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APPLICATIONS OF THEORY OF DIFFERENTIAL SUBORDINATION OF FUNCTIONS WITH FIXED INITIAL COEFFICIENT

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Abstract. Open door lemma is proved for the analytic function f in the unit disc with fixed second coefficient. Conditions on f are obtained so that α-convex integral operator on f belong to certain subclasses of starlike functions. Several interesting applications are given.

1. Introduction

Let $\mathscr{H}$ denote the class of analytic functions in the unit disc $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. For a fixed positive integer n, let $\mathscr{H}[a,n]$ be the subset of $\mathscr{H}$ consisting of functions p of the form

$$p(z) = a + p_n z^n + p_{n+1} z^{n+1} + \cdots.$$

Let $\mathscr{S}$ be the subclass of $\mathscr{H}[0,1]$ consisting of univalent functions of the form $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$. In the theory of univalent functions, the second coefficient in the Taylor series expansion of functions in the class $\mathscr{S}$ plays an important role and influences many properties. For example, a bound for the second coefficient of functions in the class $\mathscr{S}$ yields growth and distortion estimates [4] as well as the Koebe constant. The investigation of univalent functions with fixed second coefficients were initiated by Gronwall [6]. For a brief survey of these developments, see [1]. Works in this direction include those of [2, 5, 7, 8, 13–15].

The theory of differential subordination developed by Miller and Mocanu (see their monograph [11]) was extended to analytic functions with fixed initial coefficient by Ali *et al.* [3]. Nagpal and Ravichandran [12] applied the results in [3] to obtain several generalization of classical results in univalent function theory. In this paper, we investigate conditions on analytic function f in $\mathbb{D}$ with fixed second coefficient so that α-convex integral operator on f belongs to certain subclasses of starlike functions. We also obtain open door lemma for such type of functions f. Certain interesting applications are given.

For a fixed constant $\beta \in \mathbb{C}$, let $\mathscr{H}_\beta[a,n]$ denote the subset of $\mathscr{H}[a,n]$ consisting of functions p of the form

$$p(z) = a + \beta z^n + p_{n+1} z^{n+1} + \cdots.$$

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Without loss of generality, we assume that β is a nonnegative real number. Let Q be the class of functions q that are analytic and injective in $\overline{\mathbb{D}}\setminus E(q)$, where

$$E(q) := \{y \in \partial\mathbb{D} : \lim_{z\to y} q(z) = \infty\},$$

and are such that $q'(y) \neq 0$ for $y \in \partial\mathbb{D}\setminus E(q)$. The following fundamental lemma for functions with fixed second coefficient is required in our investigation.

LEMMA 1.1. [3] *Let $q \in Q$, with $q(0) = a$, and let $p \in \mathscr{H}_\beta[a,n]$ with $p(z) \not\equiv a$ and $n \geqslant 1$. If p is not subordinate to q, then there exist points $z_0 = r_0e^{i\theta_0} \in \mathbb{D}$ and $\zeta_0 \in \partial\mathbb{D}\setminus E(q)$ for which $p(\{z \in \mathbb{C} : |z| < r_0\}) \subset q(\mathbb{D})$,*

(i) $p(z_0) = q(\zeta_0)$,

(ii) $z_0p'(z_0) = m\zeta_0q'(\zeta_0)$,

(iii) $\operatorname{Re}(1+(z_0p''(z_0))/p'(z_0) \geqslant m\operatorname{Re}(1+(\zeta_0q''(\zeta_0))/q'(\zeta_0))$, and

(iv) $m \geqslant n+(|q'(0)|-\beta|z_0|^n)/(|q'(0)|+\beta|z_0|^n)$.

2. Main results

For a fixed $b \in \mathbb{C}$, let $\mathscr{A}_{n,b}$ denote the subset of $\mathscr{H}$ consisting of functions f of the form

$$f(z) = z + bz^{n+1} + a_{n+2}z^{n+2} + \cdots.$$

For $\alpha > 0$, let $\mathbf{A}_\alpha$ be the α-convex integral operator defined as

$$F(z) = \mathbf{A}_\alpha[f](z) := \frac{1}{\alpha}\Big(\int_0^z f^{1/\alpha}(t)t^{-1}\,dt\Big)^\alpha. \tag{2.1}$$

The name come from the fact that f is starlike if and only if F is α-convex; this can be directly seen from the equation

$$\frac{zf'(z)}{f(z)} = (1-\alpha)\frac{zF'(z)}{F(z)} + \alpha\Big(\frac{zF''(z)}{F'(z)}+1\Big) := J(\alpha,F;z).$$

In this section, we find the conditions on $f \in \mathscr{A}_{n,b}$ such that $F = \mathbf{A}_\alpha[f]$ will belong to a certain subclass of starlike functions. For this purpose, we need the following differential subordination result.

LEMMA 2.1. *Let n be a positive integer, $\alpha > 0$, $0 < \beta \leqslant 1$ and suppose that the function $P \in \mathscr{H}_{-\beta(\alpha n+1)}[1,n]$ satisfies the subordination*

$$P(z) \prec 1+z+\Big(n+\frac{1-\beta}{1+\beta}\Big)\frac{\alpha z}{1+z} =: h(z). \tag{2.2}$$

If the function $p \in \mathscr{H}_\beta[1,n]$ satisfies the differential equation

$$\alpha zp'(z) + P(z)p(z) = 1, \tag{2.3}$$

then $p(z) \prec 1/(1+z)$.

Proof. The univalent function $q:\mathbb{D}\to\mathbb{C}$ defined by $q(z)=1/(1+z)$ is related to the function h by the equation

$$h(z)=\frac{1}{q(z)}-\left(n+\frac{1-\beta}{1+\beta}\right)\alpha\frac{zq'(z)}{q(z)}.$$

Then

$$\operatorname{Re}[(1+z)h'(z)]=\operatorname{Re}\left(1+z+\left(n+\frac{1-\beta}{1+\beta}\right)\frac{\alpha}{1+z}\right)>0,\quad z\in\mathbb{D}.$$

Therefore, h is close-to-convex, and hence univalent in $\mathbb{D}$. Clearly, the region $h(\mathbb{D})$ is symmetric with respect to the real axis. For $t\in[-\pi,\pi]$, set

$$w=h(e^{it})=\left(2\cos(t/2)+\left(n+\frac{1-\beta}{1+\beta}\right)\frac{\alpha}{2\cos(t/2)}\right)e^{it/2}. \tag{2.4}$$

Using (2.2) and (2.3), we get

$$P(z)=\frac{1}{p(z)}-\alpha\frac{zp'(z)}{p(z)}\prec h(z). \tag{2.5}$$

If we assume that p is not subordinate to q, then by Lemma 1.1, there exist points $z_0\in\mathbb{D}$ and $\zeta_0\in\partial\mathbb{D}$ and an

$$m\geqslant n+\frac{1-\beta|z_0|^n}{1+\beta|z_0|^n}, \tag{2.6}$$

such that $p(z_0)=q(\zeta_0)$ and $z_0p'(z_0)=m\zeta_0q'(\zeta_0)$. Hence from (2.5), we get that

$$P(z_0)=\frac{1}{q(\zeta_0)}-\alpha\frac{mq'(\zeta_0)}{q(\zeta_0)}=1+\zeta_0+\frac{m\alpha\zeta_0}{1+\zeta_0}.$$

If we set $\zeta_0=e^{it}$, then using (2.6) and (2.4), we get $P(z_0)=Re^{it/2}$ where

$$\begin{aligned}R&=2\cos(t/2)+\frac{m\alpha}{2\cos(t/2)}\\&\geqslant 2\cos(t/2)+\left(n+\frac{1-\beta|z_0|^n}{1+\beta|z_0|^n}\right)\frac{\alpha}{2\cos(t/2)}\\&\geqslant 2\cos(t/2)+\left(n+\frac{1-\beta}{1+\beta}\right)\frac{\alpha}{2\cos(t/2)}=|h(e^{it})|.\end{aligned}$$

This shows that $P(z_0)\notin h(\mathbb{D})$, which contradicts (2.2). Hence we must have $p(z)\prec 1/(1+z)$. □

REMARK 2.2. If $\beta=1$, then Lemma 2.1 reduces to [11, Lemma 5.1a, p. 253]. Since

$$1+z+\frac{n\alpha z}{1+z}\prec 1+z+\left(n+\frac{1-\beta}{1+\beta}\right)\frac{\alpha z}{1+z}\quad(0<\beta\leqslant 1),$$

it is clear that Lemma 2.1 extends [11, Lemma 5.1a, p. 253] for functions $p\in\mathcal{H}_\beta[1,n]$.

THEOREM 2.3. *Let n be a positive integer, $\alpha > 0$, $0 < -nb/(\alpha n+1) \leqslant 1$. Let $f \in \mathscr{A}_{n,b}$ and $F = \mathbf{A}_\alpha[f]$, where $\mathbf{A}_\alpha$ is defined by* (2.1). *If*

$$\frac{zf'(z)}{f(z)} \prec 1+z+\left(n+\frac{\alpha n+1+nb}{\alpha n+1-nb}\right)\frac{\alpha z}{1+z}, \tag{2.7}$$

then $zF'(z)/F(z) \prec 1+z$ or $|zF'(z)/F(z)-1| < 1$.

Proof. Suppose that the function $f \in \mathscr{A}_{n,b}$ satisfy (2.7). Let

$$p(z) = \frac{1}{\alpha f^{1/\alpha(z)}} \int_0^z f^{1/\alpha}(t)t^{-1}\,dt. \tag{2.8}$$

By [11, Lemma 1.2c, p. 11], p is well defined. A calculation shows that $p \in \mathscr{H}_\beta[1,n]$, where $\beta = -nb/(\alpha n+1)$. Therefore, by the given condition, $0 < \beta \leqslant 1$. The function P defined by $P(z) = zf'(z)/f(z)$ belongs to $\mathscr{H}_{-\beta(\alpha n+1)}[1,n]$ and on differentiating (2.8), we see that p satisfies the differential equation

$$\alpha z p'(z) + P(z)p(z) = 1. \tag{2.9}$$

Therefore, by Lemma 2.1 and (2.7), it follows that $p(z) \prec 1/(1+z)$. Since $p(\mathbb{D}) \subseteq \{w : \operatorname{Re} w > 1/2\}$, $p(z) \neq 0$ and we can define the analytic function $F \in \mathscr{A}_{n,b/(\alpha n+1)}$ by

$$F(z) = f(z)(p(z))^\alpha. \tag{2.10}$$

It can be easily verified that this function F coincides with the function given in (2.1). Using (2.9) and (2.10), we get that

$$\frac{zF'(z)}{F(z)} = P(z) + \alpha\frac{zp'(z)}{p(z)} = \frac{1}{p(z)}.$$

Since $p(z) \prec 1/(1+z)$, we deduce that $zF'(z)/F(z) \prec 1+z$. $\square$

REMARK 2.4. If $b = -(\alpha + 1/n)$, then Theorem 2.3 reduces to [11, Theorem 5.1b, p. 255]. Since, for $0 < \frac{-nb}{\alpha n+1} \leqslant 1$,

$$1+z+\frac{n\alpha z}{1+z} \prec 1+z+\left(n+\frac{\alpha n+1+nb}{\alpha n+1-nb}\right)\frac{\alpha z}{1+z}$$

Theorem 2.3 extends [11, Theorem 5.1b, p. 255] for functions $f \in \mathscr{A}_{n,b}$.

THEOREM 2.5. *Let n be a positive integer, $\alpha > 0$, $-1/n \leqslant c < 0$. Suppose that $F \in \mathscr{A}_{n,c}$ and*

$$J(\alpha,F;z) \prec 1+z+\left(n+\frac{1+nc}{1-nc}\right)\frac{\alpha z}{1+z} := h(z)$$

then $zF'(z)/F(z) \prec 1+z$ and $|zF'(z)/F(z)-1| < 1$.

Proof. Let $f \in \mathscr{A}_{n,b}$ satisfy

$$\frac{zf'(z)}{f(z)} = J(\alpha,F;z) := (1-\alpha)\frac{zF'(z)}{F(z)} + \alpha\left(\frac{zF''(z)}{F'(z)}+1\right).$$

A simple calculation shows that $b = c(\alpha n + 1)$. Therefore, by the given condition, we get $0 < -nb/(\alpha n + 1) \leqslant 1$. Hence the result follows from Theorem 2.3. □

REMARK 2.6. If $c = -1/n$, then Theorem 2.5 reduces to [11, Theorem 5.1c, p. 255]. Since, for $-1/n \leqslant c < 0$,

$$1 + z + \frac{n\alpha z}{1+z} \prec 1 + z + \left(n + \frac{1+nc}{1-nc}\right)\frac{\alpha z}{1+z},$$

it is clear that Theorem 2.5 extends [11, Theorem 5.1c, p. 255] for functions $F \in \mathscr{A}_{n,c}$.

REMARK 2.7. For $F \in \mathscr{A}_{n,c}$, $\alpha > 0$ and $-1/n \leqslant c < 0$, Theorem 2.5 can be restated in the following symmetric form

$$J(\alpha, F; z) \prec J\left(\alpha\left(n + \frac{1+nc}{1-nc}\right), k; z\right)$$

implies

$$J(0, F; z) \prec J(0, k; z),$$

where $k(z) = ze^z$.

Let us next formulate the Theorem 2.5 to the two important particular cases of $h(\mathbb{D})$ containing a circle centered at origin and a circle centered at the point $(1,0)$.

Case 1. Set $w = u + iv = h(e^{it}) = re^{it/2}$, where h is given by Theorem 2.5 and $t \in [-\pi, \pi]$. Then,

$$r = r(t) = 2\cos(t/2) + \left(n + \frac{1+nc}{1-nc}\right)\frac{\alpha}{2\cos(t/2)}, \tag{2.11}$$

$$u = 2\cos^2(t/2) + \left(n + \frac{1+nc}{1-nc}\right)\frac{\alpha}{2}, \tag{2.12}$$

and

$$v = u\tan(t/2). \tag{2.13}$$

Using (2.11), we get

$$M(\alpha, n, c) = \min r(\theta) = \begin{cases} 2\sqrt{\left(n + \frac{1+nc}{1-nc}\right)\alpha}, & 0 < \left(n + \frac{1+nc}{1-nc}\right)\alpha \leqslant 4, \\ 2 + \left(n + \frac{1+nc}{1-nc}\right)\frac{\alpha}{2}, & \left(n + \frac{1+nc}{1-nc}\right)\alpha > 4. \end{cases} \tag{2.14}$$

The set of points $|w| \leqslant M(\alpha, n, c)$ is contained in $h(\mathbb{D})$. In this particular case, Theorem 2.5 can be expressed in the following form:

COROLLARY 2.8. *If $F \in \mathscr{A}_{n,c}$, $\alpha > 0$ and $-1/n \leqslant c < 0$ and $|J(\alpha,F;z)| < M(\alpha,n,c)$, where $M(\alpha,n,c)$ is given by (2.14) then $|zF'(z)/F(z) - 1| < 1$.*

Case 2. In this case, we consider the particular case of Theorem 2.5 by taking a circle centered at the point $(1,0)$ which is contained in $h(\mathbb{D})$. For this, we evaluate the minimum distance from $(1,0)$ to the boundary $h(e^{it})$. From (2.12) and (2.13), we get

$$d^2 = (u-1)^2 + v^2 = \frac{4u^2}{2u - (n + (1+nc)/(1-nc))\alpha} - 2u + 1,$$

for

$$\left(n + \frac{1+nc}{1-nc}\right)\frac{\alpha}{2} < u \leqslant 2 + \left(n + \frac{1+nc}{1-nc}\right)\frac{\alpha}{2}.$$

A simple computation shows that

$$\min d = 1 + \left(n + \frac{1+nc}{1-nc}\right)\frac{\alpha}{2}.$$

In this particular case, Theorem 2.5 can be rephrased in the following form.

COROLLARY 2.9. *If $F \in \mathscr{A}_{n,c}$, $\alpha > 0$ and $-1/n \leqslant c < 0$ and*

$$|J(\alpha,F;z) - 1| < 1 + \left(n + \frac{1+nc}{1-nc}\right)\frac{\alpha}{2}$$

then $|zF'(z)/F(z) - 1| < 1$.

REMARK 2.10. If $c = -1/n$, then Corollary 2.9 reduces to [11, Corollary 5.1c.2, p. 257]. For $-1/n \leqslant c < 0$,

$$1 + \frac{n\alpha}{2} \leqslant 1 + \left(n + \frac{1+nc}{1-nc}\right)\frac{\alpha}{2}.$$

Therefore, Corollary 2.9 extends [11, Corollary 5.1c.2, p. 257]. Similarly, [11, Corollary 5.1c.1, p. 256] is extended by Corollary 2.8.

A sufficient condition for a function $p \in \mathscr{H}[1,n]$ to be a function with positive real part is that $p(z) + zp'(z)/p(z) \prec \mathscr{R}(z)$, where $\mathscr{R}$ is the open door mapping given by

$$\mathscr{R}(z) := \frac{1+z}{1-z} + \frac{2z}{1-z^2}.$$

Similarly, we will next investigate the sufficient condition for a function $p \in \mathscr{H}_b[1,1]$ to be a function with positive real part. To find this, we will first prove the following lemma.

LEMMA 2.11. *Let n be a positive integer, $0 < \beta \leqslant 2$ and suppose that the function $P \in \mathscr{H}_{-\beta(n+1)}[1,n]$ satisfies the subordination*

$$P(z) \prec \frac{1+z}{1-z} + 2\left(n + \frac{2-\beta}{2+\beta}\right)\frac{z}{1-z^2} =: h(z). \tag{2.15}$$

If the function $p \in \mathscr{H}_\beta[1,n]$ *satisfies the differential equation*

$$zp'(z) + P(z)p(z) = 1, \tag{2.16}$$

then $\operatorname{Re} p(z) > 0$ *in* $\mathbb{D}$.

Proof. Let q be the convex univalent function defined by $q(z) = (1+z)/(1-z)$. Then it is clear that $q(0) = 1$ and q maps $\mathbb{D}$ onto the right-half plane. If we assume that p is not subordinate to q, then by Lemma 1.1, there exist points $z_0 \in \mathbb{D}$ and $\zeta_0 \in \partial\mathbb{D}$ and an

$$m \geqslant n + \frac{2-\beta|z_0|^n}{2+\beta|z_0|^n}, \tag{2.17}$$

such that

$$p(z_0) = q(\zeta_0) \tag{2.18}$$

and

$$z_0 p'(z_0) = m\zeta_0 q'(\zeta_0) = -\frac{m}{2}|1-q(\zeta_0)|^2. \tag{2.19}$$

Now, $q(\zeta_0) = ik$ and $m\zeta_0 q'(\zeta_0) = l$, for some real number k and l. Then, using (2.17), (2.18) and (2.19), we get

$$p(z_0) = ik \tag{2.20}$$

and

$$\begin{aligned} z_0 p'(z_0) = l &= -\frac{m}{2}(1+k^2 - 2\operatorname{Re}(ik)) \\ &= -\frac{m}{2}(1+k^2) \leqslant -\frac{1+k^2}{2}\left(n + \frac{2-\beta|z_0|^n}{2+\beta|z_0|^n}\right) < 0. \end{aligned} \tag{2.21}$$

Substituting $z = z_0$ in (2.16) and then using (2.20) and (2.21), we get

$$l + ikP(z_0) = 1.$$

Taking $P(z) = C(z) + iD(z)$ and then using the above equation, we get

$$l - kD(z_0) = 1 \quad \text{and} \quad kC(z_0) = 0. \tag{2.22}$$

Since by using (2.21), we obtain $l < 0$, we can use (2.22) to deduce that $k \neq 0$ and $C(z_0) = 0$. Therefore, we have the two following cases:

Case 1: $k > 0$. In this case, by using (2.21) and (2.22), we get

$$D(z_0) = \frac{l-1}{k} \leqslant -\frac{1}{2k}\left(2 + (1+k^2)\left(n + \frac{2-\beta|z_0|^n}{2+\beta|z_0|^n}\right)\right) := F(k).$$

It can be easily verified that

$$\max F(k) = -\left(n + \frac{2-\beta|z_0|^n}{2+\beta|z_0|^n}\right) \times \left(1 + 2\Big/\left(n + \frac{2-\beta|z_0|^n}{2+\beta|z_0|^n}\right)\right)^{\frac{1}{2}} := -B_n.$$

Therefore, $D(z_0) \leqslant -B_n$.

Case 2: $k < 0$. Similarly, in this case, we get $D(z_0) \geqslant B_n$.

Note that $h(\mathbb{D})$ is the complex plane with the slits along the half-lines $\operatorname{Re} w = 0$ and $|\operatorname{Im} w| \geqslant B_n$. Hence, in both the cases, we have

$$P(z_0) = C(z_0) + iD(z_0) \notin h(\mathbb{D}),$$

which contradicts (2.15). Hence we must have $\operatorname{Re} p(z) > 0$ in $\mathbb{D}$. $\square$

REMARK 2.12. If $\beta = 2$, then Lemma 2.11 reduces to [11, Lemma 2.5b, p. 46] with $c = 1$. Lemma 2.11 extends [11, Lemma 2.5b, p. 46] with $c = 1$.

THEOREM 2.13. *Let $-2 \leqslant b < 0$ and suppose that the function $p \in \mathscr{H}_b[1,1]$ satisfies the subordination*

$$p(z) + \frac{zp'(z)}{p(z)} \prec \frac{1+z}{1-z} + \left(\frac{8}{2-b}\right)\frac{z}{1-z^2} =: h(z) \tag{2.23}$$

then $\operatorname{Re} p(z) > 0$ *in* $\mathbb{D}$.

Proof. If the function p satisfies (2.23), then we are assuming that $p(z) \neq 0$ for $z \in \mathbb{D}$. Set $f(z) = 1/p(z)$. Then $f \in \mathscr{H}_{-b}[1,1]$ and from (2.23), we get

$$\frac{1}{f(z)} - \frac{zf'(z)}{f(z)} \prec h(z).$$

Define

$$P(z) = \frac{1}{f(z)} - \frac{zf'(z)}{f(z)}.$$

Then clearly, $P \in \mathscr{H}_{2b}[1,1]$, $P(z) \prec h(z)$ and $P(z)f(z) + zf'(z) = 1$. Therefore, by Lemma 2.11, we get $\operatorname{Re} f(z) > 0$ and hence, $\operatorname{Re} p(z) > 0$. $\square$

Taking $p(z) = zf'(z)/f(z)$ in Theorem 2.13, we get the following result.

COROLLARY 2.14. *Let $-2 \leqslant b < 0$ and suppose that the function $f \in \mathscr{A}_{1,b}$ satisfies the subordination*

$$1 + \frac{zf''(z)}{f'(z)} \prec \frac{1+z}{1-z} + \left(\frac{8}{2-b}\right)\frac{z}{1-z^2}$$

then f is starlike function in $\mathbb{D}$.

REMARK 2.15. If $b = -2$, then Corollary 2.14 reduces to [11, Theorem 2.6 d, p. 61]. Corollary 2.14 extends [11, Theorem 2.6 d, p. 61] for functions $f \in \mathscr{A}_{1,b}$.

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