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EXTENSION OF WEYL–HEISENBERG WAVE PACKET BESSEL SEQUENCES TO DUAL FRAMES IN $L^2(\mathbb{R})$

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Abstract. Christensen et al. proved in [Extensions of Bessel sequences to dual pairs of frames, Appl. Comput. Harmon. Anal., 34 (2013), 224–233] that in any separable Hilbert space, any pairs of Bessel sequences (even if the given Bessel system is Gabor system in $L^2(\mathbb{R})$) can be extended to a pair of dual frames. In this paper, we extend results by Christensen et al. to a pair of Bessel sequences in $L^2(\mathbb{R})$ having wave packet structure to a pair of dual frames such that extension have wave packet structure as well. We present sufficient conditions for the extension of a pair of Bessel sequences to wave packet type dual frames for $L^2(\mathbb{R})$. Several examples and counter-examples are given to illustrate our results.

1. Introduction and preliminaries

Duffin and Schaeffer introduced frames in [14] to address some deep problems in nonharmonic Fourier series. More precisely, Duffin and Schaeffer abstracted the fundamental approach by Gabor [15] for decomposition of a signal into *elementary signals* (or *atoms*). For some reason the work of Duffin and Schaeffer was not continued until 1986, when the fundamental work of Daubechies, Grossmann and Meyer [11] brought this all back to life, right at the dawn of the “wavelet era”. Frames are generalization of the concept of orthonormal systems in Hilbert spaces. Frames provides an unconditional series representation of each vector in a Hilbert space. Application of frames in applied mathematics in different directions can be found in recent books [1, 3, 4], as well as in [12].

Let $\mathscr{H}$ be a separable real (or complex) Hilbert space with inner product $\langle .,. \rangle$. A countable sequence $\{f_k\}_{k\in\Omega} \subset \mathscr{H}$ is called a *frame* (or *Hilbert frame*) for $\mathscr{H}$ if there exist numbers $0 < A \leqslant B < \infty$ such that

$$A\|f\|^2 \leqslant \sum_{k\in\Omega} |\langle f, f_k\rangle|^2 \leqslant B\|f\|^2 \text{ for all } f \in \mathscr{H}. \tag{1}$$

The numbers A and B are called *lower* and *upper frame bounds*, respectively. They are not unique. If it is possible to choose $A = B$, then the frame $\{f_k\}_{k\in\Omega}$ is called the A*-Parseval frame* (or A*-tight frame*). If $\{f_k\}_{k\in\Omega} \subset \mathscr{H}$ satisfies the upper inequality in (1), then we say that $\{f_k\}_{k\in\Omega}$ is a *Bessel sequence* in $\mathscr{H}$ with *Bessel bound* B.

Let $\{f_k\}_{k\in\Omega}$ be a frame for $\mathscr{H}$. There are three important bounded linear operators associated with the frame $\{f_k\}_{k\in\Omega}$:

$$\textit{pre-frame operator} \quad V : \ell^2(\Omega) \to \mathscr{H}, \quad V\{c_k\}_{k\in\Omega} = \sum_{k\in\Omega} c_k f_k,\ \{c_k\}_{k\in\Omega} \in \ell^2(\Omega),$$

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$$\textit{analysis operator} \quad V^*: \mathcal{H} \to \ell^2(\Omega), \quad V^*f = \{\langle f, f_k\rangle\}_{k\in\Omega}, \ f \in \mathcal{H},$$
$$\textit{frame operator} \quad S = VV^*: \mathcal{H} \to \mathcal{H}, \quad Sf = \sum_{k\in\Omega}\langle f, f_k\rangle f_k, \ f \in \mathcal{H}.$$

The frame operator S is a positive, self-adjoint and invertible operator on $\mathcal{H}$. Thus, we have the *reconstruction formula* for all $f \in \mathcal{H}$,

$$f = SS^{-1}f = \sum_{k\in\Omega}\langle S^{-1}f, f_k\rangle f_k \quad \Big(= \sum_{k\in\Omega}\langle f, S^{-1}f_k\rangle f_k \Big). \tag{2}$$

The scalars $\{\langle f, S^{-1}f_k\rangle\}_{k\in\Omega}$ are called *frame coefficients* of the vector $f \in \mathcal{H}$. The series in (2) converges unconditionally.

Let $\{f_k\}_{k\in\Omega}$ be a frame for $\mathcal{H}$. A frame $\{g_k\}_{k\in\Omega}$ for $\mathcal{H}$ satisfying

$$f = \sum_{k\in\Omega}\langle f, g_k\rangle f_k \text{ for all } f \in \mathcal{H} \tag{3}$$

is called a *dual frame* of $\{f_k\}_{k\in\Omega}$. If $\{g_k\}_{k\in\Omega}$ is a dual frame of $\{f_k\}_{k\in\Omega}$, then $\{f_k\}_{k\in\Omega}$ is also a dual of $\{g_k\}_{k\in\Omega}$, see [3, 4]. We call $\{f_k\}_{k\in\Omega}$ and $\{g_k\}_{k\in\Omega}$ a *pair of dual frames* or a *dual frame pair*, when (3) holds. For any frame $\{f_k\}_{k\in\Omega}$ there exist at least one dual frame $\{S^{-1}f_k\}_{k\in\Omega}$ which is called the *canonical dual frame* of $\{f_k\}_{k\in\Omega}$, see (2). If $\{f_k\}$ is a tight frame then $\{f_k\}$ has a dual of the form $g_k = Cf_k$ $(k \in \Omega)$ for some constant $C > 0$. Furthermore, if $\{f_k\}_{k\in\Omega}$ is a tight frame with frame bounds $A = B = 1$, then we can take $g_k = f_k$, $k \in \Omega$ and elements of $\mathcal{H}$ has representation of the form

$$f = \sum_{k\in\Omega}\langle f, f_k\rangle f_k \text{ for all } f \in \mathcal{H}.$$

Thus, tight frames provides a series expansion of each vector in $\mathcal{H}$ in terms of "pure" frame elements.

1.1. Why extension of Bessel systems to frames?

It is well known that a Bessel sequence in a Hilbert space $\mathcal{H}$ need not be a frame for $\mathcal{H}$. The problem of extension of Bessel systems to frames (in particular tight frames) is one of the attractive problem in frame theory. In fact, the extension problems in frame theory have a long history. Recall that a dual frame pair provides a series expansion of each vector in $\mathcal{H}$, see (3). If $\{f_k\}_{k\in\Omega}$ is a tight frame for $\mathcal{H}$ with bound A, then $\{g_k\}_{k\in\Omega} = \{\frac{1}{A}f_k\}_{k\in\Omega}$ is a dual frame for $\{f_k\}_{k\in\Omega}$. Hence computation of a dual frame for a tight frame is very convenient. So extension of Bessel sequences to tight frames is of major importance. It is known that in the general Hilbert space setting, it is possible to extend Bessel sequence with bound B to tight frames with same bound B. But it may possible that a Bessel sequence with bound B having some specific structure can not be extended to a tight frame with same structure and with same bound B. In such cases extension to a pair of dual frames might work. It was shown by Han in [16] that there exist Bessel sequences having wavelet structure with bound B that can only be extended to a tight frame with a bound that is strictly larger than B. In such cases, we can work out by extending a pair of Bessel sequences in a Hilbert space $\mathscr{H}$ to a dual frame pair. Computation of a dual frame is not required in such cases. Thus, a dual frame pair is a more flexible tool than tight frames. Hence extension to dual pair of frames might be more efficient. Casazza and Leonhard [2], Li and Sun [22] proved that for any Bessel sequence $\{f_k\}_{k\in\Omega}$ in a separable Hilbert space $\mathscr{H}$ there exists a sequence $\{g_k\}_{k\in\Omega}$ such that $\{f_k\}_{k\in\Omega}\bigcup\{g_k\}_{k\in\Omega}$ is a tight frame for $\mathscr{H}$. In [22], Li and Sun proved sufficient conditions for the extension of Gabor Bessel systems to tight Gabor frames for $L^2(\mathbb{R})$. Christensen et al. provide a natural generalization to construction of dual frame pairs in [8]. More precisely, for a given pair of Bessel sequences $\{f_k\}_{k\in\Omega}$ and $\{g_k\}_{k\in\Omega}$ in a Hilbert space $\mathscr{H}$, they proved results regarding existence of another pair of Bessel sequences $\{p_k\}_{k\in\Omega}$ and $\{q_k\}_{k\in\Omega}$ in $\mathscr{H}$ such that

$$f = \sum_{k\in\Omega}\langle f,g_k\rangle f_k + \sum_{k\in\Omega}\langle f,q_k\rangle p_k \text{ for all } f\in\mathscr{H}.$$

Furthermore, they studied the extension problem of Gabor Bessel systems and wavelet Bessel systems to Gabor frames and wavelet frames, respectively. Christensen et al. in [7] studied the extension of wavelet systems via the unitary extension principle.

1.2. Wave packet systems

The wave packet system was introduced by Cordoba and Fefferman in [9]. They obtained the wave packet system by applying certain collection of dilations, modulations and translations to the Gaussian function in the study of some classes of singular integral operators. Later, Labate et al. in [19] characterized wave packet systems which constitute normalized tight frames for $L^2(\mathbb{R}^d)$. They examined in detail both the continuous and discrete versions of wave packet in [19]. Note that Gabor systems, wavelet systems and the Fourier transform of wavelet systems are special cases of wave packet systems. Lacey and Thiele [20, 21] gave applications of wave packet systems in boundedness of the Hilbert transforms. In [5, 6, 10, 13, 17, 18], authors gave some fundamental results about wave packet systems and related frame properties. By $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}^+$ and $\mathbb{R}$ we denote the set of all natural numbers, integers, positive real numbers and real numbers, respectively. Let $a,b\in\mathbb{R}$ and $c\in\mathbb{R}\setminus\{0\}$. Define operators T_a, E_b, $D_c : L^2(\mathbb{R})\to L^2(\mathbb{R})$ by

$$T_a f(t) = f(t-a) \quad \text{(Translation of } f \text{ by } a)$$
$$E_b f(t) = e^{2\pi i b t} f(t) \quad \text{(Modulation of } f \text{ by } b)$$
$$D_c f(t) = |c|^{\frac{1}{2}} f(ct) \quad \text{(Dilation of } f \text{ by } c).$$

A system of the form

$$\{D_{a_j} T_{bk} E_{c_m}\psi\}_{j,k,m\in\mathbb{Z}},$$

where $\psi\in L^2(\mathbb{R})$, $\{a_j\}_{j\in\mathbb{Z}}\subset\mathbb{R}^+$, $b\in\mathbb{R}\setminus\{0\}$ and $\{c_m\}_{m\in\mathbb{Z}}\subset\mathbb{R}$ is called *irregular Weyl-Heisenberg wave packet system* (or simply *wave packet system*) in $L^2(\mathbb{R})$. If such a system forms a frame for $L^2(\mathbb{R})$, then it is called a *wave packet frame* (or an *irregular Weyl-Heisenberg wave packet frame*) for $L^2(\mathbb{R})$.

1.3. Outline of the paper

Let $\psi \in L^2(\mathbb{R})$, $b \in \mathbb{R}\setminus\{0\}$, $c \in \mathbb{R}$ and $a \in \mathbb{R}^+$ be fixed. In this paper we consider a system of the form

$$\{D_{a^j} T_{bk} E_{cm} \psi\}_{j\in\Lambda_1, k\in\Lambda_2, m\in\Lambda_3},$$

where Λ_1, Λ_2 and Λ_3 are countable sets and call it *regular Weyl-Heisenberg wave packet system* (in short, RWH *wave packet system*). Christensen et al. proved in [8] that if the given Bessel sequences have Gabor structure, it is always possible to extend to a dual frame pair by adding one extra Gabor system to each Bessel sequence. They also analyze the corresponding problem for wavelet systems. In this paper, we present sufficient conditions under which a pair of Bessel sequences having wave packet structure can be extended to a pair of dual frames for $L^2(\mathbb{R})$ by adding systems having wave packet structure as well. Several examples are provided to defend our results.

To conclude the section, we recall that the *Fourier transform* of a function f, denoted by $\hat{f}$, is defined as

$$\hat{f}(\gamma) = \int_{\mathbb{R}} f(x) e^{-2\pi i x\gamma} dx, \ \gamma \in \mathbb{R}.$$

2. Wave packet Bessel sequences and dual frames in $L^2(\mathbb{R})$

DEFINITION 1. Let $\psi \in L^2(\mathbb{R})$, $b \in \mathbb{R}\setminus\{0\}$, $c \in \mathbb{R}$ and $a \in \mathbb{R}^+$ be fixed. Let Λ_1, Λ_2 and Λ_3 be countable sets. A frame for $L^2(\mathbb{R})$ of the form $\{D_{a^j} T_{bk} E_{cm} \psi\}_{j\in\Lambda_1, k\in\Lambda_2, m\in\Lambda_3}$ is called a *regular Weyl-Heisenberg wave packet frame* (in short, RWH *wave packet frame*).

If there exists a positive real number B such that

$$\sum_{j\in\Lambda_1, k\in\Lambda_2, m\in\Lambda_3} |\langle f, D_{a^j} T_{bk} E_{cm} \psi\rangle|^2 \leqslant B\|f\|^2 \text{ for all } f \in L^2(\mathbb{R}),$$

then $\{D_{a^j} T_{bk} E_{cm} \psi\}_{j\in\Lambda_1, k\in\Lambda_2, m\in\Lambda_3}$ is called a RWH *wave packet Bessel sequence* in $L^2(\mathbb{R})$ with Bessel bound B.

Regarding the existence of RWH wave packet frames we have the following examples.

EXAMPLE 1. Fix $n \in \mathbb{N}$. Let $\Lambda_1 = \{\alpha_1, \alpha_2, \ldots, \alpha_n\} \subset \mathbb{Z}, \Lambda_2 = \Lambda_3 = \mathbb{Z}$.

Choose $a = 2$, $b = 1$, $c = 1$ and $\phi = \chi_{[0,1]}$. Then, $\{D_{2^j} T_k E_m \phi\}_{j\in\Lambda_1, k\in\Lambda_2, m\in\Lambda_3}$ is a RWH wave packet frame for $L^2(\mathbb{R})$. To prove this, first we note that the system $\{E_m T_k \phi\}_{m,k\in\mathbb{Z}}$ and hence $\{T_k E_m \phi\}_{k,m\in\mathbb{Z}}$ is an orthonormal basis for $L^2(\mathbb{R})$, see [4, p. 196]. By using the fact that the dilation is a unitary operator, we compute

$$\begin{aligned}\sum_{j\in\Lambda_1,k\in\Lambda_2,m\in\Lambda_3}|\langle f,D_{2^j}T_kE_m\phi\rangle|^2 &= \sum_{j\in\Lambda_1,k\in\Lambda_2,m\in\Lambda_3}|\langle D_{2^j}^*f,T_kE_m\phi\rangle|^2\\ &= \sum_{j\in\Lambda_1,k\in\Lambda_2,m\in\Lambda_3}|\langle D_{2^{-j}}f,T_kE_m\chi_{[0,1]}\rangle|^2\\ &= \sum_{j\in\Lambda_1}\|D_{2^{-j}}f\|^2 = n\|f\|^2 \text{ for all } f\in L^2(\mathbb{R}).\end{aligned}$$

Therefore, $\{D_{2^j}T_kE_m\phi\}_{j\in\Lambda_1,k\in\Lambda_2,m\in\Lambda_3}$ is a RWH wave packet frame for $L^2(\mathbb{R})$ with frame bounds $A=B=n$.

EXAMPLE 2. Let $a=2$, $b=c=1$ and $\phi=\chi_{[0,1]}$. Then, the wave packet system $\{D_{2^j}T_kE_m\phi\}_{j,k,m\in\mathbb{Z}}$ does not constitute a RWH wave packet frame for $L^2(\mathbb{R})$. Indeed, let B be an upper frame bound for $\{D_{2^j}T_kE_m\phi\}_{j,k,m\in\mathbb{Z}}$. Since the system $\{E_mT_k\chi_{[0,1]}\}_{m,k\in\mathbb{Z}}$ and hence $\{T_kE_m\chi_{[0,1]}\}_{k,m\in\mathbb{Z}}$ is an orthonormal basis for $L^2(\mathbb{R})$.

Choose $f_0=\chi_{[0,1]}$.

We compute

$$\begin{aligned}\sum_{j,k,m\in\mathbb{Z}}|\langle f_0,D_{2^j}T_kE_m\phi\rangle|^2 &= \sum_{j,k,m\in\mathbb{Z}}|\langle D_{2^j}^*f_0,T_kE_m\chi_{[0,1]}\rangle|^2\\ &= \sum_{j,k,m\in\mathbb{Z}}|\langle D_{2^{-j}}f_0,T_kE_m\chi_{[0,1]}\rangle|^2\\ &= \sum_{j\in\mathbb{Z}}\|D_{2^{-j}}f_0\|^2 = \sum_{j\in\mathbb{Z}}\|f_0\|^2 > B\|f_0\|^2,\end{aligned}$$

which contradicts the fact that B is an upper frame bound for $\{D_{2^j}T_kE_m\phi\}_{j,k,m\in\mathbb{Z}}$.

A wave packet Bessel sequence in $L^2(\mathbb{R})$ is, in general, not a frame for $L^2(\mathbb{R})$. It would be interesting to extend wave packet Bessel systems to frames. The following theorem provides sufficient conditions for the extension of a pair of RWH wave packet Bessel sequences to dual RWH wave packet frames for $L^2(\mathbb{R})$. This is an extension of Lemma 4.1 in [8].

THEOREM 1. *Let $\{D_{a^j}T_{bk}E_{cm}\psi_1\}_{j,k,m\in\mathbb{Z}}$ and $\{D_{a^j}T_{bk}E_{cm}\widetilde{\psi_1}\}_{j,k,m\in\mathbb{Z}}$ be* RWH *wave packet Bessel sequences in $L^2(\mathbb{R})$ with pre-frame operators T and U and analysis operators T^* and U^*, respectively. Let I be the identity operator on $L^2(\mathbb{R})$. Assume there exist $\phi\in L^2(\mathbb{R})$ such that*

(i) $\{D_{a^j}T_{bk}E_{cm}\phi\}_{j,k,m\in\mathbb{Z}}$ is a RWH *wave packet frame with a dual $\{D_{a^j}T_{bk}E_{cm}\widetilde{\phi}\}_{j,k,m\in\mathbb{Z}}$.*
(ii) $TU^*T_{bk}E_{cm}\phi = T_{bk}E_{cm}TU^*\phi$.

Let $\psi_2,\widetilde{\psi_2}\in L^2(\mathbb{R})$ be such that $\psi_2=\Xi^*\phi$ and $\widetilde{\psi_2}=\widetilde{\phi}$, where $\Xi=I-UT^*$. Then, $\{D_{a^j}T_{bk}E_{cm}\psi_1\}_{j,k,m\in\mathbb{Z}} \cup \{D_{a^j}T_{bk}E_{cm}\psi_2\}_{j,k,m\in\mathbb{Z}}$ and $\{D_{a^j}T_{bk}E_{cm}\widetilde{\psi_1}\}_{j,k,m\in\mathbb{Z}} \cup \{D_{a^j}T_{bk}E_{cm}\widetilde{\psi_2}\}_{j,k,m\in\mathbb{Z}}$ form dual RWH *wave packet frames for $L^2(\mathbb{R})$.*

Proof. For all $f \in L^2(\mathbb{R})$, we have

$$\begin{aligned} UT^* f &= U(\{\langle f, D_{a^j} T_{bk} E_{cm} \psi_1 \rangle\}_{j,k,m\in\mathbb{Z}}) \\ &= \sum_{j,k,m\in\mathbb{Z}} \langle f, D_{a^j} T_{bk} E_{cm} \psi_1 \rangle D_{a^j} T_{bk} E_{cm} \widetilde{\psi_1}. \end{aligned}$$

Define a bounded linear operator $\Xi : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ as $\Xi = I - UT^*$.

Then, since $\{D_{a^j} T_{bk} E_{cm} \phi\}_{j,k,m\in\mathbb{Z}}$ and $\{D_{a^j} T_{bk} E_{cm} \widetilde{\phi}\}_{j,k,m\in\mathbb{Z}}$ form a pair of dual frames for $L^2(\mathbb{R})$, we have

$$\begin{aligned} \Xi f &= \sum_{j,k,m\in\mathbb{Z}} \langle \Xi f, D_{a^j} T_{bk} E_{cm} \phi \rangle D_{a^j} T_{bk} E_{cm} \widetilde{\phi} \\ &= \sum_{j,k,m\in\mathbb{Z}} \langle f, \Xi^*(D_{a^j} T_{bk} E_{cm} \phi) \rangle D_{a^j} T_{bk} E_{cm} \widetilde{\phi}, \ f \in L^2(\mathbb{R}). \end{aligned} \tag{4}$$

It can be verified easily that D_{a^j} commutes with TU^*.

By using condition (ii), we compute

$$\begin{aligned} \Xi^* D_{a^j} T_{bk} E_{cm} \phi = (I - TU^*) D_{a^j} T_{bk} E_{cm} \phi &= D_{a^j} (I - TU^*) T_{bk} E_{cm} \phi \\ = D_{a^j} T_{bk} E_{cm} (I - TU^*) \phi &= D_{a^j} T_{bk} E_{cm} \Xi^* \phi. \end{aligned} \tag{5}$$

Furthermore, by definition of Ξ, we have

$$\Xi f = f - \sum_{j,k,m\in\mathbb{Z}} \langle f, D_{a^j} T_{bk} E_{cm} \psi_1 \rangle D_{a^j} T_{bk} E_{cm} \widetilde{\psi_1}. \tag{6}$$

Recall that by definition $\psi_2 = \Xi^* \phi$ and $\widetilde{\psi_2} = \widetilde{\phi}$. Therefore, by using (4), (5) and (6) for all $f \in L^2(\mathbb{R})$, we have

$$f = \sum_{j,k,m\in\mathbb{Z}} \langle f, D_{a^j} T_{bk} E_{cm} \psi_1 \rangle D_{a^j} T_{bk} E_{cm} \widetilde{\psi_1} + \sum_{j,k,m\in\mathbb{Z}} \langle f, D_{a^j} T_{bk} E_{cm} \psi_2 \rangle D_{a^j} T_{bk} E_{cm} \widetilde{\psi_2}.$$

Hence the system $\{D_{a^j} T_{bk} E_{cm} \psi_1\}_{j,k,m\in\mathbb{Z}} \bigcup \{D_{a^j} T_{bk} E_{cm} \psi_2\}_{j,k,m\in\mathbb{Z}}$ and $\{D_{a^j} T_{bk} E_{cm} \widetilde{\psi_1}\}_{j,k,m\in\mathbb{Z}} \bigcup \{D_{a^j} T_{bk} E_{cm} \widetilde{\psi_2}\}_{j,k,m\in\mathbb{Z}}$ form dual RWH wave packet frames for $L^2(\mathbb{R})$. □

We now demonstrate by a concrete example that the condition given in Theorem 1 is sufficient but not necessary.

EXAMPLE 3. Let $a = 2$, $b = \frac{1}{4}$, $c = 1$, and $\psi_1 = \chi_{[0,\frac{3}{4})}$.

For $x \in [0, \frac{1}{4}]$, we compute

$$\begin{aligned} &\sum_{n\in\mathbb{Z}} |\psi_1(x - \frac{n}{4})|^2 - \sum_{k\neq 0} |\sum_{n\in\mathbb{Z}} \psi_1(x - \frac{n}{4}) \overline{\psi_1(x - \frac{n}{4} - k)}| \\ &= \sum_{n\in\mathbb{Z}} |\chi_{[0,\frac{3}{4})}(x - \frac{n}{4})|^2 - \sum_{k\neq 0} |\sum_{n\in\mathbb{Z}} \chi_{[0,\frac{3}{4})}(x - \frac{n}{4}) \overline{\chi_{[0,\frac{3}{4})}(x - \frac{n}{4} - k)}| \end{aligned}$$

$$= \sum_{n\in\mathbb{Z}} |\chi_{[\frac{n}{4},\frac{3}{4}+\frac{n}{4})}(x)|^2 - \sum_{k\neq 0} |\sum_{n\in\mathbb{Z}} \chi_{[\frac{n}{4},\frac{3}{4}+\frac{n}{4})}(x)\overline{\chi_{[\frac{n}{4}+k,\frac{3}{4}+\frac{n}{4}+k)}(x)}|$$
$$= \sum_{n\in\mathbb{Z}} |\chi_{[\frac{n}{4},\frac{3}{4}+\frac{n}{4})}(x)|^2 \geqslant 1.$$

This gives

$$\inf_{x\in[0,\frac{1}{4}]} \left[\sum_{n\in\mathbb{Z}} |\psi_1(x-\frac{n}{4})|^2 - \sum_{k\neq 0} |\sum_{n\in\mathbb{Z}} \psi_1(x-\frac{n}{4})\overline{\psi_1(x-\frac{n}{4}-k)}| \right] \geqslant 1 > 0. \tag{7}$$

Next, we compute

$$\sum_{k\in\mathbb{Z}} |\sum_{n\in\mathbb{Z}} \psi_1(x-\frac{n}{4})\overline{\psi_1(x-\frac{n}{4}-k)}| = \sum_{k\in\mathbb{Z}} |\sum_{n\in\mathbb{Z}} \chi_{[0,\frac{3}{4})}(x-\frac{n}{4})\overline{\chi_{[0,\frac{3}{4})}(x-\frac{n}{4}-k)}|$$
$$= \sum_{k\in\mathbb{Z}} |\sum_{n\in\mathbb{Z}} \chi_{[\frac{n}{4},\frac{3}{4}+\frac{n}{4})}(x)\overline{\chi_{[\frac{n}{4}+k,\frac{3}{4}+\frac{n}{4}+k)}(x)}|$$
$$= |\sum_{n\in\mathbb{Z}} |\chi_{[\frac{n}{4},\frac{3}{4}+\frac{n}{4})}(x)|^2| \leqslant 5.$$

Therefore

$$\sup_{x\in[0,\frac{1}{4}]} \sum_{k\in\mathbb{Z}} |\sum_{n\in\mathbb{Z}} \psi_1(x-\frac{n}{4})\overline{\psi_1(x-\frac{n}{4}-k)}| \leqslant 5 < \infty. \tag{8}$$

Therefore, by using (7), (8) and Theorem 9.1.5 in [4, p. 201], the system $\{E_m T_{\frac{1}{4}k}\psi_1\}_{k,m\in\mathbb{Z}}$ and hence $\{T_{\frac{1}{4}k}E_m\psi_1\}_{k,m\in\mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$. Thus, by using the fact that the dilation operator is unitary, the system $\{D_2 T_{\frac{1}{4}k}E_m\psi_1\}_{k,m\in\mathbb{Z}}$ is a RWH wave packet frame for $L^2(\mathbb{R})$.

Let S be the frame operator for $\{T_{\frac{1}{4}k}E_m\psi_1\}_{k,m\in\mathbb{Z}}$. Then, for any $k',m'\in\mathbb{Z}$, we have

(*i*) $ST_{\frac{1}{4}k'}f = T_{\frac{1}{4}k'}Sf$ for all $f\in L^2(\mathbb{R})$, that is, S and $T_{\frac{1}{4}k'}$ commute.

(*ii*) $SE_{m'}f = E_{m'}Sf$ for all $f\in L^2(\mathbb{R})$, that is, S and $E_{m'}$ commute.

Indeed, for all $f\in L^2(\mathbb{R})$, we have

$$\begin{aligned}
ST_{\frac{1}{4}k'}f &= T_{\frac{1}{4}k'}(T_{\frac{1}{4}k'})^{-1}ST_{\frac{1}{4}k'}f \\
&= T_{\frac{1}{4}k'}T_{-\frac{1}{4}k'} \sum_{k,m\in\mathbb{Z}} \langle T_{\frac{1}{4}k'}f, T_{\frac{1}{4}k}E_m\psi_1\rangle T_{\frac{1}{4}k}E_m\psi_1 \\
&= T_{\frac{1}{4}k'} \sum_{k,m\in\mathbb{Z}} \langle f, T_{\frac{1}{4}(k-k')}E_m\psi_1\rangle T_{\frac{1}{4}(k-k')}E_m\psi_1 \\
&= T_{\frac{1}{4}k'} \sum_{k,m\in\mathbb{Z}} \langle f, T_{\frac{1}{4}k}E_m\psi_1\rangle T_{\frac{1}{4}k}E_m\psi_1 \\
&= T_{\frac{1}{4}k'}Sf.
\end{aligned}$$

Similarly, we can show that $SE_{m'}f = E_{m'}Sf$ for all $f \in L^2(\mathbb{R})$. Hence (i) and (ii) holds.

By using (i) and (ii), we compute

$$S^{-1}T_{\frac{1}{4}k}E_m\psi_1 = (E_{-m}T_{-\frac{1}{4}k}S)^{-1}\psi_1 = (SE_{-m}T_{-\frac{1}{4}k})^{-1}\psi_1 = T_{\frac{1}{4}k}E_mS^{-1}\psi_1.$$

Choose $\widetilde{\psi_1} = S^{-1}\psi_1$.

Then, $\{D_2T_{\frac{1}{4}k}E_m\psi_1\}_{k,m\in\mathbb{Z}}$ is a RWH wave packet frame for $L^2(\mathbb{R})$ with a dual frame $\{D_2T_{\frac{1}{4}k}E_m\widetilde{\psi_1}\}_{k,m\in\mathbb{Z}} = \{D_2T_{\frac{1}{4}k}E_mS^{-1}\psi_1\}_{k,m\in\mathbb{Z}}$. Indeed, for any $f \in L^2(\mathbb{R})$, we have

$$\begin{aligned} f &= D_2D_{2^{-1}}f \\ &= D_2\sum_{k,m\in\mathbb{Z}}\langle D_{2^{-1}}f, T_{\frac{1}{4}k}E_m\widetilde{\psi_1}\rangle T_{\frac{1}{4}k}E_m\psi_1 \\ &= \sum_{k,m\in\mathbb{Z}}\langle f, D_2T_{\frac{1}{4}k}E_m\widetilde{\psi_1}\rangle D_2T_{\frac{1}{4}k}E_m\psi_1. \end{aligned}$$

Thus, in particular, we have a pair of RWH wave packet Bessel sequences $\{D_2T_{\frac{1}{4}k}E_m\psi_1\}_{k,m\in\mathbb{Z}}$ and $\{D_2T_{\frac{1}{4}k}E_m\widetilde{\psi}_1\}_{k,m\in\mathbb{Z}}$ in $L^2(\mathbb{R})$ with pre-frame operators T and U (say), respectively. Furthermore, for any $f \in L^2(\mathbb{R})$, we have

$$\begin{aligned} TU^*(f) &= T(\{\langle f, D_2T_{\frac{1}{4}k}E_m\widetilde{\psi_1}\rangle\}_{k,m\in\mathbb{Z}}) \\ &= \sum_{k,m\in\mathbb{Z}}\langle f, D_2T_{\frac{1}{4}k}E_m\widetilde{\psi_1}\rangle D_2T_{\frac{1}{4}k}E_m\psi_1 \\ &= f. \end{aligned}$$

Thus, $TU^* = I_{L^2(\mathbb{R})}$, i. e., $\Xi = I - UT^* = 0$.

Choose $\psi_2 = \chi_{[0,2]}, \widetilde{\psi}_2 = 0$. Then, $\{D_2T_{\frac{1}{4}k}E_m\psi_1\}_{k,m\in\mathbb{Z}} \bigcup \{D_2T_{\frac{1}{4}k}E_m\psi_2\}_{k,m\in\mathbb{Z}}$ and $\{D_2T_{\frac{1}{4}k}E_m\widetilde{\psi_1}\}_{k,m\in\mathbb{Z}} \bigcup \{D_2T_{\frac{1}{4}k}E_m\widetilde{\psi_2}\}_{k,m\in\mathbb{Z}}$ constitute dual RWH wave packet frames for $L^2(\mathbb{R})$. But, for any $\phi \in L^2(\mathbb{R})$, we have $\psi_2 \neq \Xi^*(\phi) = 0$. Hence there is no $\phi \in L^2(\mathbb{R})$ which satisfies all the conditions in Theorem 1 such that $\psi_2 = \Xi^*(\phi)$.

The next two results gives sufficient conditions for the extension of a pair of RWH wave packet Bessel sequences to RWH wave packet dual frames for $L^2(\mathbb{R})$, where the window function ϕ belongs to a special class of functions. First we recall basic symbols and notations. For $\mathscr{Y} \subset L^2(\mathbb{R})$, define $S_b(\mathscr{Y}) = \overline{\text{span}}\{T_{bk}\phi : \phi \in \mathscr{Y}, k \in \mathbb{Z}\}$ and $W_c(\mathscr{Y}) = \overline{\text{span}}\{E_{cm}\phi : \phi \in \mathscr{Y}, m \in \mathbb{Z}\}$.

THEOREM 2. *Let* $\{D_{a^j}T_{bk}E_{cm}\psi_1\}_{j,k,m\in\mathbb{Z}}$ *and* $\{D_{a^j}T_{bk}E_{cm}\widetilde{\psi_1}\}_{j,k,m\in\mathbb{Z}}$ *be* RWH *wave packet Bessel sequences in* $L^2(\mathbb{R})$ *with pre-frame operators* T *and* U *and analysis operators* T^* *and* U^**, respectively. Assume there exist* $\phi \in L^2(\mathbb{R})$ *such that*

(i) $\{D_{a^j}T_{bk}E_{cm}\phi\}_{j,k,m\in\mathbb{Z}}$ *is a* RWH *wave packet frame with a dual* $\{D_{a^j}T_{bk}E_{cm}\widetilde{\phi}\}_{j,k,m\in\mathbb{Z}}$,

(ii) $TU^*T_{bk}\phi = T_{bk}TU^*\phi$,

(*iii*) $\phi \in L^2(\mathbb{R}) \ominus W_c(\{D_{a^j}T_{bk}E_{cm}\widetilde{\psi_1} : j > 0, k, m \in \mathbb{Z}\})$.

Then, there exist $\psi_2, \widetilde{\psi_2} \in L^2(\mathbb{R})$ *such that* $\{D_{a^j}T_{bk}E_{cm}\psi_1\}_{j,k,m\in\mathbb{Z}} \bigcup \{D_{a^j}T_{bk}E_{cm}\psi_2\}_{j,k,m\in\mathbb{Z}}$ *and* $\{D_{a^j}T_{bk}E_{cm}\widetilde{\psi_1}\}_{j,k,m\in\mathbb{Z}} \bigcup \{D_{a^j}T_{bk}E_{cm}\widetilde{\psi_2}\}_{j,k,m\in\mathbb{Z}}$ *form dual* RWH *wave packet frames for* $L^2(\mathbb{R})$.

Proof. Since $\phi \in L^2(\mathbb{R}) \ominus W_c(\{D_{a^j}T_{bk}E_{cm}\widetilde{\psi_1} : j > 0, k, m \in \mathbb{Z}\})$, we have

$$\langle \phi, E_{cm'}D_{a^j}T_{bk}E_{cm}\widetilde{\psi_1}\rangle = 0 \text{ for all } j \in \mathbb{Z}^+, k, m, m' \in \mathbb{Z}. \tag{9}$$

By using (9), we compute

$$\begin{aligned}
TU^*E_{cm'}\phi &= \sum_{j,k,m\in\mathbb{Z}} \langle E_{cm'}\phi, D_{a^j}T_{bk}E_{cm}\widetilde{\psi_1}\rangle D_{a^j}T_{bk}E_{cm}\psi_1 \\
&= \sum_{j,k,m\in\mathbb{Z}} \langle \phi, E_{-cm'}D_{a^j}T_{bk}E_{cm}\widetilde{\psi_1}\rangle D_{a^j}T_{bk}E_{cm}\psi_1 \\
&= \sum_{j\leqslant 0,k,m\in\mathbb{Z}} \langle \phi, E_{-cm'}D_{a^j}T_{bk}E_{cm}\widetilde{\psi_1}\rangle D_{a^j}T_{bk}E_{cm}\psi_1 \\
&= E_{cm'}E_{-cm'} \sum_{j\leqslant 0,k,m\in\mathbb{Z}} \langle \phi, E_{-cm'}D_{a^j}T_{bk}E_{cm}\widetilde{\psi_1}\rangle D_{a^j}T_{bk}E_{cm}\psi_1 \\
&= E_{cm'} \sum_{j\leqslant 0,k,m\in\mathbb{Z}} \langle \phi, E_{-cm'}D_{a^j}T_{bk}E_{cm}\widetilde{\psi_1}\rangle E_{-cm'}D_{a^j}T_{bk}E_{cm}\psi_1 \\
&= E_{cm'} \sum_{j\leqslant 0,k,m\in\mathbb{Z}} \langle \phi, D_{a^j}E_{\frac{-cm'}{a^j}}T_{bk}E_{cm}\widetilde{\psi_1}\rangle D_{a^j}E_{\frac{-cm'}{a^j}}T_{bk}E_{cm}\psi_1 \\
&= E_{cm'} \sum_{j\leqslant 0,k,m\in\mathbb{Z}} \left\langle \phi, \exp\left(2\pi i\left(\frac{-cm'}{a^j}\right)bk\right)D_{a^j}T_{bk}E_{\frac{-cm'}{a^j}}E_{cm}\widetilde{\psi_1}\right\rangle \\
&\quad \times \exp\left(2\pi i\left(\frac{-cm'}{a^j}\right)bk\right)D_{a^j}T_{bk}E_{\frac{-cm'}{a^j}}E_{cm}\psi_1 \\
&= E_{cm'} \sum_{j\leqslant 0,k,m\in\mathbb{Z}} \langle \phi, D_{a^j}T_{bk}E_{\frac{-cm'}{a^j}}E_{cm}\widetilde{\psi_1}\rangle D_{a^j}T_{bk}E_{\frac{-cm'}{a^j}}E_{cm}\psi_1 \\
&= E_{cm'} \sum_{j\leqslant 0,k,m\in\mathbb{Z}} \langle \phi, D_{a^j}T_{bk}E_{c(m-\frac{m'}{a^j})}\widetilde{\psi_1}\rangle D_{a^j}T_{bk}E_{c(m-\frac{m'}{a^j})}\psi_1 \\
&= E_{cm'} \sum_{j,k,m\in\mathbb{Z}} \langle \phi, D_{a^j}T_{bk}E_{cm}\widetilde{\psi_1}\rangle D_{a^j}T_{bk}E_{cm}\psi_1 \\
&= E_{cm'}TU^*\phi.
\end{aligned}$$

The result follows from Theorem 1. □

The conditions given in Theorem 2 are only sufficient but not necessary. This is justified in the following example. First we note that for all $n, m \in \mathbb{Z}$, we have $T_nE_m = E_mT_n$. Indeed, let $f \in L^2(\mathbb{R})$ and $\gamma \in \mathbb{R}$ be arbitrary. Then, we have

$$(T_nE_mf)(\gamma) = (E_mf)(\gamma - n) = \exp^{2\pi i m(\gamma-n)} f(\gamma - n) = \exp^{2\pi i m\gamma} f(\gamma - n) = (E_mT_nf)(\gamma).$$

Hence $T_nE_m = E_mT_n$ for all $n, m \in \mathbb{Z}$.

EXAMPLE 4. Let $a=b=c=1$, $\psi_1=\chi_{[\frac{1}{2},1]}$ and $\widetilde{\psi_1}=\chi_{[0,1]}$.

Then, $\{D_1T_kE_m\psi_1\}_{k,m\in\mathbb{Z}}=\{T_kE_m\psi_1\}_{k,m\in\mathbb{Z}}$, $\{D_1T_kE_m\widetilde{\psi}_1\}_{k,m\in\mathbb{Z}}=\{T_kE_m\widetilde{\psi}_1\}_{k,m\in\mathbb{Z}}$ is a pair of RWH wave packet Bessel sequences in $L^2(\mathbb{R})$, see [4, p. 204]. Similarly, if we choose $\psi_2=\chi_{[0,\frac{1}{2})}$, $\widetilde{\psi}_2=\chi_{[0,1]}$, then $\{D_1T_kE_m\psi_2\}_{k,m\in\mathbb{Z}}=\{T_kE_m\psi_2\}_{k,m\in\mathbb{Z}}$, $\{D_1T_kE_m\widetilde{\psi}_2\}_{k,m\in\mathbb{Z}}=\{T_kE_m\widetilde{\psi}_2\}_{k,m\in\mathbb{Z}}$ is a pair of RWH wave packet Bessel sequences in $L^2(\mathbb{R})$.

We compute

$$\begin{aligned}
f &= \sum_{k,m\in\mathbb{Z}}\langle f,T_kE_m\chi_{[0,1]}\rangle T_kE_m\chi_{[0,1]}\\
&= \sum_{k,m\in\mathbb{Z}}\langle f,T_kE_m(\chi_{[0,\frac{1}{2})}+\chi_{[\frac{1}{2},1]})\rangle T_kE_m\chi_{[0,1]}\\
&= \sum_{k,m\in\mathbb{Z}}(\langle f,T_kE_m\chi_{[0,\frac{1}{2})}\rangle+\langle f,T_kE_m\chi_{[\frac{1}{2},1]}\rangle)T_kE_m\chi_{[0,1]}\\
&= \sum_{k,m\in\mathbb{Z}}\langle f,T_kE_m\chi_{[0,\frac{1}{2})}\rangle T_kE_m\chi_{[0,1]}+\sum_{k,m\in\mathbb{Z}}\langle f,T_kE_m\chi_{[\frac{1}{2},1]}\rangle T_kE_m\chi_{[0,1]}\\
&= \sum_{k,m\in\mathbb{Z}}\langle f,T_kE_m\psi_2\rangle T_kE_m\widetilde{\psi}_2+\sum_{k,m\in\mathbb{Z}}\langle f,T_kE_m\psi_1\rangle T_kE_m\widetilde{\psi}_1.
\end{aligned}$$

Hence by Lemma 5.6.2 in [3, p. 112] the pair of Bessel sequences

$$\{T_kE_m\psi_1\}_{k,m\in\mathbb{Z}}\bigcup\{T_kE_m\psi_2\}_{k,m\in\mathbb{Z}}\quad\text{and}\quad\{T_kE_m\widetilde{\psi_1}\}_{k,m\in\mathbb{Z}}\bigcup\{T_kE_m\widetilde{\psi_2}\}_{k,m\in\mathbb{Z}}$$

constitute dual RWH wave packet frames for $L^2(\mathbb{R})$.

Next, we compute

$$\begin{aligned}
W_1(\{T_kE_m\widetilde{\psi_1}:k,m\in\mathbb{Z}\}) &= \overline{\operatorname{span}}\{E_{m'}T_kE_m\widetilde{\psi_1}:m',k,m\in\mathbb{Z}\}\\
&= \overline{\operatorname{span}}\{E_{m'}E_mT_k\widetilde{\psi_1}:m',k,m\in\mathbb{Z}\}\\
&= \overline{\operatorname{span}}\{E_{m+m'}T_k\widetilde{\psi_1}:m',k,m\in\mathbb{Z}\}\\
&= \overline{\operatorname{span}}\{E_mT_k\widetilde{\psi_1}:k,m\in\mathbb{Z}\}\\
&= \overline{\operatorname{span}}\{T_kE_m\widetilde{\psi_1}:k,m\in\mathbb{Z}\}\\
&= \overline{\operatorname{span}}\{T_kE_m\chi_{[0,1]}:k,m\in\mathbb{Z}\}\\
&= L^2(\mathbb{R}).
\end{aligned}$$

This gives

$$L^2(\mathbb{R})\ominus W_1(\{T_kE_m\widetilde{\psi_1}:k,m\in\mathbb{Z}\})=\{0\}.$$

Therefore, only choice for ϕ is 0. Thus, conditions given in Theorem 2 are not satisfied.

THEOREM 3. *Let* $\{D_{a^j}T_{bk}E_{cm}\psi_1\}_{j,k,m\in\mathbb{Z}}$ *and* $\{D_{a^j}T_{bk}E_{cm}\widetilde{\psi_1}\}_{j,k,m\in\mathbb{Z}}$ *be* RWH *wave packet Bessel sequences in* $L^2(\mathbb{R})$ *with pre-frame operators* T *and* U *and analysis operators* T^* *and* U^**, respectively. Suppose that there exist* $\phi\in L^2(\mathbb{R})$ *such that*

(i) $\{D_{a^j}T_{bk}E_{cm}\phi\}_{j,k,m\in\mathbb{Z}}$ *is a* RWH *wave packet frame with a dual* $\{D_{a^j}T_{bk}E_{cm}\widetilde{\phi}\}_{j,k,m\in\mathbb{Z}}$,

(ii) $TU^*E_{cm}\phi = E_{cm}TU^*\phi$,

(iii) $\phi \in L^2(\mathbb{R}) \ominus S_b(\{D_{a^j}T_{bk}E_{cm}\widetilde{\psi_1} : j < 0, k, m \in \mathbb{Z}\})$.

Then, there exist $\psi_2, \widetilde{\psi_2} \in L^2(\mathbb{R})$ *such that* $\{D_{a^j}T_{bk}E_{cm}\psi_1\}_{j,k,m\in\mathbb{Z}} \bigcup \{D_{a^j}T_{bk}E_{cm}\psi_2\}_{j,k,m\in\mathbb{Z}}$ *and* $\{D_{a^j}T_{bk}E_{cm}\widetilde{\psi_1}\}_{j,k,m\in\mathbb{Z}} \bigcup \{D_{a^j}T_{bk}E_{cm}\widetilde{\psi_2}\}_{j,k,m\in\mathbb{Z}}$ *form a pair of* RWH *wave packet dual frames for* $L^2(\mathbb{R})$.

Proof. By using condition (iii), we compute

$$\begin{aligned}
TU^*T_{bk'}\phi &= \sum_{j,k,m\in\mathbb{Z}} \langle T_{bk'}\phi, D_{a^j}T_{bk}E_{cm}\widetilde{\psi_1}\rangle D_{a^j}T_{bk}E_{cm}\psi_1 \\
&= \sum_{j,k,m\in\mathbb{Z}} \langle \phi, T_{-bk'}D_{a^j}T_{bk}E_{cm}\widetilde{\psi_1}\rangle D_{a^j}T_{bk}E_{cm}\psi_1 \\
&= \sum_{j\geqslant 0,k,m\in\mathbb{Z}} \langle \phi, T_{-bk'}D_{a^j}T_{bk}E_{cm}\widetilde{\psi_1}\rangle D_{a^j}T_{bk}E_{cm}\psi_1 \\
&= T_{bk'}T_{-bk'} \sum_{j\geqslant 0,k,m\in\mathbb{Z}} \langle \phi, T_{-bk'}D_{a^j}T_{bk}E_{cm}\widetilde{\psi_1}\rangle D_{a^j}T_{bk}E_{cm}\psi_1 \\
&= T_{bk'} \sum_{j\geqslant 0,k,m\in\mathbb{Z}} \langle \phi, T_{-bk'}D_{a^j}T_{bk}E_{cm}\widetilde{\psi_1}\rangle T_{-bk'}D_{a^j}T_{bk}E_{cm}\psi_1 \\
&= T_{bk'} \sum_{j\geqslant 0,k,m\in\mathbb{Z}} \langle \phi, D_{a^j}T_{-bk'a^j}T_{bk}E_{cm}\widetilde{\psi_1}\rangle D_{a^j}T_{-bk'a^j}T_{bk}E_{cm}\psi_1 \\
&= T_{bk'} \sum_{j\geqslant 0,k,m\in\mathbb{Z}} \langle \phi, D_{a^j}T_{b(k-a^jk')}E_{cm}\widetilde{\psi_1}\rangle D_{a^j}T_{b(k-a^jk')}E_{cm}\psi_1 \\
&= T_{bk'} \sum_{j,k,m\in\mathbb{Z}} \langle \phi, D_{a^j}T_{bk}E_{cm}\widetilde{\psi_1}\rangle D_{a^j}T_{bk}E_{cm}\psi_1 \\
&= T_{bk'}TU^*\phi.
\end{aligned}$$

This gives $TU^*T_{bk}\phi = T_{bk}TU^*\phi$. Hence by using condition (i), (ii) and Theorem 1, the result is proved. □

REMARK 1. The result given in Theorem 3 is an extension of Theorem 4.2 in [8].

REMARK 2. The conditions in Theorem 3 are sufficient but not necessary. Indeed, consider a system of RWH wave packet Bessel sequences $\{D_1T_kE_m\psi_1\}_{k,m\in\mathbb{Z}}$ and $\{D_1T_kE_m\widetilde{\psi_1}\}_{k,m\in\mathbb{Z}}$ given in Example 4.

First we compute

$$\begin{aligned}
S_1(\{T_kE_m\widetilde{\psi_1} : k, m \in \mathbb{Z}\}) &= \overline{\text{span}}\{T_lT_kE_m\widetilde{\psi_1} : l, k, m \in \mathbb{Z}\} \\
&= \overline{\text{span}}\{T_{l+k}E_m\widetilde{\psi_1} : l, k, m \in \mathbb{Z}\} \\
&= \overline{\text{span}}\{T_kE_m\widetilde{\psi_1} : k, m \in \mathbb{Z}\} \\
&= \overline{\text{span}}\{T_kE_m\chi_{[0,1]} : k, m \in \mathbb{Z}\} \\
&= L^2(\mathbb{R}).
\end{aligned}$$

We need a $\phi \in L^2(\mathbb{R})$ such that all three conditions $(i)-(iii)$ in Theorem 3 are satisfied.

But

$$L^2(\mathbb{R}) \ominus S_1(\{T_k E_m \widetilde{\psi_1} : k,m \in \mathbb{Z}\}) = \{0\},$$

so zero is the only possible choice for ϕ. Therefore, the condition (i) of Theorem 3 is not satisfied.

Next we observe that if $\phi \in L^2(\mathbb{R}) \ominus W_c(\{D_{a^j} T_{bk} E_{cm} \widetilde{\psi_1} : j > 0, k,m \in \mathbb{Z}\})$ and $\phi \in L^2(\mathbb{R}) \ominus S_b(\{D_{a^j} T_{bk} E_{cm} \widetilde{\psi_1} : j < 0, k,m \in \mathbb{Z}\})$, then $TU^* E_{cm}\phi = E_{cm} TU^*\phi$ and $TU^* T_{bk}\phi = T_{bk} TU^*\phi$.

Therefore

$$TU^* T_{bk} E_{cm}\phi = T_{bk} TU^* E_{cm}\phi = T_{bk} E_{cm} TU^*\phi.$$

So the condition (ii) of Theorem 1 is satisfied. Thus, we have a consequence of Theorem 2 and Theorem 3 which is given in following theorem.

THEOREM 4. *Let $\{D_{a^j} T_{bk} E_{cm}\psi_1\}_{j,k,m\in\mathbb{Z}}$ and $\{D_{a^j} T_{bk} E_{cm}\widetilde{\psi_1}\}_{j,k,m\in\mathbb{Z}}$ be* RWH *wave packet Bessel sequences in $L^2(\mathbb{R})$ with pre-frame operators T and U and analysis operators T^* and U^*, respectively. Assume there exist $\phi \in L^2(\mathbb{R})$ such that*

(i) $\{D_{a^j} T_{bk} E_{cm}\phi\}_{j,k,m\in\mathbb{Z}}$ is a RWH *wave packet frame with a dual $\{D_{a^j} T_{bk} E_{cm}\widetilde{\phi}\}_{j,k,m\in\mathbb{Z}}$,*

(ii) $\phi \in L^2(\mathbb{R}) \ominus S_b(\{D_{a^j} T_{bk} E_{cm}\widetilde{\psi_1} : j < 0, k,m \in \mathbb{Z}\})$,

(iii) $\phi \in L^2(\mathbb{R}) \ominus W_c(\{D_{a^j} T_{bk} E_{cm}\widetilde{\psi_1} : j > 0, k,m \in \mathbb{Z}\})$.

Then, there exist $\psi_2, \widetilde{\psi_2} \in L^2(\mathbb{R})$ such that $\{D_{a^j} T_{bk} E_{cm}\psi_1\}_{j,k,m\in\mathbb{Z}} \bigcup \{D_{a^j} T_{bk} E_{cm}\psi_2\}_{j,k,m\in\mathbb{Z}}$ and $\{D_{a^j} T_{bk} E_{cm}\widetilde{\psi_1}\}_{j,k,m\in\mathbb{Z}} \bigcup \{D_{a^j} T_{bk} E_{cm}\widetilde{\psi_2}\}_{j,k,m\in\mathbb{Z}}$ form dual RWH *wave packet frames for $L^2(\mathbb{R})$.*

REMARK 3. Consider the system of RWH wave packet Bessel sequences $\{D_1 T_k E_m \psi_1\}_{k,m\in\mathbb{Z}}$ and $\{D_1 T_k E_m \widetilde{\psi}_1\}_{k,m\in\mathbb{Z}}$ given in Example 4. We have,

$$L^2(\mathbb{R}) \ominus W_1(\{D_{1^j} T_k E_m \widetilde{\psi_1} : j = 1, k,m \in \mathbb{Z}\}) = \{0\}.$$

Thus, there is no ϕ such that all the conditions $(i)-(iii)$ of Theorem 4 holds. Hence conditions given in Theorem 4 are sufficient but not necessary.

To conclude the paper we provide sufficient conditions for extension problem in terms of compact support of window functions.

THEOREM 5. *For $a \geqslant 2$, assume there exists a $\phi \in L^2(\mathbb{R})$ such that*

(i) $\{D_{a^j} T_{bk} E_{cm}\phi\}_{j,k,m\in\mathbb{Z}}$ is a RWH *wave packet frame with a dual $\{D_{a^j} T_{bk} E_{cm}\widetilde{\phi}\}_{j,k,m\in\mathbb{Z}}$,*

(ii) $supp\phi \subseteq [-N_o, N_o] \setminus \left(\frac{-1}{2}, \frac{1}{2}\right)$, $supp\widehat{\phi} \subseteq [-M_o, M_o] \setminus \left(\frac{-1}{2}, \frac{1}{2}\right)$ for some $N_o, M_o \in \mathbb{N}$,

(*iii*) $|b| > 2N_o+1, |c| > 2M_o+1$.

Then, for any RWH *wave packet Bessel sequences* $\{D_{a^j}T_{bk}E_{cm}\psi_1\}_{j,k,m\in\mathbb{Z}}$ *and* $\{D_{a^j}T_{bk}E_{cm}\widetilde{\psi_1}\}_{j,k,m\in\mathbb{Z}}$ *in* $L^2(\mathbb{R})$ *with supp* $\widetilde{\psi_1} \subseteq [-1,1]$, *supp* $\widehat{\widetilde{\psi_1}} \subseteq [-1,1]$, *there exist* $\psi_2, \widetilde{\psi_2} \in L^2(\mathbb{R})$ *such that*

$$\{D_{a^j}T_{bk}E_{cm}\psi_1\}_{j,k,m\in\mathbb{Z}} \bigcup \{D_{a^j}T_{bk}E_{cm}\psi_2\}_{j,k,m\in\mathbb{Z}}$$

and

$$\{D_{a^j}T_{bk}E_{cm}\widetilde{\psi_1}\}_{j,k,m\in\mathbb{Z}} \bigcup \{D_{a^j}T_{bk}E_{cm}\widetilde{\psi_2}\}_{j,k,m\in\mathbb{Z}}$$

form dual RWH *wave packet frames for* $L^2(\mathbb{R})$.

Proof. It suffices to show $\phi \in L^2(\mathbb{R}) \ominus S_b(\{D_{a^j}T_{bk}E_{cm}\widetilde{\psi_1} : j<0, k,m\in\mathbb{Z}\})$ and $\phi \in L^2(\mathbb{R}) \ominus W_c(\{D_{a^j}T_{bk}E_{cm}\widetilde{\psi_1} : j>0, k,m\in\mathbb{Z}\})$. The result then follows from Theorem 4.

We denote the Fourier transform of $T_{bk'}D_{a^j}T_{bk}E_{cm}\widetilde{\psi_1}$ by $\mathscr{F}(T_{bk'}D_{a^j}T_{bk}E_{cm}\widetilde{\psi_1})$. Consider

$$\begin{aligned}\mathscr{F}(T_{bk'}D_{a^j}T_{bk}E_{cm}\widetilde{\psi_1})(\gamma) &= E_{-bk'}D_{a^{-j}}E_{-bk}T_{cm}\widehat{\widetilde{\psi_1}}(\gamma)\\ &= a^{\frac{-j}{2}}\exp^{-2\pi ib\gamma(k'+a^{-j}k)}\,\widehat{\widetilde{\psi_1}}(a^{-j}\gamma - cm).\end{aligned}$$

For $j<0,\ k',k,m\in\mathbb{Z}$, we have

$$\begin{aligned}\operatorname{supp}(\mathscr{F}(T_{bk'}D_{a^j}T_{bk}E_{cm}\widetilde{\psi_1}))(\bullet) &= \operatorname{supp}\widehat{\widetilde{\psi_1}}\left(\frac{\bullet}{a^j} - cm\right)\\ &\subseteq [a^j(-1+cm), a^j(1+cm)]\\ &\subseteq \left[\frac{-1+cm}{2}, \frac{1+cm}{2}\right].\end{aligned}$$

Therefore, for $m=0$ we have

$$\operatorname{supp}(\mathscr{F}(T_{bk'}D_{a^j}T_{bk}E_{cm}\widetilde{\psi_1})) \subseteq \left[\frac{-1}{2}, \frac{1}{2}\right]. \tag{10}$$

If $m \geqslant 1$, then for $c > 2M_o+1$ (which gives $\frac{-1+cm}{2} > \frac{-1+2M_o+1}{2} = M_o$), we have

$$\operatorname{supp}(\mathscr{F}(T_{bk'}D_{a^j}T_{bk}E_{cm}\widetilde{\psi_1})) \subseteq (M_o, \infty).$$

Similarly, for $c < -(2M_o+1)$, we have $\frac{1+cm}{2} < \frac{1-(2M_o+1)}{2} = -M_o$.

This gives

$$\operatorname{supp}(\mathscr{F}(T_{bk'}D_{a^j}T_{bk}E_{cm}\widetilde{\psi_1})) \subseteq (-\infty, -M_o).$$

Therefore

$$\mathrm{supp}(\mathscr{F}(T_{bk'}D_{a^j}T_{bk}E_{cm}\widetilde{\psi_1})) \subseteq \mathbb{R}\setminus(-M_o, M_o). \tag{11}$$

Similarly for $m \leqslant -1$, we can show that

$$\mathrm{supp}(\mathscr{F}(T_{bk'}D_{a^j}T_{bk}E_{cm}\widetilde{\psi_1})) \subseteq \mathbb{R}\setminus(-M_o, M_o). \tag{12}$$

By using (10), (11), (12) and the Plancheral's equation, we have

$$\begin{aligned}\langle \phi, T_{bk'}D_{a^j}T_{bk}E_{cm}\widetilde{\psi_1}\rangle &= \langle \widehat{\phi}, \mathscr{F}(T_{bk'}D_{a^j}T_{bk}E_{cm}\widetilde{\psi_1})\rangle \\ &= 0 \ \text{ for all } \ j<0, m, k, k' \in \mathbb{Z}.\end{aligned}$$

Therefore, $\phi \in L^2(\mathbb{R}) \ominus S_b(\{D_{a^j}T_{bk}E_{cm}\widetilde{\psi_1} : j<0, k, m \in \mathbb{Z}\})$.

Note that $(E_{cm'}D_{a^j}T_{bk}E_{cm}\widetilde{\psi_1})(\gamma) = a^{\frac{j}{2}}\exp^{2\pi ic(\gamma(m'+a^jm)-mbk)}\widehat{\widetilde{\psi_1}}(a^j\gamma - bk)$.

For $j>0$, $k, m, m' \in \mathbb{Z}$, we compute

$$\begin{aligned}\mathrm{supp}(E_{cm'}D_{a^j}T_{bk}E_{cm}\widetilde{\psi_1})(\bullet) &= \mathrm{supp}\widetilde{\psi_1}(a^j\bullet - bk) \\ &\subseteq \left[\frac{-1+bk}{a^j}, \frac{1+bk}{a^j}\right] \\ &\subseteq \left[\frac{-1+bk}{2}, \frac{1+bk}{2}\right].\end{aligned}$$

This gives

$$\mathrm{supp}(E_{cm'}D_{a^j}T_{bk}E_{cm}\widetilde{\psi_1}) \subseteq \left[\frac{-1}{2}, \frac{1}{2}\right], \text{ for } k=0, \tag{13}$$

and

$$\mathrm{supp}(E_{cm'}D_{a^j}T_{bk}E_{cm}\widetilde{\psi_1}) \subseteq \mathbb{R}\setminus(-N_o, N_o), \text{ for } k\neq 0. \tag{14}$$

The outcomes in (13) and (14) gives

$$\langle \phi, E_{cm'}D_{a^j}T_{bk}E_{cm}\widetilde{\psi_1}\rangle = 0 \text{ for all } j>0, k, m, m' \in \mathbb{Z}.$$

Hence $\phi \in L^2(\mathbb{R}) \ominus W_c(\{D_{a^j}T_{bk}E_{cm}\widetilde{\psi_1} : j>0, k, m \in \mathbb{Z}\})$. The theorem is proved. □

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REFERENCES

[1] P. G. CASAZZA AND G. KUTYNIOK, *Finite Frames: Theory and Applications*, Birkhäuser, 2012.

[2] P. G. CASAZZA AND N. LEONHARD, *Classes of finite equal norm Parseval frames*, Contemporary Mathematics, **451** (2008), 11–13.

[3] O. CHRISTENSEN, *An Introduction to Frames and Riesz Bases*, Birkhäuser, Boston, 2002.

[4] O. CHRISTENSEN, *Frames and Bases: An Introductory Course*, Birkhäuser, Boston, 2008.

[5] O. CHRISTENSEN AND A. RAHIMI, *Frame properties of wave packet systems in $L^2(\mathbb{R}^d)$*, Advances in Computational Mathematics, **29** (2008), 101–111.

[6] O. CHRISTENSEN AND A. RAHIMI, *An introduction to wave packet Systems in $L^2(\mathbb{R})$*, Indian Journal of Industrial and Applied Mathematics, **1** (2008), 42–57.

[7] O. CHRISTENSEN, H. O. KIM AND R. Y. KIM, *On Parseval wavelet frames with two or three generators via the unitary extension principle*, Canadian Mathematical Bulletin, **57** (2) (2014), 254–263.

[8] O. CHRISTENSEN, H. O. KIM AND R. Y. KIM, *Extensions of Bessel sequences to dual pairs of frames*, Applied and Computational Harmonic Analysis, **34** (2013), 224–233.

[9] A. CORDOBA AND C. FEFFERMAN, *Wave packets and Fourier integral operators*, Communications in Partial Differential Equations, **3** (11) (1978), 979–1005.

[10] W. CZAJA, G. KUTYNIOK AND D. SPEEGLE, *The geometry of sets of prameters of wave packets*, Applied and Computational Harmonic Analysis, **20** (2006), 108–125.

[11] I. DAUBECHIES, A. GROSSMANN AND Y. MEYER, *Painless nonorthogonal expansions*, Journal of Mathematical Physics, **27** (1986), 1271–1283.

[12] I. DAUBECHIES, *Ten Lectures on Wavelets*, SIAM, Philadelphia, 1992.

[13] DEEPSHIKHA AND L. K. VASHISHT, *Extension of Bessel sequences to dual frames in Hilbert spaces*, submitted.

[14] R. J. DUFFIN AND A. C. SCHAEFFER, *A class of nonharmonic Fourier series*, Transactions of the American Mathematical Society, **72** (1952), 341–366.

[15] D. GABOR, *Theory of communications*, Journal of Inst. Electrical Engineering, **93** (3) (1946), 429–457.

[16] D. HAN, *Dilations and completions for Gabor systems*, Journal of Fourier Analysis and Applications, **15** (2) (2009), 201–217.

[17] E. HERNÁNDEZ, D. LABATE AND G. WEISS, *A unified characterization of reproducing systems generated by a finite family II*, Journal of Geometric Analysis, **12** (4) (2002), 615–662.

[18] E. HERNÁNDEZ, D. LABATE, G. WEISS AND E. WILSON, *Oversampling, quasi-affine frames and wave packets*, Applied and Computational Harmonic Analysis, **16** (2004), 111–147.

[19] D. LABATE, G. WEISS AND E. WILSON, *An approach to the study of wave packet systems*, Contemporary Mathematics, **345** (2004), 215–235.

[20] M. LACEY AND C. THIELE, *L^p estimates on the bilinear Hilbert transform for $2 < p < \infty$*, Annals of Mathematics, **146** (1997), 693–724.

[21] M. LACEY AND C. THIELE, *On Calderón's conjecture*, Annals of Mathematics, **149** (1999), 475–496.

[22] D. F. LI AND W. SUN, *Expansion of frames to tight frames*, Acta Mathematica Sinica (English Series), **25** (2009), 287–292.

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