obtained by Aziz [1] who proved that if $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leqslant k$ where $k \leqslant 1$, then for each $q \geqslant 0$

$$n\left\{\int_0^{2\pi}\left|\frac{P(e^{i\theta})}{P'(e^{i\theta})}\right|^q d\theta\right\}^{1/q} \leqslant \left\{\int_0^{2\pi}\left|k+e^{i\theta}\right|^q d\theta\right\}^{1/q}. \tag{4}$$

The estimate (4) is best possible and equality in (4) holds for $P(z) = (\alpha z + \beta k)^n$ where $|\alpha| = |\beta|$.

Since $\left|P(e^{i\theta})\right| \leqslant \max_{|z|=1}|P(z)|$, $0 \leqslant \theta < 2\pi$, it follows from (4) that if $P(z)$ has all its zeros in $|z| \leqslant k$ where $k \leqslant 1$, then

$$n\left\{\int_0^{2\pi}\left|P\left(e^{i\theta}\right)\right|^q d\theta\right\}^{\frac{1}{q}} \leqslant \left\{\int_0^{2\pi}\left|1+ke^{i\theta}\right|^q d\theta\right\}^{\frac{1}{q}} \max_{|z|=1}|P'(z)|. \tag{5}$$

Inequality (5) reduces to the inequality (2) by letting $q \to \infty$.

Let $D_\alpha P(z)$ denote the polar derivative of a polynomial $P(z)$ of degree n with respect to a point $\alpha \in \mathbb{C}$, then (see [7])

$$D_\alpha P(z) = nP(z) + (\alpha - z)P'(z).$$

The polynomial $D_\alpha P(z)$ is of degree at most $n-1$ and it generalizes the ordinary derivative in the sense that

$$\lim_{\alpha\to\infty}\frac{D_\alpha P(z)}{\alpha} = P'(z)$$

uniformly with respect to z for $|z| \leqslant R, R > 0$.

As an extension of (2) to the polar derivative, Aziz and Rather [2] proved that if all the zeros of $P(z)$ lie in $|z| \leqslant k$ where $k \leqslant 1$, then for $\alpha \in \mathbb{C}$ with $|\alpha| \geqslant k$,

$$n(|\alpha| - k)\max_{|z|=1}|P(z)| \leqslant (1+k)\max_{|z|=1}|D_\alpha P(z)|. \tag{6}$$

For the class of lacunary type polynomials $P(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu}$, $1 \leqslant \mu \leqslant n$, of degree n having all their zeros in $|z| \leqslant k$ where $k \leqslant 1$, Aziz and Rather [3] also proved that if for $\alpha \in \mathbb{C}$ with $|\alpha| \geqslant k^\mu$,

$$n(|\alpha| - k^\mu)\max_{|z|=1}|P(z)| \leqslant (1+k^\mu)\max_{|z|=1}|D_\alpha P(z)|. \tag{7}$$

In this paper, we first extend inequality (5) to the polar derivative and prove the following generalization of inequality (4).

THEOREM 1. *If $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leqslant k$ where $k \leqslant 1$, then for $\alpha \in \mathbb{C}$ with $|\alpha| \geqslant k$, and for each $p > 0$,*

$$n(|\alpha| - k)\left\{\int_0^{2\pi}\left|\frac{P(e^{i\theta})}{D_\alpha P(e^{i\theta})}\right|^p d\theta\right\}^p \leqslant \left\{\int_0^{2\pi}|1+ke^{i\theta}|^p d\theta\right\}^p \tag{8}$$

The result is best possible and equality in (8) holds for $P(z) = (z-k)^n$.

REMARK 1. By letting $|\alpha| \to \infty$ in (8), we obtain (4).

Next, we consider the class of lacunary polynomials $P(z) = a_n z^n + \sum_{j=\mu}^{n} a_{n-j} z^{n-j}$, $1 \leqslant \mu \leqslant n$, of degree n having all their zeros in $|z| \leqslant k$, $k \leqslant 1$ and prove the following result from which Theorem 1 follows by taking $\mu = 1$.

THEOREM 2. *If* $P(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu}$, $1 \leqslant \mu \leqslant n$, *is a polynomial of degree* n *having all its zeros in* $|z| \leqslant k$ *where* $k \leqslant 1$, *then for* $\alpha \in \mathbb{C}$ *with* $|\alpha| \geqslant k^\mu$ *and for each* $p > 0$,

degree n *having all its zeros in* $|z| \leqslant k$ *where* $k \leqslant 1$, *then for* $\alpha, \beta \in \mathbb{C}$ *with* $|\alpha| \geqslant k^\mu$, $|\beta| \leqslant 1$ *and for each* $p > 0$,

$$n(|\alpha| - k^\mu) \left\{ \int_0^{2\pi} \left| \frac{P(e^{i\theta})}{D_\alpha P(e^{i\theta})} \right|^p d\theta \right\}^p \leqslant \left\{ \int_0^{2\pi} |1 + k^\mu e^{i\theta}|^p d\theta \right\}^p . \tag{9}$$

Instead of proving Thereon 2, we prove the following more general result.

THEOREM 3. *If* $P(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu}$, $1 \leqslant \mu \leqslant n$, *is a polynomial of degree* n *having all its zeros in* $|z| \leqslant k$ *where* $k \leqslant 1$, *then for* $\alpha, \beta \in \mathbb{C}$ *with* $|\alpha| \geqslant k^\mu$, $|\beta| \leqslant 1$ *and for each* $p > 0$,

$$n(|\alpha| - k^\mu) \left\{ \int_0^{2\pi} \left| \frac{P(e^{i\theta}) + \beta \dfrac{m}{k^{n-\mu}}}{|D_\alpha P(e^{i\theta})| - \frac{mn}{k^{n-\mu}}} \right|^p d\theta \right\}^p \leqslant \left\{ \int_0^{2\pi} |1 + k^\mu e^{i\theta}|^p d\theta \right\}^p \tag{10}$$

where $m = \min_{|z|=k} |P(z)|$.

For taking $\beta = 0$ in (10), we obtain the folowing result of which Theorem 2 is a special case.

COROLLARY 1. *If* $P(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu}$, $1 \leqslant \mu \leqslant n$, *is a polynomial of degree* n *having all its zeros in* $|z| \leqslant k$ *where* $k \leqslant 1$, *then for* $\alpha \in \mathbb{C}$ *with* $|\alpha| \geqslant k^\mu$ *and for each* $p > 0$,

$$n(|\alpha| - k^\mu) \left\{ \int_0^{2\pi} \left| \frac{P(e^{i\theta})}{|D_\alpha P(e^{i\theta})| - \frac{mn}{k^{n-\mu}}} \right|^p d\theta \right\}^p \leqslant \left\{ \int_0^{2\pi} |1 + k^\mu e^{i\theta}|^p d\theta \right\}^p \tag{11}$$

where $m = \min_{|z|=k} |P(z)|$.

Since $\left|D_\alpha P(e^{i\theta})\right| \leqslant \max_{|z|=1} |D_\alpha P(z)|$, $0 \leqslant \theta < 2\pi$, therefore, we obtain the following result due to Rather, Bhat and Gulzar [9] from Theorem 3.

If $P(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu}$, $1 \leqslant \mu \leqslant n$, is a polynomial of degree n having all its zeros in $|z| \leqslant k$ where $k \leqslant 1$, then for $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \geqslant k^\mu$, $|\beta| \leqslant 1$ and for each $p > 0$,

$$n(|\alpha| - k^{\mu})\left\{\int_0^{2\pi}\left|P(e^{i\theta}) + \frac{\beta m}{k^{n-\mu}}\right|^p d\theta\right\}^{\frac{1}{p}}$$
$$\leqslant \left\{\int_0^{2\pi}|1 + k^{\mu}e^{i\theta}|^p d\theta\right\}^{\frac{1}{p}}\left\{\max_{|z|=1}|D_{\alpha}P(z)| - \frac{mn}{k^{n-\mu}}\right\} \tag{12}$$

where $m = \min_{|z|=k}|P(z)|$.

2. Lemma

For the proof of Theorem 3, we need the following Lemma.

LEMMA 1. *If $P(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\nu}z^{n-\nu}$, $1 \leqslant \mu \leqslant n$, is a polynomial of degree almost n having all its zeros in in $|z| \leqslant k$ $k \leqslant 1$ then for $|z| = 1$,*

$$|Q'(z)| + \frac{nm}{k^{n-\mu}} \leqslant k^{\mu}|P'(z)| \tag{13}$$

where $Q(z) = z^n\overline{P(1/\overline{z})}$ and $m = \min_{|z|=k}|P(z)|$.

The above Lemma is due to N. A. Rather [8].

3. Proof of Theorem 3

Proof. Let $Q(z) = z^n\overline{P(1/\overline{z})}$, then $P(z) = z^n\overline{Q(1/\overline{z})}$ and it can be easily verified that for $|z| = 1$,

$$|Q'(z)| = |nP(z) - zP'(z)| \text{ and } |P'(z)| = |nQ(z) - zQ'(z)|. \tag{14}$$

By Lemma 1, we have for every β with $|\beta| \leqslant 1$ and $|z| = 1$,

$$\left|Q'(z) + \bar{\beta}\frac{nmz^{n-1}}{k^{n-\mu}}\right| \leqslant |Q'(z)| + \frac{nm}{k^{n-\mu}} \leqslant k^{\mu}|P'(z)|. \tag{15}$$

Using (14) in (15), we get for $|z| = 1$,

$$\left|Q'(z) + \bar{\beta}\frac{nmz^{n-1}}{k^{n-\mu}}\right| \leqslant k^{\mu}|nQ(z) - zQ'(z)|. \tag{16}$$

Again, by Lemma 1 for every real or complex number α with $|\alpha| \geqslant k$ and $|z| = 1$, we have

$$|D_{\alpha}P(z)| \geqslant |\alpha||P'(z)| - |Q'(z)|$$
$$\geqslant (|\alpha| - k^{\mu})|P'(z)| + \frac{mn}{k^{n-\mu}},$$

so that

$$|D_\alpha P(z)| - \frac{mn}{k^{n-\mu}} \geqslant (|\alpha| - k^\mu)|P'(z)|. \tag{17}$$

Since $P(z)$ has all its zeros in $|z| \leqslant k \leqslant 1$, it follows by Gauss-Lucas Theorem that all the zeros of $P'(z)$ also lie in $|z| \leqslant k \leqslant 1$. This implies that the polynomial

$$z^{n-1}\overline{P'(1/\overline{z})} \equiv nQ(z) - zQ'(z)$$

does not vanish in $|z| < 1$. Therefore, it follows from (16) that the function

$$w(z) = \frac{z\left(Q'(z) + \bar{\beta}\dfrac{nmz^{n-1}}{k^{n-\mu}}\right)}{k^\mu\left(nQ(z) - zQ'(z)\right)}$$

is analytic for $|z| \leqslant 1$ and $|w(z)| \leqslant 1$ for $|z| = 1$. Furthermore, $w(0) = 0$. Thus the function $1 + k^\mu w(z)$ is subordinate to the function $1 + k^\mu z$ for $|z| \leqslant 1$. Hence by a well known property of subordination [4], we have

$$\int_0^{2\pi} \left|1 + k^\mu w(e^{i\theta})\right|^p d\theta \leqslant \int_0^{2\pi} \left|1 + k^\mu e^{i\theta}\right|^p d\theta, \; p > 0. \tag{18}$$

Now

$$1 + k^\mu w(z) = \frac{n\left(Q(z) + \bar{\beta}\dfrac{mz^n}{k^{n-\mu}}\right)}{nQ(z) - zQ'(z)},$$

and

$$|P'(z)| = |z^{n-1}\overline{P'(1/\overline{z})}| = |nQ(z) - zQ'(z)| \text{ for } |z| = 1,$$

therefore for $|z| = 1$,

$$n\left|Q(z) + \bar{\beta}\frac{mz^n}{k^{n-\mu}}\right| = |1 + k^\mu w(z)||nQ(z) - zQ'(z)| = |1 + k^\mu w(z)||P'(z)|.$$

Equivalently,

$$n\left|z^n\overline{P(1/\overline{z})} + \bar{\beta}\frac{mz^n}{k^{n-\mu}}\right| = |1 + k^\mu w(z)||P'(z)|.$$

This implies

$$n\left|P(z) + \beta\frac{m}{k^{n-\mu}}\right| = |1 + k^\mu w(z)||P'(z)| \text{ for } |z| = 1. \tag{19}$$

Combining (17) and (19), we get for $|\alpha| \geqslant k^\mu$ and $|z| = 1$.

$$n(|\alpha| - k^\mu)\left|P(z) + \beta\frac{m}{k^{n-\mu}}\right| = |1 + k^\mu w(z)|\left(|D_\alpha P(z)| - \frac{mn}{k^{n-\mu}}\right). \tag{20}$$

From (18) and (20), we deduce for each $p > 0$,

$$n^p(|\alpha| - k^\mu)^p \int_0^{2\pi} \left| \frac{P(e^{i\theta}) + \beta \frac{m}{k^{n-\mu}}}{|D_\alpha P(e^{i\theta})| - \frac{mn}{k^{n-\mu}}} \right|^p d\theta \leqslant \int_0^{2\pi} |1 + k^\mu e^{i\theta}|^p d\theta$$

which gives

$$n(|\alpha| - k^\mu) \left\{ \int_0^{2\pi} \left| \frac{P(e^{i\theta}) + \beta \frac{m}{k^{n-\mu}}}{|D_\alpha P(e^{i\theta})| - \frac{mn}{k^{n-\mu}}} \right|^p d\theta \right\}^p \leqslant \left\{ \int_0^{2\pi} |1 + k^\mu e^{i\theta}|^p d\theta \right\}^p$$

This completes the proof of Theorem 3. □

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