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UNIQUENESS SETS FOR FOURIER SERIES

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Abstract. This article explores some of the sets of uniqueness for Fourier series. The case, when these sets have zero Lebesgue measure is considered.

1. Introduction

In this article the following problem is discussed: to find conditions on a set $E \subseteq [-\pi,\pi]$, such that Fourier series of a function $f(x)$, $-\pi < x < \pi$, from the given classis, which converges to zero at each point of the set E, is identically zero.

The first nontrivial result of this type was proved by G. Cantor and W. Young, see [1], p. 191.

THEOREM 1. *Let* $c_k \to 0$ *and for each point* $x \in [-\pi,\pi] \setminus F$ *we have*

$$\lim_{n\to\infty} \sum_{k=-n}^{n} c_k e^{ikx} = 0,$$

where F *is a countable set. Then* $c_k = 0$ *for each* $k \in \mathbb{Z}$.

D. Menshov, see [1], p. 806, has constructed a nonzero measure $d\mu$ which has the zero Lebesgue measure support and such that its Fourier coefficients go to zero. The partial sums, of its Fourier series converge to zero outside of the set $\operatorname{supp}(d\mu)$.

2. Auxiliary definitions and results

More information, about the following quantities, related with Hausdorff's measure and the capacities, one can find in [3], pp. 13–46. For convenience of the reader here we give some definitions.

DEFINITION 1. Let $h(x) \geqslant 0$, $0 \leqslant x \leqslant 1$ be a non-negative, increasing function and $h(0) = 0$. Let the subset $E \subseteq \{z; |z| = 1\}$ be covered by arcs $\{S_k\}_{k=1}^{\infty}$, i.e.

$$E \subseteq \bigcup_{k=1}^{\infty} S_k.$$

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Then we put

$$M_h(E) = \inf\left(\sum_{k=1}^{\infty} h(|S_k|)\right),$$

where $|S|$ is the length of the arc S and the infimum is taken over all covers.

DEFINITION 2. Let $0 < \alpha < 1$ and E be bounded Borel set. The $C_\alpha(E)$ – capacity of the set E is defined by the formula

$$C_\alpha(E) = \left(\inf_{d\mu \prec E} \int_E \int_E \frac{d\mu(x)d\mu(y)}{|x-y|^\alpha}\right)^{-1},$$

where $d\mu \prec E$ means that $d\mu$ is a probability measure with support in E.

For each number $0 < \alpha < 1$ from Parseval's equality it follows that there is a constant M such that,

$$\sum_{k=-\infty}^{\infty} |\hat{f}_k|^2 |k|^\alpha \leqslant M \int_{-\pi}^{\pi} |f(x)|^2 dx + M \int_{-\pi}^{\pi} \int_{\pi}^{\pi} \frac{|f(x)-f(y)|^2}{|x-y|^{1+\alpha}} dxdy.$$

The following A. Zigmund's statement one can find in [6], p. 22. Let $g(-\pi) = g(\pi)$ and the function $g(x)$, $-\pi \leqslant x \leqslant \pi$ have bounded variation. If

$$|g(x) - g(y)| \leqslant M \cdot h(|x-y|)$$

then there is a constant B such that the Fourier coefficients of the function $g(x)$ satisfy the inequalities

$$\sum_{2^j \leqslant |k| < 2^{j+1}} |\hat{g}_k|^2 \leqslant B 2^{-j} h\left(\frac{\pi}{3} 2^{1-j}\right).$$

DEFINITION 3. Denote $\Lambda(n)$ the function of von Mangoldt

$$\Lambda(p^n) = \ln p,$$

where p is prime number and n is a natural number. For all other natural numbers m

$$\Lambda(m) = 0.$$

It is known, that for an arbitrary natural number $n \geqslant 2$ the following equality

$$\ln n = \sum_{d/n} \Lambda(d)$$

is valid, where the sum is taken over all positive divisors d of the number n. In the following theorem of A. Broman's see [1], p. 851, the characterization for exceptional closed sets is given.

THEOREM 2. *Let* $0 < \alpha < 1$ *and*

$$\sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} |c_n|^2 |n|^{-\alpha} < \infty.$$

Let f *be closed set and*

$$\lim_{r \to 1-0} \sum_{k=-\infty}^{\infty} r^{|k|} c_k e^{ikx} = 0,$$

for an arbitrary point $x \in [-\pi, \pi] \setminus F$. *Then* $c_k = 0$ *for arbitrary* $k \in \mathbb{Z}$*, if and only if*

$$C_{1-\alpha}(F) = 0.$$

A. Zygmund, see [7], proved the next nontrivial result.

THEOREM 3. *Let* $\varepsilon > 0$ *and* $\varepsilon_n > 0$*,* $n = 1, 2, \ldots$ *be an arbitrary decreasing sequence, tending to zero. Let* $|c_n| \leqslant \varepsilon_n$*,* $n = 1, 2, \ldots$*. Then there is a subset* $E \subseteq [-\pi, \pi]$ *with* $m(E) > 2\pi - \varepsilon$*, such that, if for each point* $x \in [-\pi, \pi] \setminus E$ *we have*

$$\lim_{n \to \infty} \sum_{k=-n}^{n} c_k e^{ikx} = 0,$$

then $c_k = 0$ *for arbitrary* $k \in \mathbb{Z}$.

The proof of the next theorem can be found in [5].

THEOREM 4. *Let* $0 \leqslant \alpha < 1$ *and*

$$\int_{-\pi}^{\pi} |f(x)| dx + \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{|f(x) - f(y)|}{|x - y|^{1+\alpha}} dx dy < \infty.$$

Let $E \subset [-\pi, \pi]$ *be a subset such that for almost all points* $x_0 \in [-\pi, \pi]$ *we have*

$$\sum_{n=1}^{\infty} 2^{n(1-\alpha)} C_{1-\alpha}(E_n(x_0)) = \infty,$$

where

$$E_n(x_0) = \{x \notin E; 2^{-n-1} \leqslant |x - x_0| < 2^{-n}\}.$$

If

$$\lim_{n \to \infty} \sum_{k=-n}^{n} \hat{f}_k e^{ikx} = 0, \quad x \in E,$$

where $\hat{f}_n$ *are Fourier coefficiens of* f*, then* $f(x) = 0$*,* $x \in [-\pi, \pi]$.

In this paper is proved a new result of this type for other classes od functions.

3. New uniqueness result

The next result, in different form, one can find in the article [4].

THEOREM 5. *Let $f(-\pi)=f(\pi)$ be a differentiable function. Then*

$$\sum_{p\in P}\ln p\left(\sum_{n=1}^{\infty}\left[\frac{1}{p_n}\sum_{k=1}^{p^n}\left(\frac{2\pi k}{p^n}\right)-\hat{f}_0\right]\right)=\sum_{\substack{j\neq 0\\ j=-\infty}}^{\infty}\hat{f}_j\ln|j|,$$

where P denotes the set of primes.

Proof. For $z=re^{ix}$, $0<r<1$, we have

$$\begin{aligned}
A&=\sum_{p\in P}\ln p\sum_{n=1}^{\infty}\left(\frac{1}{p^n}\sum_{k=1}^{p^n}\frac{1-|z|^2}{|1-z\exp\{-\frac{2\pi ik}{p^n}\}|^2}-1\right)\\
&=\sum_{n=1}^{\infty}\Lambda(n)\left(\frac{1}{n}\sum_{k=1}^{n}\frac{1-|z|^2}{|1-z\exp\{-\frac{2\pi ik}{n}\}|^2}-1\right)\\
&=\sum_{n=1}^{\infty}\Lambda(n)\frac{1}{n}\sum_{k=1}^{n}\left(\sum_{\substack{j\neq 0\\ j=-\infty}}^{n}r^{|j|}\exp\left\{ixj-\frac{2\pi ikj}{n}\right\}\right)\\
&=\sum_{n=1}^{\infty}\Lambda(n)\sum_{\substack{j\neq 0\\ j=-\infty}}^{n}r^{|j|}e^{ixj}\left(\frac{1}{n}\sum_{k=1}^{n}\exp\left\{-\frac{2\pi ikj}{n}\right\}\right)\\
&=\sum_{\substack{j\neq 0\\ j=-\infty}}^{\infty}r^{|j|}e^{ixj}\left(\sum_{j=0\,(\mathrm{mod}\,n)}\Lambda(n)\right).
\end{aligned}$$

Since

$$\begin{aligned}
&\sum_{n=1}^{\infty}\Lambda(n)\sum_{\substack{j\neq 0\\ j=0\,(\mathrm{mod}\,n)}}r^{|j|}\left|\frac{1}{n}\sum_{k=1}^{n}\exp\left\{ixj-\frac{2\pi ikj}{n}\right\}\right|\\
&\qquad\leqslant 2\sum_{n=1}^{\infty}\Lambda(n)\sum_{q=1}^{\infty}r^{qn}\leqslant\frac{2}{1-r}\sum_{n=1}^{\infty}r^n\ln n<\infty.
\end{aligned}$$

So, we have

$$A=\sum_{\substack{j\neq 0\\ j=-\infty}}^{\infty}r^{|j|}e^{ixj}\ln|j|.$$

Multiplying absolutely covergent series by the function $\frac{1}{2\pi}f(x)$ and intergrating we have

$$\sum_{n=1}^{\infty}\Lambda(n)\left(\frac{1}{n}\sum_{k=1}^{n}\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1-r^2}{|1-r\exp\{ix-\frac{2\pi ik}{n}\}|^2}f(x)dx-\hat{f}_0\right)=\sum_{\substack{j\neq 0\\ j=-\infty}}^{\infty}r^{|j|}\hat{f}_j\ln|j|.$$

Passing to the limit if $r \to 1-0$ we get the required equality. $\square$

REMARK. This result we can write in the form

$$\sum_{p\in P} \ln p \left(\sum_{n=1}^{\infty} \left[\frac{1}{p^n} \sum_{k=1}^{p^n} \delta \left(x - \frac{2\pi k}{p^n} \right) - 1 \right] \right) = \sum_{\substack{j=-\infty \\ j\neq 0}}^{\infty} e^{ijx} \ln |j|, \quad 0 \leqslant x < 2\pi.$$

This equality is a generalization of the following Poisson's formula:

$$2\pi \sum_{n=-\infty}^{\infty} \delta(x - 2\pi n) = \sum_{j=-\infty}^{\infty} e^{ijx}, \quad -\infty < x < \infty.$$

THEOREM 6. *Let $0 < \alpha < 1$ and a non-negative function $h(k)$ satisfy the following condition:*

1. $h(x) \geqslant x$, $0 \leqslant x < 1$;

2.

$$\int_0^1 \frac{h(x)}{x^{2-\alpha}} \ln^2 \frac{e}{x} dx < \infty.$$

Let for the function $f(x)$ we have

$$\int_{-\pi}^{\pi} |f(x)|^2 dx + \int_{-\pi}^{\pi} \int_{\pi}^{\pi} \frac{|f(x) - f(y)|^2}{|x-y|^{1+\alpha}} dxdy < \infty.$$

Let $E \subset [-\pi, \pi]$ be a set satisfying the following conditions:

3. $M_h(E) > 0$,

4. if $x \in E$ then every point

$$x + \frac{2\pi k}{p^n},$$

where $k \in \mathbb{Z}$, $n \in \mathbb{N}$ and p is prime number, belongs to the set E in the sense mod (2π).

If for every $x \in E$ we have

$$\lim_{r\to 1-0} \sum_{k=-\infty}^{\infty} r^{|k|} \hat{f}_k e^{ikx} = 0,$$

then the function $f(x)$ is identicaly zero.

Proof. By O. Frostman's theorem, see [3], p. 7, there is a probability measure $d\mu$ such that $\operatorname{supp}(d\mu) \subseteq E$,

$$1. \quad M_h(\operatorname{supp}(d\mu)) > 0,$$

and for each $0 < \delta$ the inequality

$$\int_{[x,x+\delta]} d\mu \leqslant Ah(\delta)$$

valid. Let us assume that the function $\mu(t)$ be continuous at the points 0 and 2π and $\mu(0)+1=\mu(2\pi)$. We know that

$$f(re^{ix})=\sum_{n=-\infty}^{\infty} r^{|n|}\hat{f}_n e^{inx}.$$

Then we have

$$\begin{aligned}\sum_{p\in P}\ln p&\left(\sum_{n=1}^{\infty}\left[\frac{1}{p^n}\sum_{k=1}^{p^n}\int_E f\left(r\exp\left\{\frac{2\pi ik}{p^n}+ix\right\}\right)d\mu(x)-\hat{f}_0\right]\right)\\ &=\sum_{n=2}^{\infty} r^n\left[\hat{f}_n\int_E e^{inx}d\mu(x)+\hat{f}_{-n}\int_E e^{-inx}d\mu(x)\right]\ln n\\ &=2\pi i\sum_{n=2}^{\infty}(\hat{f}_n\hat{g}_{-n}+\hat{f}_{-n}\hat{g}_{-n})r^n\ln n.\end{aligned}$$

where

$$g(x)=\mu(x)-\frac{x}{2\pi}.$$

Indeed, for $n\neq 0$ we have

$$\begin{aligned}\hat{g}_n&=\frac{1}{2\pi}\int_0^{2\pi}e^{-inx}\left(\mu(x)-\frac{x}{2\pi}\right)dx\\ &=\frac{1}{2\pi ni}\int_0^{2\pi}e^{-inx}d\mu(x)=\frac{1}{2\pi ni}\int_E e^{-inx}d\mu(x).\end{aligned}$$

Since the function $f(e^{ix})$ vanishes on the set E we have

$$\left|\sum_{p\in P}\hat{f}_0\ln p\right|\leqslant 2\sum_{n=2}^{\infty}(|\hat{f}_{-n}||\hat{g}_n|+|\hat{f}_n||\hat{g}_{-n}|)n\ln n.$$

Using, the given above Zygmund's estimation, we get

$$\begin{aligned}\sum_{n=2}^{\infty}|\hat{f}_n||\hat{g}_{-n}|n\ln n&\leqslant\left(\sum_{n=1}^{\infty}|\hat{f}_n|^2 n^{\alpha}\right)^{\frac{1}{2}}\left(\sum_{n=1}^{\infty}|\hat{g}_{-n}|^2 n^{2-\alpha}\ln^2 n\right)^{\frac{1}{2}}\\ &\leqslant\left(\sum_{n=1}^{\infty}|\hat{f}_n|^2 n^{\alpha}\right)^{\frac{1}{2}}\left(\sum_{j=1}^{\infty}j^2 2^{(2-\alpha)j}\sum_{n=2^j}^{2^{j+1}-1}|\hat{g}_{-n}|^2\right)^{\frac{1}{2}}\\ &\leqslant M\left(\sum_{j=1}^{\infty}j^2 2^{(1-\alpha)j}h(2^{-j})\right)^{\frac{1}{2}}<\infty.\end{aligned}$$

The inequality

$$\left|\sum_{p\in P}\hat{f}_0\ln p\right|<\infty$$

valid only if $\hat{f}_0=0$. Instead of $f(x)$ by examing the functions $e^{inx}f(x)$, $n\in\mathbb{Z}$ we get

$$\hat{f}_n=0,\quad n\in\mathbb{Z}.\quad\square$$

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