Journal of Classical Analysis
Volume 8, Number 2 (2016), 123–130 doi:10.7153/jca-08-11

HARMONIC SERIES WITH POLYGAMMA FUNCTIONS

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Abstract. The paper is about evaluating in closed form the following classes of series involving the product of the nth harmonic number and the polygamma functions

$$S_k=\sum_{n=1}^{\infty}H_n\left(\zeta(k)-1-\frac{1}{2^k}-\cdots-\frac{1}{n^k}\right)=\frac{(-1)^k}{(k-1)!}\sum_{n=1}^{\infty}H_n\psi^{(k-1)}(n+1),\ \ k\geqslant 3,$$

$$T_k=\sum_{n=1}^{\infty}nH_n\left(\zeta(k)-1-\frac{1}{2^k}-\cdots-\frac{1}{n^k}\right)=\frac{(-1)^k}{(k-1)!}\sum_{n=1}^{\infty}nH_n\psi^{(k-1)}(n+1),\ \ k\geqslant 4,$$

and

$$R_k=\sum_{n=1}^{\infty}H_n^2\left(\zeta(k)-1-\frac{1}{2^k}-\cdots-\frac{1}{n^k}\right)=\frac{(-1)^k}{(k-1)!}\sum_{n=1}^{\infty}H_n^2\psi^{(k-1)}(n+1),\ \ k\geqslant 3,$$

where k is an integer.

1. Introduction and the main results

The celebrated Riemann zeta function ζ is a function of a complex variable (see [10, p. 265]) defined by

$$\zeta(z)=\sum_{n=1}^{\infty}\frac{1}{n^z}=1+\frac{1}{2^z}+\frac{1}{3^z}+\cdots+\frac{1}{n^z}+\cdots,\qquad \Re(z)>1.$$

When $z=k\geqslant 2$ is an integer, one has that the Riemann zeta function value $\zeta(k)$ is defined by the series formula

$$\zeta(k)=\sum_{n=1}^{\infty}\frac{1}{n^k}=1+\frac{1}{2^k}+\frac{1}{3^k}+\cdots+\frac{1}{n^k}+\cdots.$$

The polygamma functions $\psi^{(k)}$ are defined (see [7, p. 22]) by

$$\psi^{(k)}(z)=\frac{d^{k+1}}{dz^{k+1}}\log\Gamma(z)=\frac{d^k}{dz^k}\psi(z)\quad k\geqslant 0,\ \ z\notin\{0,-1,-2,\ldots\},$$

or, in terms of the generalized (or Hurwitz) zeta function $\zeta(\cdot,\cdot)$

$$\psi^{(k)}(z)=(-1)^{k+1}k!\sum_{i=0}^{\infty}\frac{1}{(i+z)^{k+1}}=(-1)^{k+1}k!\zeta(k+1,z)\quad k\geqslant 1,\ z\notin\{0,-1,-2,\ldots\}.$$

Mathematics subject classification (2010): 40G10, 40A05, 33B15, 11M35.
Keywords and phrases: Abel's summation formula, harmonic numbers, polygamma function, Riemann zeta function.

Series involving closed form evaluation of $\zeta(k)$ are collected in [7] and, more recently, in [8]. Other series, involving the Riemann zeta function and harmonic numbers, that evaluate to special constants can be found in [5].

The nth harmonic number H_n is defined, for $n \geqslant 1$, by $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$. A famous sum, due to Euler, in which the nth harmonic number is involved is given below ([4], [7, p. 103], [8, p. 228]):

$$E_q = \sum_{k=1}^{\infty} \frac{H_k}{n^q} = (1 + \frac{q}{2})\zeta(q+1) - \frac{1}{2}\sum_{k=1}^{q-2} \zeta(q-k)\zeta(k+1), \quad q \in \mathbb{N} \setminus \{1\}, \tag{1}$$

where an empty sum is understood to be nil.

For a proof of (1) the reader is referred to [7, pp. 103–105]. We mention that other series of Euler type can be found in [9].

In this paper we evaluate three classes of series involving the product of the nth harmonic number and the tail of $\zeta(k)$. More precisely, we calculate in closed form the infinite series

$$S_k = \sum_{n=1}^{\infty} H_n \left(\zeta(k) - 1 - \frac{1}{2^k} - \cdots - \frac{1}{n^k} \right) = \frac{(-1)^k}{(k-1)!} \sum_{n=1}^{\infty} H_n \psi^{(k-1)}(n+1), \tag{2}$$

where $k \geqslant 3$ is an integer. We also consider the series

$$T_k = \sum_{n=1}^{\infty} nH_n \left(\zeta(k) - 1 - \frac{1}{2^k} - \cdots - \frac{1}{n^k} \right) = \frac{(-1)^k}{(k-1)!} \sum_{n=1}^{\infty} nH_n \psi^{(k-1)}(n+1), \ \ k \geqslant 4, \tag{3}$$

and

$$R_k = \sum_{n=1}^{\infty} H_n^2 \left(\zeta(k) - 1 - \frac{1}{2^k} - \cdots - \frac{1}{n^k} \right) = \frac{(-1)^k}{(k-1)!} \sum_{n=1}^{\infty} H_n^2 \psi^{(k-1)}(n+1), \ \ k \geqslant 3, \tag{4}$$

where k is an integer.

The main results of this paper are the following theorems.

THEOREM 1. *Let $k \geqslant 3$ be an integer and let S_k be the series in (2). Then,*

$$S_k = \frac{k+1}{2}\zeta(k) - \zeta(k-1) - \frac{1}{2}\sum_{i=1}^{k-3} \zeta(k-1-i)\zeta(i+1),$$

where the last sum is missing when $k = 3$.

In particular when $k = 3$ or $k = 4$ we have the following result.

COROLLARY 1. *The following equalities hold:*

(a) $\displaystyle\sum_{n=1}^{\infty} H_n \left(\zeta(3) - 1 - \frac{1}{2^3} - \cdots - \frac{1}{n^3} \right) = 2\zeta(3) - \zeta(2);$

(b) $\sum_{n=1}^{\infty} H_n \left(\zeta(4) - 1 - \frac{1}{2^4} - \cdots - \frac{1}{n^4}\right) = \frac{5}{4}\zeta(4) - \zeta(3).$

THEOREM 2. *Let $k \geqslant 4$ be an integer and let T_k be the series in (3). Then,*

$$T_k = \frac{1}{2}E_{k-2} - \frac{1}{2}E_{k-1} - \frac{1}{4}\zeta(k-2) + \frac{1}{4}\zeta(k-1),$$

where E_{k-2} and E_{k-1} are the series defined in (1).

The following particular cases are worth mentioning.

COROLLARY 2. *The following equalities hold:*

(a) $\sum_{n=1}^{\infty} nH_n \left(\zeta(4) - 1 - \frac{1}{2^4} - \cdots - \frac{1}{n^4}\right) = -\frac{1}{4}\zeta(2) + \frac{5}{4}\zeta(3) - \frac{5}{8}\zeta(4);$

(b) $\sum_{n=1}^{\infty} nH_n \left(\zeta(5) - 1 - \frac{1}{2^5} - \cdots - \frac{1}{n^5}\right) = \frac{7}{8}\zeta(4) - \frac{1}{4}\zeta(3) - \frac{3}{2}\zeta(5) + \frac{1}{2}\zeta(2)\zeta(3).$

THEOREM 3. (A quadratic harmonic sum)

(a) *The following equality holds:*

$$\sum_{n=1}^{\infty} H_n^2 \left(\zeta(3) - 1 - \frac{1}{2^3} - \cdots - \frac{1}{n^3}\right) = 3\zeta(4) - 4\zeta(3) + 2\zeta(2).$$

(b) *Let $k \geqslant 3$ be an integer and let R_k be the sum in (4). Then,*

$$R_k = \sum_{m=1}^{\infty} \frac{H_m^2}{m^{k-1}} - E_k - E_{k-1} - S_k + \zeta(k-1),$$

where E_{k-1} and E_k are the series defined in (1).

REMARK 1. We mention that, the quadratic series $\sum_{m=1}^{\infty} \frac{H_m^2}{m^{k-1}}$ has been evaluated in terms of products of Riemann zeta function values, for all even integer k, in [3].

We need in our analysis Abel's summation formula ([1, p. 55], [5, p. 258]) which states that if $(a_n)_{n\geqslant 1}$ and $(b_n)_{n\geqslant 1}$ are two sequences of real numbers and $A_n = \sum_{k=1}^{n} a_k$, then

$$\sum_{k=1}^{n} a_k b_k = A_n b_{n+1} + \sum_{k=1}^{n} A_k (b_k - b_{k+1}). \tag{5}$$

We will also be using, in our calculations, the infinite version of the preceding formula

$$\sum_{k=1}^{\infty} a_k b_k = \lim_{n\to\infty} (A_n b_{n+1}) + \sum_{k=1}^{\infty} A_k (b_k - b_{k+1}). \tag{6}$$

2. Some lemmas and the proofs of the main results

Before we prove the main results of this paper we need the following lemmas.

LEMMA 1. *Let $n \geqslant 1$ be an integer. The following equalities hold:*

(a) $\sum_{k=1}^{n} H_k = (n+1)H_{n+1} - (n+1) = (n+1)H_n - n$;

(b) $\sum_{k=1}^{n} kH_k = \frac{n(n+1)}{2}H_{n+1} - \frac{n(n+1)}{4}$.

Proof. The lemma can be proved by induction or by an application of formula (5). □

LEMMA 2. *Let $k \geqslant 3$ be an integer. Then,*

$$\sum_{n=1}^{\infty}\left(\zeta(k) - 1 - \frac{1}{2^k} - \cdots - \frac{1}{n^k}\right) = \zeta(k-1) - \zeta(k).$$

Proof. We apply formula (6), with $a_n = 1$ and $b_n = \zeta(k) - 1 - \frac{1}{2^k} - \cdots - \frac{1}{n^k}$, and we have

$$\begin{aligned}\sum_{n=1}^{\infty}\left(\zeta(k)-1-\frac{1}{2^k}-\cdots-\frac{1}{n^k}\right) &= \lim_{n\to\infty} n\left(\zeta(k)-1-\frac{1}{2^k}-\cdots-\frac{1}{(n+1)^k}\right) + \sum_{n=1}^{\infty}\frac{n}{(n+1)^k}\\ &= \sum_{n=1}^{\infty}\frac{n}{(n+1)^k}\\ &= \zeta(k-1) - \zeta(k),\end{aligned}$$

and Lemma 2 is proved. □

Now we are ready to prove Theorem 1.

Proof. We apply formula (6), with $a_n = H_n$ and $b_n = \zeta(k) - 1 - \frac{1}{2^k} - \cdots - \frac{1}{n^k}$, combined to part (a) of Lemma 1 and we have, since

$$b_n - b_{n+1} = \frac{1}{(n+1)^k},$$

that

$$\begin{aligned}S_k = \lim_{n\to\infty}(H_1 + H_2 + \cdots + H_n)&\left(\zeta(k) - 1 - \frac{1}{2^k} - \cdots - \frac{1}{(n+1)^k}\right)\\ &+ \sum_{n=1}^{\infty}\frac{H_1 + H_2 + \cdots + H_n}{(n+1)^k}\end{aligned}$$

$$
\begin{aligned}
&= \lim_{n\to\infty} ((n+1)H_{n+1} - n - 1)\left(\zeta(k) - 1 - \frac{1}{2^k} - \cdots - \frac{1}{(n+1)^k}\right) \\
&\quad + \sum_{n=1}^{\infty} \frac{(n+1)H_{n+1} - (n+1)}{(n+1)^k} \\
&= \sum_{n=1}^{\infty} \left(\frac{H_{n+1}}{(n+1)^{k-1}} - \frac{1}{(n+1)^{k-1}}\right) \\
&= \sum_{m=2}^{\infty} \left(\frac{H_m}{m^{k-1}} - \frac{1}{m^{k-1}}\right) \\
&= \sum_{m=1}^{\infty} \left(\frac{H_m}{m^{k-1}} - \frac{1}{m^{k-1}}\right) \\
&= E_{k-1} - \zeta(k-1) \\
&= \frac{k+1}{2}\zeta(k) - \zeta(k-1) - \frac{1}{2}\sum_{i=1}^{k-3} \zeta(k-1-i)\zeta(i+1),
\end{aligned}
$$

where the last equality follows based on formula (1). We also used that

$$
\begin{aligned}
&\lim_{n\to\infty} ((n+1)H_{n+1} - n - 1)\left(\zeta(k) - 1 - \frac{1}{2^k} - \cdots - \frac{1}{(n+1)^k}\right) \\
&= \lim_{n\to\infty} \frac{(n+1)H_{n+1} - n - 1}{(n+1)^{k-1}} \cdot \lim_{n\to\infty} (n+1)^{k-1}\left(\zeta(k) - 1 - \frac{1}{2^k} - \cdots - \frac{1}{(n+1)^k}\right) = 0,
\end{aligned}
$$

and Theorem 1 is proved. □

Now we prove Theorem 2.

Proof. We apply formula (6), with $a_n = nH_n$ and $b_n = \zeta(k) - 1 - \frac{1}{2^k} - \cdots - \frac{1}{n^k}$, combined to part (b) of Lemma 1 and we have that

$$
\begin{aligned}
T_k &= \lim_{n\to\infty} (H_1 + 2H_2 + \cdots + nH_n)\left(\zeta(k) - 1 - \frac{1}{2^k} - \cdots - \frac{1}{(n+1)^k}\right) \\
&\quad + \sum_{n=1}^{\infty} \frac{H_1 + 2H_2 + \cdots + nH_n}{(n+1)^k} \\
&= \lim_{n\to\infty} \left(\frac{n(n+1)}{2}H_{n+1} - \frac{n(n+1)}{4}\right)\left(\zeta(k) - 1 - \frac{1}{2^k} - \cdots - \frac{1}{(n+1)^k}\right) \\
&\quad + \sum_{n=1}^{\infty} \left(\frac{n(n+1)}{2}H_{n+1} - \frac{n(n+1)}{4}\right)\frac{1}{(n+1)^k} \\
&= \frac{1}{2}\sum_{n=1}^{\infty} \frac{nH_{n+1}}{(n+1)^{k-1}} - \frac{1}{4}\sum_{n=1}^{\infty} \frac{n}{(n+1)^{k-1}} \\
&= \frac{1}{2}\sum_{n=1}^{\infty} \left(\frac{H_{n+1}}{(n+1)^{k-2}} - \frac{H_{n+1}}{(n+1)^{k-1}}\right) - \frac{1}{4}(\zeta(k-2) - \zeta(k-1)) \\
&= \frac{1}{2}\sum_{m=2}^{\infty} \left(\frac{H_m}{m^{k-2}} - \frac{H_m}{m^{k-1}}\right) - \frac{1}{4}(\zeta(k-2) - \zeta(k-1))
\end{aligned}
$$

$$= \frac{1}{2}\sum_{m=1}^{\infty}\left(\frac{H_m}{m^{k-2}}-\frac{H_m}{m^{k-1}}\right)-\frac{1}{4}\left(\zeta(k-2)-\zeta(k-1)\right)$$
$$= \frac{1}{2}E_{k-2}-\frac{1}{2}E_{k-1}-\frac{1}{4}\zeta(k-2)+\frac{1}{4}\zeta(k-1),$$

and Theorem 2 is proved. □

Next we give the proof of Theorem 3.

Proof. (b) We apply formula (6) with $a_n = H_n$ and $b_n = H_n\left(\zeta(k)-1-\frac{1}{2^k}-\cdots-\frac{1}{n^k}\right)$ combined to part (a) of Lemma 1 and we have, since

$$b_n - b_{n+1} = \frac{H_n}{(n+1)^k}-\frac{1}{n+1}\left(\zeta(k)-1-\frac{1}{2^k}-\cdots-\frac{1}{(n+1)^k}\right),$$

that

$$R_k = \lim_{n\to\infty}(H_1+H_2+\cdots+H_n)H_{n+1}\left(\zeta(k)-1-\frac{1}{2^k}-\cdots-\frac{1}{(n+1)^k}\right)$$
$$+\sum_{n=1}^{\infty}(H_1+H_2+\cdots+H_n)\left[\frac{H_n}{(n+1)^k}-\frac{1}{n+1}\left(\zeta(k)-1-\frac{1}{2^k}-\cdots-\frac{1}{(n+1)^k}\right)\right]$$
$$= \lim_{n\to\infty}\left((n+1)H_{n+1}-(n+1)\right)H_{n+1}\left(\zeta(k)-1-\frac{1}{2^k}-\cdots-\frac{1}{(n+1)^k}\right)$$
$$+\sum_{n=1}^{\infty}\left((n+1)H_{n+1}-(n+1)\right)\left[\frac{H_n}{(n+1)^k}-\frac{1}{n+1}\left(\zeta(k)-1-\frac{1}{2^k}-\cdots-\frac{1}{(n+1)^k}\right)\right].$$

It is an exercise in classical analysis to show that

$$\lim_{n\to\infty}(n+1)^{k-1}\left(\zeta(k)-1-\frac{1}{2^k}-\cdots-\frac{1}{(n+1)^k}\right)=\frac{1}{k-1},$$

and this implies, since $k \geqslant 3$, that

$$\lim_{n\to\infty}\left((n+1)H_{n+1}-(n+1)\right)H_{n+1}\left(\zeta(k)-1-\frac{1}{2^k}-\cdots-\frac{1}{(n+1)^k}\right)$$
$$=\lim_{n\to\infty}\frac{H_{n+1}^2-H_{n+1}}{(n+1)^{k-2}}\cdot(n+1)^{k-1}\left(\zeta(k)-1-\frac{1}{2^k}-\cdots-\frac{1}{(n+1)^k}\right)=0.$$

It follows that

$$R_k = \sum_{n=1}^{\infty}\left((n+1)H_{n+1}-(n+1)\right)\left[\frac{H_n}{(n+1)^k}-\frac{1}{n+1}\left(\zeta(k)-1-\frac{1}{2^k}-\cdots-\frac{1}{(n+1)^k}\right)\right].$$

Let

$$x_n = \left((n+1)H_{n+1}-(n+1)\right)\left[\frac{H_n}{(n+1)^k}-\frac{1}{n+1}\left(\zeta(k)-1-\frac{1}{2^k}-\cdots-\frac{1}{(n+1)^k}\right)\right].$$

A calculation shows that

$$x_n = \frac{H_n H_{n+1}}{(n+1)^{k-1}} - H_{n+1}\left(\zeta(k) - 1 - \frac{1}{2^k} - \cdots - \frac{1}{(n+1)^k}\right) - \frac{H_n}{(n+1)^{k-1}} + \zeta(k) - 1 - \frac{1}{2^k} - \cdots - \frac{1}{(n+1)^k},$$

and since

$$\frac{H_n H_{n+1}}{(n+1)^{k-1}} = \frac{H_{n+1}^2}{(n+1)^{k-1}} - \frac{H_{n+1}}{(n+1)^k} \quad \text{and} \quad \frac{H_n}{(n+1)^{k-1}} = \frac{H_{n+1}}{(n+1)^{k-1}} - \frac{1}{(n+1)^k},$$

we get that

$$\begin{aligned} x_n &= \frac{H_{n+1}^2}{(n+1)^{k-1}} - \frac{H_{n+1}}{(n+1)^k} - H_{n+1}\left(\zeta(k) - 1 - \frac{1}{2^k} - \cdots - \frac{1}{(n+1)^k}\right) \\ &\quad - \frac{H_{n+1}}{(n+1)^{k-1}} + \frac{1}{(n+1)^k} + \zeta(k) - 1 - \frac{1}{2^k} - \cdots - \frac{1}{(n+1)^k} \\ &= y_{n+1}. \end{aligned}$$

Thus,

$$\begin{aligned} R_k &= \sum_{n=1}^{\infty} x_n = \sum_{n=1}^{\infty} y_{n+1} = \sum_{m=2}^{\infty} y_m \overset{y_1=0}{=} \sum_{m=1}^{\infty} y_m \\ &= \sum_{m=1}^{\infty} \frac{H_m^2}{m^{k-1}} - \sum_{m=1}^{\infty} \frac{H_m}{m^k} - \sum_{m=1}^{\infty} H_m \left(\zeta(k) - 1 - \frac{1}{2^k} - \cdots - \frac{1}{m^k}\right) \\ &\quad - \sum_{m=1}^{\infty} \frac{H_m}{m^{k-1}} + \sum_{m=1}^{\infty} \frac{1}{m^k} + \sum_{m=1}^{\infty} \left(\zeta(k) - 1 - \frac{1}{2^k} - \cdots - \frac{1}{m^k}\right) \\ &\overset{\text{Lemma 2}}{=} \sum_{m=1}^{\infty} \frac{H_m^2}{m^{k-1}} - E_k - S_k - E_{k-1} + \zeta(k) + \zeta(k-1) - \zeta(k) \\ &= \sum_{m=1}^{\infty} \frac{H_m^2}{m^{k-1}} - E_k - S_k - E_{k-1} + \zeta(k-1). \end{aligned}$$

(a) When $k = 3$ we get, based on part (b) of the theorem, that

$$\begin{aligned} &\sum_{n=1}^{\infty} H_n^2 \left(\zeta(3) - 1 - \frac{1}{2^3} - \cdots - \frac{1}{n^3}\right) \\ &= \sum_{m=1}^{\infty} \frac{H_m^2}{m^2} - E_3 - S_3 - E_2 + \zeta(2) \\ &= \frac{17}{4}\zeta(4) - \frac{5}{4}\zeta(4) - (2\zeta(3) - \zeta(2)) - 2\zeta(3) + \zeta(2) \\ &= 3\zeta(4) - 4\zeta(3) + 2\zeta(2), \end{aligned}$$

since

$$E_3 = \frac{5}{2}\zeta(4) - \frac{1}{2}\zeta^2(2) = \frac{5}{4}\zeta(4), \quad E_2 = 2\zeta(3) \quad \text{and} \quad \sum_{m=1}^{\infty} \frac{H_m^2}{m^2} = \frac{17}{4}\zeta(4).$$

We mention that, the identity $\sum_{m=1}^{\infty}\frac{H_m^2}{m^2}=\frac{17}{4}\zeta(4)$ was discovered numerically by Enrico Au-Yeung and proved rigorously by David Borwein and Jonathan Borwein in [2] who used Fourier series techniques combined to Parseval's formula for proving it and a recent proof involving integrals of polylogarithm functions was given in [6]. $\square$

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(Received July 30, 2015)

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