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HIGHER ORDER CORRECTED TRAPEZOIDAL RULES IN LEBESGUE AND ALEXIEWICZ SPACES

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Abstract. If $f:[a,b]\to\mathbb{R}$ such that $f^{(n)}$ is integrable then integration by parts gives the formula

$$\int_a^b f(x)\,dx = \frac{(-1)^n}{n!}\sum_{k=0}^{n-1}(-1)^{n-k-1}\left[\phi_n^{(n-k-1)}(a)f^{(k)}(a)-\phi_n^{(n-k-1)}(b)f^{(k)}(b)\right]+E_n(f),$$

where ϕ_n is a monic polynomial of degree n and the error is given by

$$E_n(f)=\frac{(-1)^n}{n!}\int_a^b f^{(n)}(x)\phi_n(x)\,dx.$$

This then gives a quadrature formula for $\int_a^b f(x)\,dx$. The polynomial ϕ_n is chosen to optimize the error estimate under the assumption that $f^{(n)}\in L^p([a,b])$ for some $1\leqslant p\leqslant\infty$ or if $f^{(n)}$ is integrable in the distributional or Henstock–Kurzweil sense. Sharp error estimates are obtained. It is shown that this formula is exact for all such ϕ_n if f is a polynomial of degree at most $n-1$. If ϕ_n is a Legendre polynomial then the formula is exact for f a polynomial of degree at most $2n-1$.

1. Introduction

This paper is based on the following observation. Suppose we wish to approximate the integral $\int_a^b f(x)\,dx$. If the nth derivative of function f is integrable then repeated integration by parts yields the formula,

$$\int_a^b f(x)\,dx = \frac{(-1)^n}{n!}\sum_{k=0}^{n-1}(-1)^{n-k-1}\left[\phi_n^{(n-k-1)}(a)f^{(k)}(a)-\phi_n^{(n-k-1)}(b)f^{(k)}(b)\right] +E_n(f), \tag{1.1}$$

where ϕ_n is a monic polynomial of degree n and $E_n(f)=\frac{(-1)^n}{n!}\int_a^b f^{(n)}(x)\phi_n(x)\,dx$. This then gives a quadrature formula for the integral of f with error term $E_n(f)$.

If $f^{(n)}\in L^p([a,b])$ then the Hölder inequality gives the error estimate $|E_n(f)|\leqslant \|f^{(n)}\|_p\|\phi_n\|_q/n!$, where p and q are conjugate exponents. (If $p,q\in(1,\infty)$ then $p^{-1}+q^{-1}=1$. If $p=1$ then $q=\infty$. If $p=\infty$ then $q=1$. Hence, we define $1/\infty=0$.) A natural question is then how to choose ϕ_n to minimize this error.

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For $n=1$ this problem is completely solvable. It is easy to see that the unique minimizing polynomial is $\phi_1(x)=x-c$, where c is the midpoint of $[a,b]$. See Corollary 2.2. The case $n=2$ was considered in [25]. It was shown there that the polynomial that minimizes $\|\phi_2\|_q$ is unique. An explicit minimizing polynomial was found for $p=1,2,4/3,\infty$. This gave sharp estimates on the error for these values of p, which improved on a number of error estimates that appear in the literature. For other values of $p\in[1,\infty]$ good estimates were found for the minimizing value of $\|\phi_2\|_q$. See also [13].

In the present paper we examine the case $n\geqslant 3$. The minimizing polynomial is shown to be unique and to have n simple zeros in $[a,b]$. When $p=1$ it is the Chebychev polynomial of the first kind, T_n. When $p=2$ it is the Legendre polynomial P_n. When $p=\infty$ it is the Chebychev polynomial of the second kind, U_n. Each of these is multiplied by a normalising factor so as to have leading coefficient one. These polynomials are orthogonal on $[-1,1]$. Each of our polynomials is composed with a linear function that maps the interval $[a,b]$ onto the interval $[-1,1]$.

We also consider the case when $\int_a^b f^{(n)}(x)\,dx$ exists as a Henstock–Kurzweil integral. This allows conditional convergence in the error term and includes the case of convergence as an improper Riemann integral or as a Cauchy–Lebesgue integral. A suitable norm is then the Alexiewicz norm, given as $\|g\|=\sup_{a\leqslant x\leqslant b}|\int_a^x g(t)\,dt|$. The polynomial that minimizes the error is again shown to be unique and to have n simple zeroes in $[a,b]$. It is given by $\phi_n(x)=2^{1-n}(T_n(x)-1)$, suitably modified by a linear transformation as above. The same formulas hold when $f^{(n-1)}$ is merely assumed to be continuous. Then $f^{(n)}$ exists as a distribution and the error integral exists as a continuous primitive integral. For a discussion of the Henstock–Kurzweil integral and Alexiewicz norm, see [14] or [23]. The continuous primitive integral is discussed in [24].

The final section of the paper discusses the degree of exactness. If f is a polynomial of degree at most $n-1$ then $E_n(f)=0$ for all $\phi_n\in\mathscr{P}_n$. If ϕ_n is a normalized Legendre polynomial of degree n, composed with a linear transformation as above, then $E_n(f)=0$ for all polynomials f of degree at most $2n-1$.

Several other authors have considered modified trapezoidal rules under conditions on $f^{(n)}$. Cerone and Dragomir [2] assume $f^{(n)}\in L^p$ and obtain formulas like (2.1) but with larger error coefficients than in this theorem or in Corollaries 2.3, 2.4, 2.5. Similarly with Dedić, Matić and Pečarić in [6]. Liu [16] assumes the condition $f^{(n-1)}\in C([a,b])\cap BV([a,b])$ and has a quadrature formula with degree of exactness equal to $n-1$. The problem is tackled using the Peano kernel by Dubeau [9] and Pečarić and Ujević [20]. Ding, Ye and Yang [8] estimate the remainder when f'' is Henstock–Kurzweil integrable.

2. $f^{(n)}\in L^p([a,b])$

Let $\mathscr{P}_m$ denote the monic polynomials of degree m. For $1\leqslant p<\infty$ let $L^p([a,b])$ be the Lebesgue measurable functions such that $\|g\|_p=(\int_a^b|g(x)|^p\,dx)^{1/p}<\infty$. Let $L^\infty([a,b])$ be the essentially bounded functions, with norm $\|g\|_\infty=\operatorname{ess\,sup}_{x\in[a,b]}|g(x)|$.

All measure-theoretic statements are with respect to Lebesgue measure.

THEOREM 2.1. *Let* $n \in \mathbb{N}$. *Let* $p,q \in [1,\infty]$ *be conjugate exponents. Let* $f : [a,b] \to \mathbb{R}$ *such that* $f^{(n-1)}$ *is absolutely continuous and* $f^{(n)} \in L^p([a,b])$. *Let* $\phi_n \in \mathscr{P}_n$. *Write*

$$\int_a^b f(x)\,dx = \frac{(-1)^n}{n!}\sum_{k=0}^{n-1}(-1)^{n-k-1}\left[\phi_n^{(n-k-1)}(a)f^{(k)}(a) - \phi_n^{(n-k-1)}(b)f^{(k)}(b)\right] + E_n(f), \tag{2.1}$$

where $E_n(f) = \frac{(-1)^n}{n!}\int_a^b f^{(n)}(x)\phi_n(x)\,dx$. *Then*

$$|E_n(f)| \leqslant \frac{\|f^{(n)}\|_p \|\phi_n\|_q}{n!} \leqslant K_{n,p}\|f^{(n)}\|_p (b-a)^{n+1/q} \tag{2.2}$$

for a constant $K_{n,p}$ *that depends on* n, p *and* ϕ_n *but is independent of* f *and* $b-a$. *There is a unique polynomial* $\tilde{\phi}_n \in \mathscr{P}_n$ *that minimizes* $K_{n,p}$. *The estimate on* $|E_n(f)|$ *is then sharp in the sense that the coefficient of* $\|f^{(n)}\|_p$ *cannot be reduced. The minimum value of* $K_{n,p}$ *is* $\tilde{K}_{n,p} = 2^{-n-1/q}\|\tilde{\phi}_n\|_q/n!$ *where the norm of* $\tilde{\phi}_n$ *is taken over* $[-1,1]$.

Proof. Integration by parts establishes (2.1). The Hölder inequality gives (2.2). Define $\tilde{\phi}_n \in \mathscr{P}_n$ by $\phi_n(x) = [(b-a)/2]^n \tilde{\phi}_n([2x-a-b]/[b-a])$. And write $\|\tilde{\phi}_n\|_q = (\int_{-1}^1 |\tilde{\phi}_n(x)|^q\,dx)^{1/q}$ for $1 \leqslant q < \infty$ and $\|\tilde{\phi}_n\|_\infty = \max_{|x|\leqslant 1}|\tilde{\phi}_n(x)|$. Then for $1 \leqslant q < \infty$,

$$\|\phi_n\|_q = \left(\frac{b-a}{2}\right)^n \left(\int_a^b \left|\tilde{\phi}_n\left(\frac{2x-a-b}{b-a}\right)\right|^q dx\right)^{1/q} = \left(\frac{b-a}{2}\right)^{n+1/q}\|\tilde{\phi}_n\|_q.$$

Similarly when $q = \infty$. This produces the factor $(b-a)^{n+1/q}$ in (2.2).

Existence of a unique minimizing polynomial for $\|\phi_n\|_q$ is proved in Lemma 3.1.

To show the coefficient of $\|f^{(n)}\|_p$ in (2.2) cannot be made any smaller, note that for $1 < p < \infty$ there is equality in the Hölder inequality when

$$f^{(n)}(x) = d\,\mathrm{sgn}[\phi_n(x)]|\phi_n(x)|^{1/(p-1)}$$

for some $d \in \mathbb{R}$ and almost all $x \in [a,b]$. See [15, p. 46]. Integrate to get

$$\begin{aligned} f(x) &= d\int_a^x \cdots \int_a^{x_{i+1}} \cdots \int_a^{x_2} \mathrm{sgn}[\phi_n(x_1)]|\phi_n(x_1)|^{1/(p-1)}dx_1 \cdots dx_i \cdots dx_n \\ &= \frac{d}{(n-1)!}\int_a^x (x-t)^{n-1}\mathrm{sgn}[\phi_n(t)]|\phi_n(t)|^{1/(p-1)}\,dt, \end{aligned}$$

modulo a polynomial of degree at most $n-1$. When $p = \infty$ the condition for equality in the Hölder inequality is that $f^{(n)}(x) = d\,\mathrm{sgn}[\phi_n(x)]$ for some $d \in \mathbb{R}$ and almost all $x \in [a,b]$. See [15, p. 46]. We can integrate as before to get

$$f(x) = \frac{d}{(n-1)!}\int_a^x (x-t)^{n-1}\mathrm{sgn}[\phi_n(t)]\,dt,$$

modulo a polynomial of degree at most $n-1$.

When $p=1$ the condition for equality in the Hölder inequality, $|\int_{-1}^{1} f^{(n)}\phi_n| = \|f^{(n)}\|_1\|\phi_n\|_\infty$, is that $\phi_n(x)=d\,\mathrm{sgn}[f^{(n)}(x)]$ for some $d\in\mathbb{R}$ and almost all $x\in[-1,1]$. See [15, p. 46]. (Because of the scaling argument above, it suffices to work on $[-1,1]$.) In general, this condition cannot be satisfied. Take $\alpha\in[-1,1]$ such that $\|\phi_n\|_\infty = |\phi_n(\alpha)|$. Let δ be the Dirac distribution. If $\alpha\in(-1,1)$ and $f^{(n)}(x)=\delta(x-\alpha)$ then $\int_{-1}^{1} f^{(n)}(x)\phi_n(x)\,dx=\phi_n(\alpha)$. Now, $\delta(x-\alpha)\notin L^1([-1,1])$ so use a δ-sequence. Let $\psi_m\colon[-1,1]\to[0,\infty)$ be continuous with support in $(\alpha-1/m,\alpha+1/m)$ such that $\int_{\alpha-1/m}^{\alpha+1/m}\psi_m(x)\,dx=1$. Let

$$
\begin{aligned}
f_m(x) &= \int_{-1}^{x}\cdots\int_{-1}^{x_{i+1}}\cdots\int_{-1}^{x_2}\psi_m(x_1)\,dx_1\cdots dx_i\cdots dx_n\\
&= \frac{1}{(n-1)!}\int_{-1}^{x}(x-t)^{n-1}\psi_m(t)\,dt.
\end{aligned}
$$

Then $f_m\in L^1([-1,1])$. Note that

$$
\|f_m^{(n)}\|_1 = \int_{-1}^{1}\left|f_m^{(n)}(x)\,dx\right| = \int_{-1}^{1}|\psi_m(x)|\,dx = 1.
$$

And, since ϕ_n is continuous, we get

$$
\lim_{m\to\infty}\left|\int_{-1}^{1} f_m^{(n)}(x)\phi_n(x)\,dx\right| = |\phi_n(\alpha)| = \|\phi_n\|_\infty,
$$

thus showing that the coefficient of $\|f^{(n)}\|_1$ in (2.2) cannot be reduced. If $|\alpha|=1$ then for each $\varepsilon>0$ there is $\beta\in(-1,1)$ such that $\|\phi_n\|_\infty<\varepsilon+|\phi_n(\beta)|$. □

Now we look at some special cases that can be solved completely.

When $n=1$ we get the usual trapezoidal rule. See [4].

COROLLARY 2.2. *If $n=1$ the approximation becomes $\int_a^b f(x)\,dx \doteq \frac{b-a}{2}[f(a)+f(b)]$ with sharp error estimate*

$$
|E_1(f)| \leqslant \begin{cases} \frac{1}{2}\|f'\|_1(b-a), & p=1\\ \frac{1}{2}\left(\frac{1}{q+1}\right)^{1/q}\|f'\|_p(b-a)^{1+1/q}, & 1<p<\infty\\ \frac{1}{4}\|f'\|_\infty(b-a)^2, & p=\infty. \end{cases}
$$

Proof. The minimizing polynomial is $\phi_1(x)=x-c$, where c is the midpoint of $[a,b]$. □

The case $n=2$ is discussed in detail in [25], where ϕ_2 and the exact values of $\tilde{K}_{2,p}$ are found for $p=1,2,4/3,\infty$.

As mentioned in the Introduction, the unique minimizing polynomial for $\|\phi_n\|_p$ is known when $p=1,2,\infty$. For these cases we can get an explicit form of the approximation to the integral that minimizes the error and compute the exact value of $\tilde{K}_{n,p}$

from Theorem 2.1. Since $[a,b]$ is a compact interval we have $L^\infty([a,b]) \subseteq L^s([a,b]) \subseteq L^r([a,b]) \subseteq L^1([a,b])$ if $1 \leqslant r \leqslant s \leqslant \infty$. The estimate for $p=1$ then applies when $f \in L^r([a,b])$ for $1 \leqslant r \leqslant \infty$. The estimate for $p=2$ applies when $f \in L^r([a,b])$ for $2 \leqslant r \leqslant \infty$.

COROLLARY 2.3. *If* $f^{(n)} \in L^r([a,b])$ *for some* $1 \leqslant r \leqslant \infty$ *then*

$$\int_a^b f(x)\,dx \doteq \frac{1}{(n-1)!}\sum_{k=0}^{n-1}\left[\frac{(b-a)^{k+1}(2n-k-2)!(n-k-1)!}{2^{2k+1}(2n-2k-2)!(k+1)!}\right]\left[f^{(k)}(a)+(-1)^k f^{(k)}(b)\right] \tag{2.3}$$

with sharp error estimate

$$|E_n(f)| \leqslant \frac{\|f^{(n)}\|_1 (b-a)^n}{2^{2n-1} n!}. \tag{2.4}$$

Proof. The unique polynomial minimizing $\|\cdot\|_\infty$ over $\mathscr{P}_n$ on $[-1,1]$ is $\tilde{\phi}_n(x) = 2^{1-n}T_n(x)$, where T_n is the Chebyshev polynomial of first type. See [3, p. 63], [18, p. 39] or [21, p. 31, 45]. Since $T_n(\cos\theta) = \cos(n\theta)$ we have $\|\tilde{\phi}_n\|_\infty = 2^{1-n}$. The minimizing polynomial on $[a,b]$ is then $\phi_n(x) = (b-a)^n 2^{1-2n} T_n([2x-a-b]/[b-a])$. From the proof of the Theorem, the sharp error estimate is then

$$|E_n(f)| \leqslant \frac{\|f^{(n)}\|_1 \|2^{1-n}T_n\|_\infty}{n!}\left(\frac{b-a}{2}\right)^n = \frac{\|f^{(n)}\|_1 (b-a)^n}{2^{2n-1} n!}.$$

To compute the expansion in (2.1) we need the derivatives of ϕ_n. We have, $\phi_n^{(m)}(x) = 2^{1+m-2n}(b-a)^{n-m}T_n^{(m)}([2x-a-b]/[b-a])$. Derivatives of T_n can be computed in terms of Gegenbauer polynomials C_n^λ. See [12, 8.949.2, 8.937.4]. Then

$$\begin{aligned}\phi_n^{(m)}(b) &= 2^{1+m-2n}(b-a)^{n-m} n 2^{m-1}(m-1)! C_{n-m}^m(1) \\ &= \frac{(b-a)^{n-m} n(n+m-1)! m!}{2^{2n-2m-1}(n-m)!(2m)!}.\end{aligned}$$

Since $T_n(-x) = (-1)^n T_n(x)$ get $\phi_n^{(m)}(a) = (-1)^{n+m}\phi_n^{(m)}(b)$. This gives (2.3). □

COROLLARY 2.4. *If* $f^{(n)} \in L^r([a,b])$ *for some* $2 \leqslant r \leqslant \infty$ *then*

$$\int_a^b f(x)\,dx \doteq \frac{n!}{(2n)!}\sum_{k=0}^{n-1}\left[\frac{(b-a)^{k+1}(2n-k-1)!}{(n-k-1)!(k+1)!}\right]\left[f^{(k)}(a)+(-1)^k f^{(k)}(b)\right] \tag{2.5}$$

with sharp error estimate

$$|E_n(f)| \leqslant \frac{\|f^{(n)}\|_2 n!(b-a)^{n+1/2}}{(2n+1)^{1/2}(2n!)}. \tag{2.6}$$

Proof. The unique polynomial minimizing $\|\cdot\|_2$ over $\mathscr{P}_n$ on $[-1,1]$ is $\tilde{\phi}_n(x) = 2^n(n!)^2P_n(x)/(2n)!$, where P_n is the Legendre polynomial. See [3, p. 109], [19, p. 48] or [21, p. 62]. As in the proof of Corollary 2.3, the error estimate follows from the integral $\int_{-1}^{1} P_n^2(x)\,dx = 2/(2n+1)$ [1, 22.2.10].

Derivatives of P_n can be computed from the hypergeometric representation $P_n(x) = {}_2F_1(-n,n+1;1;(1-x)/2)$ [1, 22.5.49, 15.2.2]. We get

$$P_n^{(m)}(x) = \frac{(-n)_m(n+1)_m(-1)^m}{(1)_m 2^m}\,{}_2F_1(m-n,m+n+1;m+1;(1-x)/2),$$

where $(a)_m$ is the Pochhammer symbol. Since ${}_2F_1(a,b;c;0) = 1$ we get

$$P_n^{(m)}(1) = \frac{(n+m)!}{(n-m)!m!2^m}.$$

This gives

$$\phi_n^{(m)}(b) = \frac{(b-a)^{n-m}(n!)^2(n+m)!}{(2n)!(n-m)!m!}$$

with $\phi_n^{(m)}(a) = (-1)^{n+m}\phi_n^{(m)}(b)$. And, (2.5) follows. □

COROLLARY 2.5. *If $f^{(n)} \in L^\infty([a,b])$ then*

$$\int_a^b f(x)\,dx \doteq \frac{1}{n!}\sum_{k=0}^{n-1}\left[\frac{(b-a)^{k+1}(2n-k)!(n-k-1)!}{2^{2k+2}(2n-2k-1)!(k+1)!}\right]\left[f^{(k)}(a)+(-1)^k f^{(k)}(b)\right] \quad (2.7)$$

with sharp error estimate

$$|E_n(f)| \leqslant \frac{\|f^{(n)}\|_\infty (b-a)^{n+1}}{2^{2n}n!}. \quad (2.8)$$

Proof. The unique polynomial minimizing $\|\cdot\|_1$ over $\mathscr{P}_n$ on $[-1,1]$ is $\tilde{\phi}_n(x) = 2^{-n}U_n(x)$, where U_n is the Chebyshev polynomial of second type. See [3, p. 222], [11, p. 26] or [21, p. 72, 83]. From the formula $U_n(\cos\theta) = \sin([n+1]\theta)/\sin\theta$ we can directly compute

$$\|U_n\|_1 = \frac{1}{n+1}\int_0^{(n+1)\pi}|\sin\theta|\,d\theta = 2.$$

Then $\|\tilde{\phi}_n\|_1 = 2^{1-n}$ and the error estimate (2.8) follows as in the previous corollaries.

Derivatives of U_n can be computed in terms of Gegenbauer polynomials [12, 8.949.5, 8.937.4]. We get

$$U_n^{(m)}(1) = 2^m m!\binom{n+m+1}{n-m}.$$

And,

$$\phi_n^{(m)}(b) = \frac{(b-a)^{n-m}m!(n+m+1)!}{2^{2n-2m}(2m+1)!(n-m)!}$$

with $\phi_n^{(m)}(a) = (-1)^{n+m}\phi_n^{(m)}(b)$. Expansion (2.7) now follows. □

The minimizing polynomial for $\|\cdot\|_p$ is even or odd about the midpoint of $[a,b]$ as n is even or odd. See Lemma 3.1. We use this simplification when computing the composite rule for uniform partitions. The polynomial $\tilde{\phi}_n$ can be replaced with any of the minimizing polynomials from Corollaries 2.3, 2.4 or 2.5.

COROLLARY 2.6. *Let $\tilde{\phi}_n \in \mathscr{P}_n$ such that $\tilde{\phi}_n$ is even or odd as n is even or odd. The composite rule for a uniform partition, $x_i = a + (b-a)i/m$, $0 \leqslant i \leqslant m$, is*

$$\int_a^b f(x)\,dx$$

$$= \frac{1}{n!}\sum_{\ell=0}^{\lfloor (n-1)/2\rfloor}\left(\frac{b-a}{2m}\right)^{2\ell+1}\tilde{\phi}_n^{(n-2\ell-1)}(1)\left[f^{(2\ell)}(a) + f^{(2\ell)}(b) + 2\sum_{i=1}^{m-1} f^{(2\ell)}(x_i)\right] \tag{2.9}$$

$$+\frac{1}{n!}\sum_{\ell=1}^{\lfloor n/2\rfloor}\left(\frac{b-a}{2m}\right)^{2\ell}\tilde{\phi}_n^{(n-2\ell)}(1)\left[f^{(2\ell-1)}(a) - f^{(2\ell-1)}(b)\right] + E_n^m(f), \tag{2.10}$$

where

$$|E_n^m(f)| \leqslant K_{n,p}\|f^{(n)}\|_p (b-a)^{n+1/q} m^{-n}. \tag{2.11}$$

Proof. Write $\int_a^b f(x)\,dx \doteq \sum_{i=1}^m \int_{x_{i-1}}^{x_i} f(x)\,dx$ and use the approximation in the Theorem on each interval $[x_{i-1}, x_i]$. Scaling to $[x_{i-1}, x_i]$ gives

$$\phi_{n,i}(x) = \left(\frac{b-a}{2m}\right)^n \tilde{\phi}_n\left(\frac{2mx - 2ma - 2(b-a)i + b - a}{b-a}\right).$$

Upon changing summation order, this gives

$$\int_a^b f(x)\,dx \doteq \frac{1}{n!}\sum_{k=0}^{n-1}(-1)^{k+1}\left(\frac{b-a}{2m}\right)^{k+1}\sum_{i=1}^{m}\left[\tilde{\phi}_n^{(n-k-1)}(-1)f^{(k)}(x_{i-1}) - \tilde{\phi}_n^{(n-k-1)}(1)f^{(k)}(x_i)\right].$$

We have that $\tilde{\phi}_n^{(n-k-1)}$ is even whenever k is odd, and is odd whenever k is even. Hence, the sum on i telescopes when k is odd.

The error is written

$$|E_n^m(f)| = \frac{1}{n!}\left|\int_a^b f^{(n)}(x)\psi_n(x)\,dx\right| \leqslant \frac{\|f^{(n)}\|_p\|\psi_n\|_q}{n!},$$

where $\psi_n(x) = \phi_{n,i}(x)\chi_{(x_{i-1},x_i)}(x)$. And, with $\Delta x = (b-a)/(2m)$,

$$\|\psi_n\|_q = \left(\sum_{i=1}^m \int_{x_{i-1}}^{x_i} |\phi_{n,i}(x)|^q\,dx\right)^{1/q}$$

$$= \left(\frac{b-a}{2m}\right)^n \left(\sum_{i=1}^m \int_{x_{i-1}}^{x_i}\left|\tilde{\phi}_n\left(\frac{2x - x_{i-1} - x_i}{\Delta x}\right)\right|^q dx\right)^{1/q}$$

$$= \left(\frac{b-a}{2m}\right)^{n+1/q}\left(\sum_{i=1}^m \int_{-1}^1 |\tilde{\phi}_n(x)|^q\,dx\right)^{1/q} = \left(\frac{b-a}{2}\right)^{n+1/q}\|\tilde{\phi}_n\|_q m^{-n}.$$

The constant is proved sharp as in the Theorem. In the case $p=1$ the same method works since at each point where ψ_n attains its extrema, it is either continuous from the left or from the right and we can take a δ-sequence supported on the left or right. $\square$

When $n=1$, the sum in (2.10) is absent and we get the usual composite trapezoidal rule

$$\int_a^b f(x)\,dx \doteq \frac{b-a}{2m}\left[f(a)+f(b)+2\sum_{i=1}^{m-1} f(x_i)\right].$$

When $n=2$, the sum in (2.9) contains only the $\ell=0$ term. Derivatives of f then appear as $f'(a)$ and $f'(b)$ but not at interior nodes, cf. [25].

If $f^{(n)}\in L^p([a,b])$ for a value of p different from 1, 2, or ∞ we can use the norm $\|f^{(n)}\|_p$ in (2.2) and estimate the value of $\|\phi_n\|_q$, even though the minimizing polynomial is not known. This gives an estimate of $\tilde{K}_{n,p}$. Together with the exact values of $\tilde{K}_{n,p}$ from Corollaries 2.3, 2.4, 2.5, these give the asymptotic behaviour of $\tilde{K}_{n,p}$ as $n\to\infty$, uniformly valid for all $p\in[1,\infty]$.

PROPOSITION 2.7. *Let $\tilde{K}_{n,p}$ be the constant from Theorem 2.1. Then $\tilde{K}_{n,1} = 2^{1-2n}/n!$, $\tilde{K}_{n,2} = n!(2n+1)^{-1/2}/(2n)!$, $\tilde{K}_{n,\infty} = 2^{-2n}/n!$. For each fixed n, $K_{n,p}$ is a decreasing function of p. Hence, $K_{n,p}\leqslant K_{n,1}$.*

Proof. The values of $\tilde{K}_{n,p}$ for $p=1,2,\infty$ are given in Corollaries 2.3, 2.4, 2.5, respectively.

If $q<r<\infty$ then Jensen's inequality gives

$$\|\tilde{\phi}_n\|_q^r = \left(2\int_{-1}^1 |\tilde{\phi}_n(x)|^q\,\frac{dx}{2}\right)^{r/q} \leqslant 2^{r/q}\int_{-1}^1 |\tilde{\phi}_n(x)|^r\,\frac{dx}{2} = 2^{r/q-1}\|\tilde{\phi}_n\|_r^r.$$

And, $\|\tilde{\phi}_n\|_q \leqslant 2^{1/q-1/r}\|\tilde{\phi}_n\|_r$.

Let $1\leqslant p_1<p_2\leqslant\infty$ with corresponding conjugate exponents q_1,q_2. Let $\tilde{\phi}_n$ be the minimizing polynomial for $\|\cdot\|_{q_1}$. Then

$$K_{n,p_2}\leqslant \frac{\|\tilde{\phi}_n\|_{q_2}}{n!2^{n+1/q_2}} \leqslant \frac{2^{1/q_2-1/q_1}\|\tilde{\phi}_n\|_{q_1}}{n!2^{n+1/q_2}} = K_{n,p_1}.$$

Hence, $K_{n,p}$ is decreasing. $\square$

3. Lemma on minimizing polynomials

For each of the L^p norms there is a unique monic polynomial that minimizes the norm. Define $F_q\colon\mathscr{P}_m\to\mathbb{R}$ by $F_q(\phi)=\|\phi\|_q$ where $1\leqslant q\leqslant\infty$ and the norms are over compact interval $[a,b]$. Since $F_q(\phi)$ is bounded below for $\phi\in\mathscr{P}_m$ it has an infimum over $\mathscr{P}_m$. It also has a unique minimum at a polynomial that has m roots in $[a,b]$. As well, the error-minimizing polynomial is even or odd about the midpoint of $[a,b]$ as m is even or odd.

LEMMA 3.1. *(a) For $m \geqslant 2$, let $\phi \in \mathscr{P}_m$ with a non-real root. There exists $\psi \in \mathscr{P}_m$ with a real root such that $F_q(\psi) < F_q(\phi)$.*

(b) Let $\phi \in \mathscr{P}_m$ with a root $t \notin [a,b]$. There exists $\psi \in \mathscr{P}_m$ with a root in $[a,b]$ such that $F_q(\psi) < F_q(\phi)$.

(c) If ϕ minimizes F_q then it has m simple zeros in $[a,b]$.

(d) If F_q has a minimum in $\mathscr{P}_m$ it is unique.

(e) F_q attains its minimum over $\mathscr{P}_m$.

(f) If $\phi \in \mathscr{P}_m$ is neither even nor odd about $c := (a+b)/2$ then there is a polynomial $\psi \in \mathscr{P}_m$ that is either even or odd about c such that $F_q(\psi) < F_q(\phi)$.

(g) The minimum of F_q occurs at a polynomial $\phi \in \mathscr{P}_m$ with m simple zeros in $[a,b]$. If m is even about c then so is ϕ. If m is odd about c then so is ϕ. This minimizing polynomial is unique.

(h) Suppose $\phi \in \mathscr{P}_m$ is a minimum of F_∞. Then $\phi(x) = \prod_{i=1}^m (x - t_i)$ for $a < t_1 < t_2 < \cdots < t_m < b$. For each $1 \leqslant i \leqslant m-1$ there is $\xi_i \in (t_i, t_{i+1})$ such that $|\phi(\xi_i)| = \|\phi\|_\infty$.

This result is proved in [25]. See also [3], [5], [7], [11], [17], [18], [19], [21], [22], [26] for background on this problem and references to original papers by Bernstein, Chebyshev, Jackson, etc.

The cases $q = 1, 2, \infty$ are used in Corollaries 2.5, 2.4, 2.3, respectively. Here, the minimizing polynomials are orthogonal polynomials. No explicit solutions appear to be known for any other values of q. Gillis and Lewis [10] give a heuristic argument to show that for no other values of q are the minimizing polnomials a family of orthogonal polynomials.

4. Alexiewicz norm

The Alexiewicz norm is useful for functions or distributions for which $\int_a^b f(x)\,dx$ exists but $\int_a^b |f(x)|\,dx$ diverges. It is defined as $\|f\| = \sup_{a \leqslant x \leqslant b} |\int_a^x f(t)\,dt|$. If $F \in C([a,b])$ with $F(a) = 0$ then define $\mathscr{A}_c$ to be the Schwartz distributions, f, for which $F' = f$. The derivative is understood in the distributional sense, $\langle F', \phi \rangle = -\langle F, \phi' \rangle = -\int_a^b F(x)\phi'(x)\,dx$ where $\phi \in C_c^\infty((a,b))$ (smooth functions with compact support in (a,b)). Then $\mathscr{A}_c$ is a Banach space isometrically isomorphic to the continuous functions on $[a,b]$ that vanish at a, and $\|f\| = \|F\|_\infty$ where F is the unique primitive of f. This integration process is often called the *continuous primitive integral* and $\int_a^x f(x)\,dx = F(x)$ for all $x \in [a,b]$. See [24] for details. Since this integral uses the space of all continuous functions as primitives, it includes the Lebesgue integral (whose primitives are absolutely continuous) and the Henstock–Kurzweil integral (whose primitives are continuous but need not be absolutely continuous and are described in [14]).

An example of a function integrable in the Henstock–Kurzweil sense but not in $L^1([-1,1])$ is given by $f(x) = F'(x)$ where

$$F(x) = \begin{cases} x^2 \sin(x^{-3}), & x \neq 0 \\ 0, & x = 0. \end{cases}$$

The pointwise derivative $F'(x)$ exists at each point. If we take F to be a continuous monotonic function whose derivative is zero almost everywhere then the Lebesgue integral $\int_a^b F'(x)\,dx = 0$ while the continuous primitive integral gives $\int_a^b F' = F(b) - F(a)$. If f is a continuous function differentiable nowhere in the pointwise sense then the distributional derivative $F' \in \mathscr{A}_c$ and $\int_a^x F' = F(x) - F(a)$ for all $x \in [a,b]$, even though F' is a distribution that does not have any pointwise values and the Lebesgue integral of F' is meaningless. Additional examples are given in [24].

The integration by parts formula for $f \in \mathscr{A}_c$ with primitive F and function g of bounded variation is given in terms of a Riemann–Stieltjes integral

$$\int_a^b f(x)g(x)\,dx = F(b)g(b) - F(a)g(a) - \int_a^b F(x)dg(x). \tag{4.1}$$

The Hölder inequality is then

$$\left|\int_a^b f(x)g(x)\,dx\right| \leqslant \|f\|(|g(b)| + Vg). \tag{4.2}$$

See [14, Theorem 12.3].

THEOREM 4.1. *Let $n \in \mathbb{N}$. Let $f : [a,b] \to \mathbb{R}$ such that the pointwise derivative $f^{(n-1)}$ is continuous. Then the distributional derivative $f^{(n)} \in \mathscr{A}_c$. Write*

$$\int_a^b f(x)\,dx = \frac{1}{(n-1)!}\sum_{k=0}^{n-2}\left[\frac{(b-a)^{k+1}(2n-k-2)!(n-k-1)!}{2^{2k+1}(2n-2k-2)!(k+1)!}\right]\left[f^{(k)}(a) + (-1)^k f^{(k)}(b)\right]$$
$$+\begin{cases}\frac{(b-a)^n f^{(n-1)}(a)}{n!2^{2n-2}}, & n \text{ odd}\\ 0, & n \text{ even}\end{cases} + E_n(f). \tag{4.3}$$

Then

$$|E_n(f)| \leqslant \frac{\|f^{(n)}\|(b-a)^n}{(n-1)!2^{2n-2}}. \tag{4.4}$$

The estimate on $|E_n(f)|$ is then sharp in the sense that the coefficient of $\|f^{(n)}\|$ cannot be reduced.

Proof. Let $\phi_n \in \mathscr{P}_n$. From (2.1) and the Hölder inequality (4.2) we have $|E_n(f)| \leqslant \|f^{(n)}\|(|\phi_n(b)| + V\phi_n)$. Note that $V\phi_n = \int_a^b |\phi_n'(x)|\,dx = \|\phi_n'\|_1 = n\|\psi_{n-1}\|_1$, where $\psi_{n-1} = \phi_n'/n$. Hence, to minimize the variation over $\phi_n \in \mathscr{P}_n$ we minimize the one-norm over $\psi_{n-1} \in \mathscr{P}_{n-1}$. Adding a constant to ϕ_n does not affect the variation so the unique minimizing polynomial on $[-1,1]$ is (cf. proof of Corollary 2.5)

$$\begin{aligned}\tilde{\phi}_n(x) &= -n2^{1-n}\int_x^1 U_{n-1}(t)\,dt\\ &= -2^{1-n}\int_x^1 T_n'(t)\,dt = 2^{1-n}[T_n(x) - 1].\end{aligned} \tag{4.5}$$

The minimizing polynomial on $[a,b]$ is

$$\phi_n(x) = 2^{1-2n}(b-a)^n\left[T_n\left(\frac{2x-a-b}{b-a}\right)-1\right].$$

We have $\phi_n(b)=0$ and

$$\begin{aligned} V\phi_n &= 2^{2-2n}(b-a)^{n-1}\int_a^b \left|T_n'\left(\frac{2x-a-b}{b-a}\right)\right| dx \\ &= 2^{1-2n}(b-a)^n\|T_n'\|_1 = 2^{1-2n}(b-a)^n n\|U_{n-1}\|_1 = 2^{2-2n}(b-a)^n n. \end{aligned}$$

The Hölder inequality (4.2) now gives the estimate in (4.4).

Formula (4.3) is computed as in the proof of Corollary 2.3.

Now show there can be equality in (4.4). Using (4.5), integrate by parts (4.1) to get

$$\begin{aligned} E_n(f) &= \frac{(-1)^n}{n!}\int_{-1}^1 f^{(n)}(x)\tilde{\phi}_n(x)\,dx \\ &= \begin{cases} -\frac{2^{1-n}}{(n-1)!}\int_{-1}^1 f^{(n-1)}(x)U_{n-1}(x)\,dx, & n \text{ even} \\ -\frac{2^{2-n}f^{(n-1)}(-1)}{n!} + \frac{2^{1-n}}{(n-1)!}\int_{-1}^1 f^{(n-1)}(x)U_{n-1}(x)\,dx, & n \text{ odd.} \end{cases} \end{aligned}$$

By the usual Hölder inequality,

$$\begin{aligned} \frac{2^{1-n}}{(n-1)!}\left|\int_{-1}^1 f^{(n-1)}(x)U_{n-1}(x)\,dx\right| &\leqslant \frac{2^{1-n}}{(n-1)!}\|f^{(n-1)}\|_\infty\|U_{n-1}\|_1 \\ &= \frac{2^{2-n}}{(n-1)!}\|f^{(n-1)}\|_\infty. \end{aligned}$$

As in the proof of Theorem 2.1 there is equality when $f^{(n-1)}(x) = d\,\mathrm{sgn}[U_{n-1}(x)]$ for some $d\in\mathbb{R}$. Since the final term in (4.1) can then have any sign, we integrate to get

$$f(x) = \frac{d}{(n-2)!}\int_{-1}^x (x-t)^{n-2}\mathrm{sgn}[U_{n-1}(t)]\,dt$$

for $n\geqslant 2$, modulo a polynomial of degree at most $n-2$ that vanishes at -1. For $n=1$, $f(x)=d(x+1)$. Hence, the coefficient in (4.4) cannot be reduced. □

When $n=1$ the minimizing polynomial is $\phi_1(x)=x-b$. The approximation becomes $\int_a^b f(x)\,dx \doteq f(a)(b-a)$ with error $|E_1(f)|\leqslant\|f'\|(b-a)$.

When $n=2$ the minimizing polynomial is $\phi_2(x)=(x-a)(x-b)$. The approximation reduces to the usual trapezoidal rule, $\int_a^b f(x)\,dx \doteq (b-a)[f(a)+f(b)]/2$ with error $|E_2(f)|\leqslant\|f''\|(b-a)^2/4$. This appears as Theorem 5.1 in [25]. An alternate form of the Alexiewicz norm is also considered in this paper.

If $f^{(n)}\in\mathscr{A}_c$ then $f^{(n-1)}\in L^\infty([a,b])$ and the results of Theorem 4.1 agree with Corollary 2.5 with n reduced by one.

COROLLARY 4.2. *The composite rule for a uniform partition, $x_i = a+(b-a)i/m$, $0 \leqslant i \leqslant m$, is*

$$\begin{aligned}\int_a^b f(x)\,dx \doteq{}& \frac{1}{(n-1)!}\sum_{\ell=0}^{\lfloor (n-2)/2\rfloor}\left(\frac{b-a}{m}\right)^{2\ell+1}\frac{(2n-2\ell-2)!(n-2\ell-1)!}{2^{4\ell+1}(2n-4\ell-2)!(2\ell+1)!}\\ &\times\left[f^{(2\ell)}(a)+f^{(2\ell)}(b)+2\sum_{i=1}^{m-1}f^{(2\ell)}(x_i)\right]\\ &+\frac{1}{(n-1)!}\sum_{\ell=1}^{\lfloor (n-1)/2\rfloor}\left(\frac{b-a}{m}\right)^{2\ell}\frac{(2n-2\ell-1)!(n-2\ell)!}{2^{4\ell-1}(2n-4\ell)!(2\ell)!}\left[f^{(2\ell-1)}(a)-f^{(2\ell-1)}(b)\right]\\ &+\begin{cases}\frac{(b-a)^n}{n!2^{2n-2}m^n}\sum_{i=0}^{m-1}f^{(n-1)}(x_i), & n \text{ odd}\\ 0, & n \text{ even.}\end{cases}\end{aligned}$$

A sharp estimate for the error is

$$|E_n^m(f)| \leqslant \frac{\|f^{(n)}\|(b-a)^n}{m^{n-1}(n-1)!2^{2n-2}}. \tag{4.6}$$

Proof. Using the minimizing polynomial (4.5), the proof of the approximation formula is similar to the proof of Corollary 2.6. With the notation of that corollary,

$$\begin{aligned}V\psi_n &= \sum_{i=1}^m\int_{x_{i-1}}^{x_i}|\phi_{n,i}'(x)|\,dx = \left(\frac{b-a}{2m}\right)^{n-1}\sum_{i=1}^m\int_{x_{i-1}}^{x_i}\left|\tilde{\phi}_n'\left(\frac{2x-x_{i-1}-x_i}{\Delta x}\right)\right|dx\\ &= \left(\frac{b-a}{2m}\right)^n\sum_{i=1}^m\int_{-1}^1|\tilde{\phi}_n'(x)|\,dx = \left(\frac{b-a}{2m}\right)^n mV(2^{1-n}T_n)\\ &= \left(\frac{b-a}{2m}\right)^n m2^{1-n}n\|U_{n-1}\|_1 = \frac{(b-a)^n n}{m^{n-1}2^{2n-2}}.\end{aligned}$$

Formula (4.6) now follows from the Hölder inequality (4.2). □

5. Degree of exactness

We now show that formula (2.1) is exact for all $\phi_n \in \mathscr{P}_n$ when f is a polynomial of degree at most $n-1$ and this formula is exact for f being a polynomial of degree at most $2n-1$ when ϕ_n is the composition of a linear function and a Legendre polynomial.

THEOREM 5.1. *Let $f \in C^{n-1}([a,b])$. Write*

$$\begin{aligned}\int_a^b f(x)\,dx = \frac{(-1)^n}{n!}\sum_{k=0}^{n-1}(-1)^{n-k-1}\left[\phi_n^{(n-k-1)}(a)f^{(k)}(a)-\phi_n^{(n-k-1)}(b)f^{(k)}(b)\right]&\\ +E_n(f),&\end{aligned} \tag{5.1}$$

where $E_n(f) = \frac{(-1)^n}{n!}\int_a^b f^{(n)}(x)\phi_n(x)\,dx$ *and* $\phi_n \in \mathscr{P}_n$. *(a) If* f *is a polynomial of degree at most* $n-1$ *then* (5.1) *is exact (* $E_n(f)=0$ *) for all* $\phi_n \in \mathscr{P}_n$. *(b) If* f *is a polynomial of degree at most* $2n-1$ *then* (5.1) *is exact (* $E_n(f)=0$ *) if and only if*

$$\phi_n(x) = \frac{(b-a)^n(n!)^2}{(2n!)}P_n\left(\frac{2x-a-b}{b-a}\right),$$

where P_n *is a Legendre polynomial.*

Proof. (a) Integrate by parts. (b) First consider the interval $[-1,1]$. By (a) and linearity we need only consider $f(x)=\sum_{i=n}^{2n-1}A_ix^i$ for real numbers A_i. Let $\phi \in \mathscr{P}_n$. We then require

$$(-1)^n n! E_n(f) = \sum_{i=n}^{2n-1} A_i i(i-1)\cdots(i-n+1)\int_{-1}^{1} x^{i-n}\phi_n(x)\,dx = 0.$$

But the terms $A_i i(i-1)\cdots(i-n+1)$ are linearly independent so we require the moments $\int_{-1}^1 x^j\phi_n(x)\,dx$ to vanish for $0 \leqslant j \leqslant n-1$. The Legendre polynomial P_n satisfies this integral condition [12, 7.222.1]. Legendre polynomials are orthogonal with respect to the inner product $\langle g,h\rangle = \int_{-1}^1 f(x)g(x)\,dx$. If there was another polynomial $\psi_n \in \mathscr{P}_n$ that also satisfied the moment condition then it could be expanded as $\psi_n(x)=\sum_{k=0}^n B_kP_k(x)$ where $B_k = \langle \psi_n, P_k\rangle / \langle P_k, P_k\rangle$. The moment condition gives

$$0 = \int_{-1}^1 x^j\psi_n(x)\,dx = \sum_{k=0}^{j} B_k \int_{-1}^1 x^jP_k(x)\,dx$$

for each $0 \leqslant j \leqslant n-1$. Putting $j=0$ gives $0 = B_0\int_{-1}^1 P_0(x)\,dx$. But $\int_{-1}^1 x^mP_m(x)\,dx = (m!)^2 2^{m+1}/(2m+1)! \neq 0$ [12, 7.224.3]. So, $B_0 \neq 0$. Successively putting $j=1,2,3,\ldots$ now shows $B_k=0$ for each $0 \leqslant k \leqslant n-1$. Hence, ψ_n is a multiple of P_n. A linear transformation gives the required polynomial on $[a,b]$. □

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