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A NOTE ON DEGENERATE HERMITE POLY–BERNOULLI NUMBERS AND POLYNOMIALS

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Abstract. In this paper, we introduce a new class of degenerate Hermite poly-Bernoulli polynomials and give some identities of these polynomials related to the Stirling numbers of the second kind. Some implicit summation formulae and general symmetry identities are derived by using different analytical means and applying generating functions. These results extend some known summations and identities of degenerate Hermite poly-Bernoulli numbers and polynomials.

1. Introduction

The 2-variable Kampe de Feriet generalization of the Hermite polynomials [3] and [6] reads

$$H_n(x,y) = n!\sum_{r=0}^{[\frac{n}{2}]}\frac{y^r x^{n-2r}}{r!(n-2r)!}. \tag{1.1}$$

These polynomials are usually defined by the generating function

$$e^{xt+yt^2} = \sum_{n=0}^{\infty} H_n(x,y)\frac{t^n}{n!} \tag{1.2}$$

and reduce to the ordinary Hermite polynomials $H_n(x)$ (see [1]) when $y=-1$ and x is replaced by $2x$.

Based on the definition and generating function above, we can define degenerate Hermite polynomials by means of the generating function

$$(1+\lambda t)^{\frac{x}{\lambda}}(1+\lambda t^2)^{\frac{y}{\lambda}} = \sum_{n=0}^{\infty} H_n(x,y;\lambda)\frac{t^n}{n!} \tag{1.3}$$

where $\lambda \neq 0$. Since $(1+\lambda t)^{\frac{1}{\lambda}} \longrightarrow e^t$ as $\lambda \longrightarrow 0$, it is evident that (1.3) reduces to (1.2). That is $H_n(x,y)$ limiting case of $H_n(x,y;\lambda)$ when $\lambda \longrightarrow 0$.

By equating coefficients of t^n on both the sides of (1.3), the following representation of $H_n(x,y;\lambda)$ is obtained

$$H_n(x,y;\lambda) = n!\sum_{r=0}^{[\frac{n}{2}]}\frac{\left(-\frac{x}{\lambda}\right)_{n-2r}\left(-\frac{y}{\lambda}\right)_r(-\lambda)^{n-r}}{r!(n-2r)!}. \tag{1.4}$$

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Since $\lim_{\lambda\longrightarrow 0} H_n(x,y;\lambda) = H_n(x,y)$, (1.1) is a limiting case of (1.4).

For $\lambda \in \mathbb{C}$, Carlitz introduced the degenerate Bernoulli polynomials given by the generating function

$$\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \beta_n(x;\lambda)\frac{t^n}{n!}, \text{ (see [4, 19, 20])} \tag{1.5}$$

so that

$$\beta_n(x;\lambda) = \sum_{n=0}^{m} \binom{n}{m} \beta_m(\lambda) \left(\frac{x}{\lambda}\right)_{n-m}. \tag{1.6}$$

When $x=0$, $\beta_n(\lambda) = \beta_n(0;\lambda)$ are called the degenerate Bernoulli numbers.

From (1.5), we note that

$$\begin{aligned}\sum_{n=0}^{\infty} \lim_{\lambda\longrightarrow 0} \beta_n(x;\lambda)\frac{t^n}{n!} &= \lim_{\lambda\longrightarrow 0} \frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}} \\ &= \frac{t}{e^t-1}e^{xt} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} \end{aligned}\tag{1.7}$$

where $B_n(x)$ are called the Bernoulli polynomials (see [1–27]).

The classical polylogarithm function $\mathrm{Li}_k(z)$ is

$$\mathrm{Li}_k(z) = \sum_{m=1}^{\infty} \frac{z^m}{m^k}, \quad (k\in\mathbb{Z}) \text{ (see [15–18])} \tag{1.8}$$

so for $k \leqslant 1$,

$$\mathrm{Li}_k(z) = -\ln(1-z), \quad \mathrm{Li}_0(z) = \frac{z}{1-z}, \quad \mathrm{Li}_{-1}(z) = \frac{z}{(1-z)^2}, \cdots$$

The poly-Bernoulli polynomials are given by

$$\frac{\mathrm{Li}_k(1-e^{-t})}{e^t-1}e^{xt} = \sum_{n=0}^{\infty} B_n^{(k)}(x)\frac{t^n}{n!}, \text{ (see [2, 10, 12]).} \tag{1.9}$$

For $k=1$ in (1.9), we have

$$\frac{\mathrm{Li}_1(1-e^{-t})}{e^t-1}e^{xt} = \frac{t}{e^t-1}e^{xt} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}. \tag{1.10}$$

From (1.7) and (1.10), we have

$$B_n^{(1)}(x) = B_n(x).$$

Very recently, Pathan and Khan introduced the generalized Hermite-Bernoulli polynomials of two variables ${}_HB_n^{(\alpha)}(x,y)$ defined by

$$\left(\frac{t}{e^t-1}\right)^{\alpha} e^{xt+yt^2} = \sum_{n=0}^{\infty} {}_HB_n^{(\alpha)}(x,y)\frac{t^n}{n!}, \text{ (see [21–26])} \tag{1.11}$$

which is essentially a generalization of Bernoulli numbers, Bernoulli polynomials, Hermite polynomials and Hermite-Bernoulli polynomials ${}_HB_n(x,y)$ introduced by Dattoli et al [6, p. 386 (1.6)] in the form

$$\left(\frac{t}{e^t-1}\right)e^{xt+yt^2}=\sum_{n=0}^{\infty}{}_HB_n(x,y)\frac{t^n}{n!}. \tag{1.12}$$

The Stirling number of the first kind is given by

$$(x)_n=x(x-1)\cdots(x-n+1)=\sum_{l=0}^{n}S_1(n,l)x^l, \quad (n\geqslant 0) \tag{1.13}$$

and the Stirling number of the second kind is defined by generating function to be

$$(e^t-1)^n=n!\sum_{l=n}^{\infty}S_2(l,n)\frac{t^l}{l!}. \tag{1.14}$$

A generalized falling factorial sum $\sigma_k(n;\lambda)$ can be defined by the generating function [27]

$$\sum_{k=0}^{\infty}\sigma_k(n;\lambda)\frac{t^k}{k!}=\frac{(1+\lambda t)^{\frac{(n+1)}{\lambda}}-1}{(1+\lambda t)^{\frac{1}{\lambda}}-1} \tag{1.15}$$

where $\lim_{\lambda\longrightarrow 0}\sigma_k(n;\lambda)=S_k(n)$.

In this paper, we first give definition of the degenerate Hermite poly-Bernoulli polynomials ${}_H\beta_n^{(k)}(x,y;\lambda)$ and then extend and illustrate how, a connection between Hermite and Bernoulli polynomials can yield new expansions and representations. Some implicit summation formulae and general symmetry identities are derived. These results establish a link between these families of polynomials (namely degenerate Hermite and degenerate poly-Bernoulli polynomials).

2. Degenerate Hermite poly-Bernoulli numbers and polynomials

For $\lambda\in\mathbb{C}$, $k\in\mathbb{Z}$, we consider the degenerate Hermite poly-Bernoulli polynomials given by the generating function

$$\frac{\mathrm{Li}_k(1-e^{-t})}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}}(1+\lambda t^2)^{\frac{y}{\lambda}}=\sum_{n=0}^{\infty}{}_H\beta_n^{(k)}(x,y;\lambda)\frac{t^n}{n!} \tag{2.1}$$

so that

$${}_H\beta_n^{(k)}(x,y;\lambda)=\sum_{m=0}^{n}\binom{n}{m}\beta_m^{(k)}(\lambda)H_{n-m}(x,y;\lambda). \tag{2.2}$$

When $x=y=0$ in (2.1), ${}_H\beta_n^{(k)}(0,0;\lambda)=\beta_n^{(k)}(\lambda)$ are called the degenerate poly-Bernoulli numbers.

Note that ${}_H\beta_n^{(1)}(x,y;\lambda)={}_H\beta_n(x,y;\lambda)$ and $\lim_{\lambda\longrightarrow 0}{}_H\beta_n^{(k)}(x,y;\lambda)={}_HB_n^{(k)}(x,y)$.

For $y=0$ in (2.1), the result reduces to the degenerate poly-Bernoulli polynomials of Kim et al [18., p. 2, Eq. (2.1)] defined as

$$\frac{\mathrm{Li}_k(1-e^{-t})}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}}=\sum_{n=0}^{\infty}\beta_n^{(k)}(x;\lambda)\frac{t^n}{n!},\quad (k\in\mathbb{Z}). \tag{2.3}$$

THEOREM 2.1. *For $n\geqslant 0$, we have*

$$_H\beta_n^{(2)}(x,y;\lambda)=\sum_{m=0}^{n}\binom{n}{m}\frac{B_m}{m+1}{}_H\beta_{n-m}(x,y;\lambda). \tag{2.4}$$

Proof. Applying Definition (2.1), we have

$$\begin{aligned}&\sum_{n=0}^{\infty}{}_H\beta_n^{(k)}(x,y;\lambda)\frac{t^n}{n!}\\&=\frac{\mathrm{Li}_k(1-e^{-t})}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}}(1+\lambda t^2)^{\frac{y}{\lambda}}\\&=\frac{(1+\lambda t)^{\frac{x}{\lambda}}(1+\lambda t^2)^{\frac{y}{\lambda}}}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\int_0^t\underbrace{\frac{1}{e^z-1}\int_0^t\frac{1}{e^z-1}\cdots\frac{1}{e^z-1}\int_0^t\frac{z}{e^z-1}}_{(k-2)-\text{times}}dz\cdots dz\end{aligned} \tag{2.5}$$

For $k=2$ in (2.5), we have

$$\begin{aligned}\sum_{n=0}^{\infty}{}_H\beta_n^{(2)}(x,y;\lambda)\frac{t^n}{n!}&=\frac{(1+\lambda t)^{\frac{x}{\lambda}}(1+\lambda t^2)^{\frac{y}{\lambda}}}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\int_0^t\frac{z}{e^z-1}dz\\&=\left(\sum_{m=0}^{\infty}\frac{B_m}{m+1}\frac{t^m}{m!}\right)\frac{(1+\lambda t)^{\frac{x}{\lambda}}(1+\lambda t^2)^{\frac{y}{\lambda}}}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\\&=\left(\sum_{m=0}^{\infty}\frac{B_m}{m+1}\frac{t^m}{m!}\right)\left(\sum_{n=0}^{\infty}{}_H\beta_n(x,y;\lambda)\frac{t^n}{n!}\right)\end{aligned}$$

Replacing n by $n-m$ in above equation, we have

$$=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}\frac{B_m}{m+1}{}_H\beta_{n-m}(x,y;\lambda)\frac{t^n}{n!}.$$

On equating the coefficients of the like powers of t in the above equation, we get the result (2.4). □

REMARK 1. For $y=0$ in Theorem (2.1), the result reduces to known result of Kim et al [18., p. 3, Theorem (2.1)].

COROLLARY 1. *For $n\geqslant 0$, we have*

$$\beta_n^{(2)}(x;\lambda)=\sum_{m=0}^{n}\binom{n}{m}\frac{B_m}{m+1}\beta_{n-m}(x;\lambda). \tag{2.6}$$

THEOREM 2.2. *For $n \geqslant 0$, we have*

$$ {}_H\beta_n^{(k)}(x,y;\lambda) = \sum_{p=0}^{n} \binom{n}{p} \left(\sum_{l=1}^{p+1} \frac{(-1)^{l+p+1} l! S_2(p+1,l)}{l^k (p+1)} \right) {}_H\beta_{n-p}(x,y;\lambda). \tag{2.7} $$

Proof. From equation (2.1), we have

$$ \sum_{n=0}^{\infty} {}_H\beta_n^{(k)}(x,y;\lambda) \frac{t^n}{n!} = \left(\frac{\mathrm{Li}_k(1-e^{-t})}{t} \right) \left(\frac{t(1+\lambda t)^{\frac{x}{\lambda}} (1+\lambda t^2)^{\frac{y}{\lambda}}}{(1+\lambda t)^{\frac{1}{\lambda}} - 1} \right). \tag{2.8} $$

Now

$$ \begin{aligned} \frac{1}{t}\mathrm{Li}_k(1-e^{-t}) &= \frac{1}{t} \sum_{l=1}^{\infty} \frac{(1-e^{-t})^l}{l^k} = \frac{1}{t} \sum_{l=1}^{\infty} \frac{(-1)^l}{l^k} (1-e^{-t})^l \\ &= \frac{1}{t} \sum_{l=1}^{\infty} \frac{(-1)^l}{l^k} l! \sum_{p=l}^{\infty} (-1)^p S_2(p,l) \frac{t^p}{p!} \\ &= \frac{1}{t} \sum_{p=1}^{\infty} \sum_{l=1}^{p} \frac{(-1)^{l+p}}{l^k} l! S_2(p,l) \frac{t^p}{p!} \\ &= \sum_{p=0}^{\infty} \left(\sum_{l=1}^{p+1} \frac{(-1)^{l+p+1}}{l^k} l! \frac{S_2(p+1,l)}{p+1} \right) \frac{t^p}{p!} \end{aligned} \tag{2.9} $$

From equations (2.8) and (2.9), we have

$$ \sum_{n=0}^{\infty} {}_H\beta_n^{(k)}(x,y;\lambda) \frac{t^n}{n!} = \sum_{p=0}^{\infty} \left(\sum_{l=1}^{p+1} \frac{(-1)^{l+p+1}}{l^k} l! \frac{S_2(p+1,l)}{p+1} \right) \frac{t^p}{p!} \left(\sum_{n=0}^{\infty} {}_H\beta_n(x,y;\lambda) \frac{t^n}{n!} \right). $$

Replacing n by $n-p$ in the r.h.s of above equation and comparing the coefficients of t^n, we get the result (2.7). □

REMARK 2. For $y=0$ in Theorem (2.2), the result reduces to known result of Kim et al [18., p. 5, Theorem (2.2)].

COROLLARY 2. *For $n \geqslant 0$, we have*

$$ \beta_n^{(k)}(x;\lambda) = \sum_{p=0}^{n} \binom{n}{p} \left(\sum_{l=1}^{p+1} \frac{(-1)^{l+p+1} l! S_2(p+1,l)}{l^k (p+1)} \right) \beta_{n-p}(x;\lambda). \tag{2.10} $$

THEOREM 2.3. *For $n \geqslant 1$, we have*

$$ \begin{aligned} & {}_H\beta_n^{(k)}(x+1,y;\lambda) - {}_H\beta_n^{(k)}(x,y;\lambda) \\ &= \sum_{p=1}^{n} \binom{n}{p} \left(\sum_{l=0}^{p-1} \frac{(-1)^{l+p+1}}{(l+1)^k} (l+1)! S_2(p,l+1) \right) H_{n-p}(x,y;\lambda). \end{aligned} \tag{2.11} $$

Proof. Using the Definition (2.1), we have

$$\sum_{n=0}^{\infty} {}_H\beta_n^{(k)}(x+1,y;\lambda)\frac{t^n}{n!} - \sum_{n=0}^{\infty} {}_H\beta_n^{(k)}(x,y;\lambda)\frac{t^n}{n!}$$
$$= \frac{\mathrm{Li}_k(1-e^{-t})}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x+1}{\lambda}}(1+\lambda t^2)^{\frac{y}{\lambda}} - \frac{\mathrm{Li}_k(1-e^{-t})}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}}(1+\lambda t^2)^{\frac{y}{\lambda}}$$
$$= \mathrm{Li}_k(1-e^{-t})(1+\lambda t)^{\frac{x}{\lambda}}(1+\lambda t^2)^{\frac{y}{\lambda}}$$
$$= \sum_{l=0}^{\infty} \frac{(1-e^{-t})^{l+1}}{(l+1)^k}(1+\lambda t)^{\frac{x}{\lambda}}(1+\lambda t^2)^{\frac{y}{\lambda}}$$
$$= \sum_{p=1}^{\infty}\left(\sum_{l=0}^{p-1}\frac{(-1)^{l+p+1}}{(l+1)^k}(l+1)!S_2(p,l+1)\right)\frac{t^p}{p!}(1+\lambda t)^{\frac{x}{\lambda}}(1+\lambda t^2)^{\frac{y}{\lambda}}$$
$$= \left(\sum_{p=1}^{\infty}\left(\sum_{l=0}^{p-1}\frac{(-1)^{l+p+1}}{(l+1)^k}(l+1)!S_2(p,l+1)\right)\frac{t^p}{p!}\right)\left(\sum_{n=0}^{\infty} H_n(x,y;\lambda)\frac{t^n}{n!}\right)$$

Replacing n by $n-p$ in the above equation and comparing the coefficients of t^n, we get the result (2.11). □

REMARK 3. For $y=0$ in Theorem (2.3), the result reduces to known result of Kim et al [18., p. 5, Theorem (2.3)].

COROLLARY 3. *For $n \geqslant 1$, we have*

$$\beta_n^{(k)}(x+1;\lambda) - \beta_n^{(k)}(x;\lambda) = \sum_{p=1}^{n}\left(\sum_{l=0}^{p-1}\frac{(-1)^{l+p+1}}{(l+1)^k}(l+1)!S_2(p,l+1)\right)\binom{n}{p}(x/\lambda)_{n-p}. \tag{2.12}$$

THEOREM 2.4. *For $n \geqslant 0$, $d \in \mathbb{N}$ and $k \in \mathbb{Z}$, we have*

$${}_H\beta_n^{(k)}(x,y;\lambda) = \sum_{a=0}^{d-1}\sum_{l=0}^{n}\sum_{p=1}^{l+1}\binom{n}{l}d^{n-l-1}\frac{(-1)^{l+p+1}p!S_2(l+1,p)}{p^k l+1}{}_H\beta_{n-l}\left(\frac{l+x}{d},y;\frac{\lambda}{d}\right). \tag{2.13}$$

Proof. From equation (2.1), we can be written as

$$\sum_{n=0}^{\infty} {}_H\beta_n^{(k)}(x,y;\lambda)\frac{t^n}{n!} = \frac{\mathrm{Li}_k(1-e^{-t})}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}}(1+\lambda t^2)^{\frac{y}{\lambda}}$$
$$= \frac{\mathrm{Li}_k(1-e^{-t})}{(1+\lambda t)^{\frac{d}{\lambda}}-1}\sum_{a=0}^{d-1}(1+\lambda t)^{\frac{l+x}{\lambda}}(1+\lambda t^2)^{\frac{y}{\lambda}}$$
$$= \left(\frac{\mathrm{Li}_k(1-e^{-t})}{t}\right)\frac{1}{d}\sum_{a=0}^{d-1}\frac{dt}{(1+\lambda t)^{\frac{d}{\lambda}}-1}(1+\lambda t)^{\frac{l+x}{\lambda}}(1+\lambda t^2)^{\frac{y}{\lambda}}$$
$$= \left(\sum_{l=0}^{\infty}\left(\sum_{p=1}^{l+1}\frac{(-1)^{l+p+1}}{p^k}p!\frac{S_2(l+1,p)}{l+1}\right)\frac{t^l}{l!}\right)\left(\sum_{n=0}^{\infty}d^{n-1}\sum_{a=0}^{d-1}{}_H\beta_n\left(\frac{l+x}{d},y;\frac{\lambda}{d}\right)\frac{t^n}{n!}\right)$$

Replacing n by $n-l$ in above equation and comparing the coefficient of t^n, we get the result (2.13). □

REMARK 4. For $y=0$ in Theorem (2.4), the result reduces to known result of Kim et al [18., p. 6., Theorem (2.4)].

COROLLARY 4. *For $n \geqslant 0$, $d \in \mathbb{N}$ and $k \in \mathbb{Z}$, we have*

$$\beta_n^{(k)}(x;\lambda) = \sum_{a=0}^{d-1}\sum_{l=0}^{n}\sum_{p=1}^{l+1}\binom{n}{l} d^{n-l-1}\frac{(-1)^{l+p+1}p!S_2(l+1,p)}{p^k l+1}\beta_{n-l}\left(\frac{l+x}{d};\frac{\lambda}{d}\right). \tag{2.14}$$

3. Implicit summation formulae involving degenerate Hermite poly-Bernoulli polynomials

For the derivation of implicit formulae involving degenerate poly-Bernoulli polynomials $\beta_n^{(k)}(x;\lambda)$ and degenerate Hermite poly-Bernoulli polynomials ${}_H\beta_n^{(k)}(x,y;\lambda)$ the same considerations as developed for the ordinary Hermite and related polynomials in Khan et al [11] and Hermite-Bernoulli polynomials in Pathan and Khan [21-26] holds as well. First we prove the following results involving degenerate Hermite poly-Bernoulli polynomials ${}_H\beta_n^{(k)}(x,y;\lambda)$.

THEOREM 3.1. *Let $x,y \in \mathbb{R}$ and $n \geqslant 0$. The following implicit summation formula involving degenerate Hermite poly-Bernoulli polynomials ${}_H\beta_n^{(k)}(x,y,\lambda)$ holds true*

$$ {}_H\beta_n^{(k)}(x+u,y+w;\lambda) = \sum_{m=0}^{n}\binom{n}{m}{}_H\beta_{n-m}^{(k)}(x,y;\lambda)H_m(u,w;\lambda). \tag{3.1}$$

Proof. By the definition of degenerate poly-Bernoulli polynomials and the definition (1.3), we have

$$\frac{\mathrm{Li}_k(1-(e)^{-t})}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x+u}{\lambda}}(1+\lambda t^2)^{\frac{y+w}{\lambda}} = \left(\sum_{n=0}^{\infty}{}_H\beta_n^{(k)}(x,y;\lambda)\frac{t^n}{n!}\right)\left(\sum_{m=0}^{\infty}H_m(u,w;\lambda)\frac{t^m}{m!}\right).$$

Now replacing n by $n-m$ and comparing the coefficients of t^n, we get the result (3.1). □

THEOREM 3.2. *For $x,y \in \mathbb{R}$ and $n \geqslant 0$. The following implicit summation formula involving degenerate Hermite poly-Bernoulli polynomials ${}_H\beta_n^{(k)}(x,y,\lambda)$ holds true*

$${}_H\beta_n^{(k)}(x,y;\lambda) = \sum_{m=0}^{n-2j}\sum_{j=0}^{[\frac{n}{2}]}\beta_m^{(k)}(\lambda)\left(-\frac{x}{\lambda}\right)_{n-m-2j}(-\lambda)^{n-m-j}\left(-\frac{y}{\lambda}\right)_j\frac{n!}{m!j!(n-2j-m)!}. \tag{3.2}$$

Proof. Applying the definition (2.1) to the term $\frac{\mathrm{Li}_k(1-e^{-t})}{(1+\lambda t)^{\frac{1}{\lambda}}-1}$ and expanding the function $(1+\lambda t)^{\frac{x}{\lambda}}(1+\lambda t^2)^{\frac{y}{\lambda}}$ at $t=0$ yields

$$\begin{aligned}&\frac{\mathrm{Li}_k(1-e^{-t})}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}}(1+\lambda t^2)^{\frac{y}{\lambda}}\\&=\left(\sum_{m=0}^{\infty}\beta_m^{(k)}(\lambda)\frac{t^m}{m!}\right)\left(\sum_{n=0}^{\infty}\left(-\frac{x}{\lambda}\right)_n\frac{(-\lambda t)^n}{n!}\right)\left(\sum_{j=0}^{\infty}\left(-\frac{y}{\lambda}\right)_j\frac{(-\lambda t^2)^j}{j!}\right)\\&=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}\beta_m^{(k)}(\lambda)\left(-\frac{x}{\lambda}\right)_{n-m}(-\lambda)^{n-m}\right)\frac{t^n}{n!}\left(\sum_{j=0}^{\infty}\left(-\frac{y}{\lambda}\right)_j\frac{(-\lambda t^2)^j}{j!}\right)\end{aligned}$$

Replacing n by $n-2j$, we have

$$\sum_{n=0}^{\infty}{}_H\beta_n^{(k)}(x,y;\lambda)\frac{t^n}{n!}$$

$$=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n-2j}\sum_{j=0}^{[\frac{n}{2}]}\binom{n-2j}{m}\beta_m^{(k)}(\lambda)\left(-\frac{x}{\lambda}\right)_{n-m-2j}(-\lambda)^{n-m-j}\left(-\frac{y}{\lambda}\right)_j\right)\frac{t^n}{(n-2j)!j!}. \tag{3.3}$$

Equating their coefficients of t^n, we get the result (3.2). □

THEOREM 3.3. *Let $x,y\in\mathbb{R}$ and $n\geqslant 0$. The following implicit summation formula involving degenerate Hermite poly-Bernoulli polynomials ${}_H\beta_n^{(k)}(x,y,\lambda)$ holds true*

$${}_H\beta_n^{(k)}(x,y;\lambda)=\sum_{m=0}^{n}\binom{n}{m}\left(-\frac{z}{\lambda}\right)_{n-m}(-\lambda)^{n-m}{}_H\beta_m^{(k)}(x-z,y;\lambda). \tag{3.4}$$

Proof. By exploiting the generating function (2.1), we can write the equation

$$\begin{aligned}\sum_{n=0}^{\infty}{}_H\beta_n^{(k)}(x,y;\lambda)\frac{t^n}{n!}&=\frac{\mathrm{Li}_k(1-e^{-t})}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x-z}{\lambda}}(1+\lambda t^2)^{\frac{y}{\lambda}}(1+\lambda t)^{\frac{z}{\lambda}}\\&=\left(\sum_{m=0}^{\infty}{}_H\beta_m^{(k)}(x-z,y;\lambda)\frac{t^m}{m!}\right)\left(\sum_{n=0}^{\infty}\left(-\frac{z}{\lambda}\right)_n\frac{(-\lambda t)^n}{n!}\right)\end{aligned} \tag{3.5}$$

Replacing n by $n-m$ in above equation and equating their coefficients of t^n leads to formula (3.4). □

THEOREM 3.4. *The following implicit summation formula involving degenerate Hermite poly-Bernoulli polynomials ${}_H\beta_n^{(k)}(x,y;\lambda)$ holds true:*

$${}_H\beta_n^{(k)}(x+1,y;\lambda)=\sum_{r=0}^{n}\binom{n}{r}\left(-\frac{1}{\lambda}\right)_r(-\lambda)^r{}_H\beta_{n-r}^{(k)}(x,y;\lambda). \tag{3.6}$$

Proof. By the definition of degenerate Hermite poly-Bernoulli polynomials, we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} {}_H\beta_n^{(k)}(x+1,y;\lambda)\frac{t^n}{n!}+\sum_{n=0}^{\infty} {}_H\beta_n^{(k)}(x,y;\lambda)\frac{t^n}{n!}\\
&=\frac{\mathrm{Li}_k(1-e^{-t})}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}}(1+\lambda t^2)^{\frac{y}{\lambda}}((1+\lambda t)^{\frac{1}{\lambda}}+1)\\
&=\left(\sum_{n=0}^{\infty} {}_H\beta_n^{(k)}(x,y;\lambda)\frac{t^n}{n!}\right)\left(\sum_{r=0}^{\infty}\left(-\frac{1}{\lambda}\right)_r\frac{(-\lambda t)^r}{r!}\right)+\sum_{n=0}^{\infty} {}_H\beta_n^{(k)}(x,y;\lambda)\frac{t^n}{n!}\\
&=\sum_{n=0}^{\infty}\sum_{r=0}^{n} {}_H\beta_{n-r}^{(k)}(x,y;\lambda)\left(-\frac{1}{\lambda}\right)_r(-\lambda)^r\frac{t^n}{(n-r)!r!}+\sum_{n=0}^{\infty} {}_H\beta_n^{(k)}(x,y;\lambda)\frac{t^n}{n!}
\end{aligned}$$

Finally, equating the coefficients of the like powers of t^n, we get (3.6). □

4. General symmetry identities for degenerate Hermite poly-Bernoulli polynomials

In this section, we give general symmetry identities for the degenerate poly-Bernoulli polynomials $\beta_n^{(k)}(x;\lambda)$ and the degenerate Hermite poly-Bernoulli polynomials ${}_H\beta_n^{(k)}(x,y;\lambda)$ by applying the generating function(2.1) and (2.3). The results extend some known identities of Young [27], Khan [13–15] and Khan et al [16], Pathan et al [21–26].

THEOREM 4.1. *Let $a,b>0$ and $a\neq b$. For $x,y\in\mathbb{R}$ and $n\geqslant 0$. Then the following identity holds true:*

$$\begin{aligned}
&\sum_{m=0}^{n}\binom{n}{m} b^m a^{n-m} {}_H\beta_{n-m}^{(k)}(bx,b^2y;\lambda)\,{}_H\beta_m^{(k)}(ax,a^2y;\lambda)\\
&=\sum_{m=0}^{n}\binom{n}{m} a^m b^{n-m} {}_H\beta_{n-m}^{(k)}(ax,a^2y;\lambda)\,{}_H\beta_m^{(k)}(bx,b^2y;\lambda). \qquad (4.1)
\end{aligned}$$

Proof. Start with

$$g(t)=\left(\frac{\mathrm{Li}_k(1-e^{-at})\mathrm{Li}_k(1-e^{-bt})}{((1+\lambda t)^{\frac{a}{\lambda}}-1)((1+\lambda t)^{\frac{b}{\lambda}}-1)}\right)(1+\lambda t)^{\frac{abx}{\lambda}}(1+\lambda t^2)^{\frac{a^2b^2y}{\lambda}}. \qquad (4.2)$$

Then the expression for g(t) is symmetric in a and b and we can expand g(t) into series in two ways to obtain

$$\begin{aligned}
g(t)&=\sum_{n=0}^{\infty} {}_H\beta_n^{(k)}(bx,b^2y;\lambda)\frac{(at)^n}{n!}\sum_{m=0}^{\infty} {}_H\beta_m^{(k)}(ax,a^2y;\lambda)\frac{(bt)^m}{m!}\\
&=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m} a^{n-m}b^m {}_H\beta_{n-m}^{(k)}(bx,b^2y;\lambda)\,{}_H\beta_m^{(k)}(ax,a^2y;\lambda)\frac{t^n}{n!}.
\end{aligned}$$

On the similar lines we can show that

$$g(t)=\sum_{n=0}^{\infty}{}_H\beta_n^{(k)}(ax,a^2y;\lambda)\frac{(bt)^n}{n!}\sum_{m=0}^{\infty}{}_H\beta_m^{(k)}(bx,b^2y;\lambda)\frac{(at)^m}{m!}$$
$$=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}a^mb^{n-m}{}_H\beta_{n-m}^{(k)}(ax,a^2y;\lambda){}_H\beta_m^{(k)}(bx,b^2y;\lambda)\frac{t^n}{n!}.$$

Comparing the coefficients of t^n on the right hand sides of the last two equations we arrive the desired result. □

REMARK 1. By setting $b=1$ in Theorem 4.1, we immediately following result

COROLLARY.

$$\sum_{m=0}^{n}\binom{n}{m}a^{n-m}{}_H\beta_{n-m}^{(k)}(x,y;\lambda){}_H\beta_m^{(k)}(ax,a^2y;\lambda)$$
$$=\sum_{m=0}^{n}\binom{n}{m}a^m{}_H\beta_{n-m}^{(k)}(ax,a^2y;\lambda){}_H\beta_m^{(k)}(x,y;\lambda). \tag{4.3}$$

THEOREM 4.2. *For all integers $a>0,b>0$, and $n\geqslant 0$, the following identity holds true:*

$$\sum_{m=0}^{n}\binom{n}{m}a^{n-m}b^m{}_H\beta_{n-m}^{(k)}\left(bx,b^2z;\lambda\right)\sum_{i=0}^{m}\binom{m}{i}\sigma_i(a-1;\lambda)\beta_{m-i}^{(k)}(ay;\lambda)$$
$$=\sum_{m=0}^{n}\binom{n}{m}a^mb^{n-m}{}_H\beta_{n-m}^{(k)}\left(ax,a^2z;\lambda\right)\sum_{i=0}^{m}\binom{m}{i}\sigma_i(b-1;\lambda)\beta_{m-i}^{(k)}(by;\lambda) \tag{4.4}$$

where generalized falling factorial sum $\sigma_k(n;\lambda)$ is given by (1.15).

Proof. We now use

$$g(t)=\frac{\mathrm{Li}_k(1-e^{-at})\mathrm{Li}_k(1-e^{-bt})((1+\lambda t)^{\frac{ab}{\lambda}}-1)(1+\lambda t)^{\frac{ab(x+y)}{\lambda}}(1+\lambda t^2)^{\frac{a^2b^2z}{\lambda}}}{((1+\lambda t)^{\frac{a}{\lambda}}-1)((1+\lambda t)^{\frac{b}{\lambda}}-1)^2}$$

to find that

$$g(t)=\left(\frac{\mathrm{Li}_k(1-e^{-at})}{(1+\lambda t)^{\frac{a}{\lambda}}-1}\right)(1+\lambda t)^{\frac{abx}{\lambda}}(1+\lambda t^2)^{\frac{a^2b^2z}{\lambda}}\left(\frac{(1+\lambda t)^{\frac{ab}{\lambda}}-1}{(1+\lambda t)^{\frac{b}{\lambda}}-1}\right)$$
$$\left(\frac{\mathrm{Li}_k(1-e^{-bt})}{(1+\lambda t)^{\frac{b}{\lambda}}-1}\right)(1+\lambda t)^{\frac{aby}{\lambda}}$$
$$=\sum_{n=0}^{\infty}{}_H\beta_n^{(k)}\left(bx,b^2z;\lambda\right)\frac{(at)^n}{n!}\sum_{n=0}^{\infty}\sigma_n(a-1;\lambda)\frac{(bt)^n}{n!}\sum_{n=0}^{\infty}\beta_n^{(k)}(ay;\lambda)\frac{(bt)^n}{n!}. \tag{4.5}$$

Using a similar plan, we get

$$g(t) = \sum_{n=0}^{\infty} {}_H\beta_n^{(k)}\left(ax, a^2z; \lambda\right) \frac{(bt)^n}{n!} \sum_{n=0}^{\infty} \sigma_n(b-1;\lambda) \frac{(at)^n}{n!} \sum_{n=0}^{\infty} \beta_n^{(k)}(by;\lambda) \frac{(at)^n}{n!}. \quad (4.6)$$

By comparing the coefficients of t^n on the right hand sides of the last two equations, we arrive at the desired result. □

5. Conclusion

The definition and generating function (2.1) of the degenerate Hermite poly-Bernoulli polynomials ${}_H\beta_n^{(k)}(x,y;\lambda)$ plays a major role in obtaining new expansions,identities and representations. We can introduce and study a class of related generalized polynomials by defining degenerate Gould- Hopper poly-Bernoulli polynomials

$$\frac{\mathrm{Li}_k(1-e^{-t})}{(1+\lambda t)^{\frac{1}{\lambda}} - 1} (1+\lambda t)^{\frac{x}{\lambda}} (1+\lambda t^r)^{\frac{y}{\lambda}} = \sum_{n=0}^{\infty} {}_H\beta_n^{(k,r)}(x,y;\lambda) \frac{t^n}{n!}. \quad (5.1)$$

The equation (2.1) may be derived from (5.1) for $r = 2$.

This process can easily be extended to establish degenerate multi-variable Hermite poly-Bernoulli polynomials and Apostle type Bernoulli polynomials.

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