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A NEW PROOF FOR A CLASSICAL QUADRATIC HARMONIC SERIES

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To my parents, Ileana and Ionel

Abstract. In the following paper we intend to present a new way of calculating a series similar to the quadratic series of Au-Yeung (see [1])

$$\sum_{n=1}^{\infty}\frac{H_n^2}{n^3}=\frac{7}{2}\zeta(5)-\zeta(2)\zeta(3),$$

where H_n denotes the nth harmonic number. We will prove the result by combining a series of techniques based on the calculation of two special logarithmic integrals, the elementary manipulations of series and then the use of the Euler's identity in (1).

1. Introduction and the main result

The following result

$$\sum_{n=1}^{\infty}\frac{H_n^2}{n^3}=\frac{7}{2}\zeta(5)-\zeta(2)\zeta(3),$$

is one of many such series involving the harmonic number that Philippe Flajolet and Bruno Salvy derived in their paper *Euler Sums and Contour Integral Representations* (see [2]) by means of contour integration. Like the quadratic series of Au-Yeung that appears in [1], the present series has become a classic in the theory of nonlinear harmonic series.

In this paper we calculate a series similar to the quadratic series of Au-Yeung, but a more advanced one where the difference is that in denominator we have now n^3 instead of n^2. Our strategy will involve the calculation of two special logarithmic integrals, the calculation of an Euler sum by symmetry reasons and the use of the Euler's identity involving harmonic numbers.

We state now a classical quadratic harmonic series summation result.

THEOREM 1. *The following equality holds:*

$$\sum_{n=1}^{\infty}\frac{H_n^2}{n^3}=\frac{7}{2}\zeta(5)-\zeta(2)\zeta(3),$$

where H_n is the nth harmonic number defined, for $n \geqslant 1$, by $H_n = 1+\frac{1}{2}+\cdots+\frac{1}{n}$.

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Before we prove Theorem 1 we collect some results we need in our analysis. Next we prove the lemmas which are used in the proof of Theorem 1.

LEMMA 2. *Let $n \geqslant 1$ be an integer. The following equalities hold:*

(a) $I_n = \int_0^1 x^{n-1}\ln(1-x)dx = -\frac{H_n}{n}$;

(b) $J_n = \int_0^1 x^{n-1}\ln^2(1-x)dx = \frac{2}{n}\sum_{k=1}^{n}\frac{H_k}{k} = \frac{H_n^2}{n} + \frac{H_n^{(2)}}{n}$,

where $H_n^{(2)}$ is the nth harmonic number of order 2 defined, for $n \geqslant 1$, by $H_n^{(2)} = 1 + \frac{1}{2^2} + \cdots + \frac{1}{n^2}$.

Proof. (a) We have, using integration by parts, that

$$\begin{aligned} I_n &= \int_0^1 x^{n-1}\ln(1-x)\mathrm{d}x \\ &= (x-1)x^{n-1}\log(1-x)\Big|_{x=0}^{x=1} - \int_0^1 x^{n-1}\mathrm{d}x + (n-1)I_{n-1} - (n-1)I_n \\ &= -\frac{1}{n} + (n-1)I_{n-1} - (n-1)I_n \end{aligned}$$

which yields the recurrence relation in k,

$$kI_k - (k-1)I_{k-1} = -\frac{1}{k}.$$

Giving values to k from $k=2$ to n and using that $\int_0^1 \log(1-x)\mathrm{d}x = -1$, we obtain that

$$I_n = \int_0^1 x^{n-1}\log(1-x)\mathrm{d}x = -\frac{H_n}{n},$$

and the part (a) of the lemma is proved.

(b) We have, using integration by parts as in (a), that

$$\begin{aligned} J_n &= \int_0^1 x^{n-1}\ln^2(1-x)\mathrm{d}x \\ &= (x-1)x^{n-1}\log^2(1-x)\Big|_{x=0}^{x=1} - 2\int_0^1 x^{n-1}\log(1-x)\mathrm{d}x + (n-1)J_{n-1} - (n-1)J_n \\ &= 2\frac{H_n}{n} + (n-1)J_{n-1} - (n-1)J_n, \end{aligned}$$

where above we used the part (a) of the lemma, $I_n = \int_0^1 x^{n-1}\log(1-x)\mathrm{d}x = -\frac{H_n}{n}$.

Then, we obtain the recurrence relation in k,

$$kJ_k - (k-1)J_{k-1} = 2\frac{H_k}{k},$$

where giving values to k from $k=2$ to n and using that $\int_0^1 \log^2(1-x)\mathrm{d}x = 2$, we obtain that

$$J_n = \int_0^1 x^{n-1}\log^2(1-x)\mathrm{d}x = \frac{2}{n}\sum_{k=1}^{n}\frac{H_k}{k} = \frac{H_n^2}{n} + \frac{H_n^{(2)}}{n},$$

and the last equality follows immediately from the fact that,

$$\begin{aligned} H_n^2 = \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{ij} &= 2\sum_{i=1}^{n}\sum_{j=1}^{i}\frac{1}{ij} - \left(1+\frac{1}{2^2}+\cdots+\frac{1}{n^2}\right)\\ &= 2\sum_{i=1}^{n}\frac{H_i}{i} - H_n^{(2)}, \end{aligned}$$

whence we get that

$$\sum_{i=1}^{n}\frac{H_i}{i} = \frac{1}{2}\left(H_n^2 + H_n^{(2)}\right)$$

and the part (b) of the lemma is proved. □

We state now the result of a special Euler sum.

LEMMA 3. *The following equality holds:*

$$\sum_{n=1}^{\infty}\frac{1}{n^3}\left(1+\frac{1}{2^2}+\cdots+\frac{1}{n^2}\right) = 3\zeta(2)\zeta(3) - \frac{9}{2}\zeta(5).$$

Proof. We start with a slightly different series,

$$S = \sum_{n=1}^{\infty}\frac{1}{n^3}\left(\zeta(2)-1-\frac{1}{2^2}-\cdots-\frac{1}{n^2}\right) = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{n^3(n+k)^2},$$

where based upon symmetry reasons we have that $S = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{n^3(n+k)^2} = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k^3(n+k)^2}$.

Summing both sides, we have that

$$\begin{aligned} 2S &= \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{n^3(n+k)^2} + \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k^3(n+k)^2}\\ &= \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{k^3+n^3}{k^3n^3(n+k)^2}\\ &= \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{(k+n)^3-3kn(k+n)}{k^3n^3(n+k)^2}\\ &= \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k^2n^3} + \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k^3n^2} - 3\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k^2n^2(n+k)}\\ &= 2\zeta(2)\zeta(3) - 3\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k^2n^2(n+k)} \end{aligned}$$

whence $S=\zeta(2)\zeta(3)-\frac{3}{2}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k^2n^2(n+k)}$.

Now, we have that

$$\frac{1}{k^2(n+k)}=\frac{1}{n}\left(\frac{1}{k^2}-\frac{1}{k(n+k)}\right),$$

and multiplying both sides by $\frac{1}{n^2}$, we obtain

$$\begin{aligned}\frac{1}{k^2n^2(n+k)}&=\frac{1}{n^3}\left(\frac{1}{k^2}-\frac{1}{k(n+k)}\right)\\&=\frac{1}{n^3}\left(\frac{1}{k^2}-\frac{1}{n}\left(\frac{1}{k}-\frac{1}{n+k}\right)\right)\\&=\frac{1}{k^2n^3}-\frac{1}{n^4}\left(\frac{1}{k}-\frac{1}{n+k}\right).\end{aligned}$$

Therefore, we have that

$$\begin{aligned}S&=\zeta(2)\zeta(3)-\frac{3}{2}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k^2n^2(n+k)}\\&=\zeta(2)\zeta(3)-\frac{3}{2}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\frac{1}{k^2n^3}-\frac{1}{n^4}\left(\frac{1}{k}-\frac{1}{n+k}\right)\right)\\&=\zeta(2)\zeta(3)-\frac{3}{2}\sum_{n=1}^{\infty}\frac{1}{n^3}\sum_{k=1}^{\infty}\frac{1}{k^2}+\frac{3}{2}\sum_{n=1}^{\infty}\frac{1}{n^4}\sum_{k=1}^{\infty}\left(\frac{1}{k}-\frac{1}{n+k}\right)\\&=-\frac{1}{2}\zeta(2)\zeta(3)+\frac{3}{2}\sum_{n=1}^{\infty}\frac{H_n}{n^4}\\&=\frac{9}{2}\zeta(5)-2\zeta(2)\zeta(3)\end{aligned}$$

where we have used Euler's identity (see [3, p. 228])

$$2\sum_{k=1}^{\infty}\frac{H_k}{k^n}=(n+2)\zeta(n+1)-\sum_{k=1}^{n-2}\zeta(n-k)\zeta(k+1),\ n\in\mathbb{N},\ n\geqslant 2. \tag{1}$$

Hence, we obtain that

$$\sum_{n=1}^{\infty}\frac{1}{n^3}\left(\zeta(2)-1-\frac{1}{2^2}-\cdots-\frac{1}{n^2}\right)=\frac{9}{2}\zeta(5)-2\zeta(2)\zeta(3).$$

As a consequence of the result above, since $\zeta(2)\sum_{n=1}^{\infty}\frac{1}{n^3}=\zeta(2)\zeta(3)$, we obtain that

$$\sum_{n=1}^{\infty}\frac{1}{n^3}\left(1+\frac{1}{2^2}+\cdots+\frac{1}{n^2}\right)=3\zeta(2)\zeta(3)-\frac{9}{2}\zeta(5). \quad \square$$

Now we are ready to prove Theorem 1.

Proof. We have, based on part (b) of Lemma 2, that

$$\int_0^1 \frac{x^{n-1}}{n^2}\ln^2(1-x)\mathrm{d}x = \frac{H_n^2}{n^3} + \frac{H_n^{(2)}}{n^3},$$

and it follows that

$$\sum_{n=1}^{\infty}\int_0^1 \frac{x^{n-1}}{n^2}\ln^2(1-x)\mathrm{d}x = \sum_{n=1}^{\infty}\left(\frac{H_n^2}{n^3} + \frac{H_n^{(2)}}{n^3}\right).$$

Then, we obtain that

$$\begin{aligned}\sum_{n=1}^{\infty}\left(\frac{H_n^2}{n^3} + \frac{H_n^{(2)}}{n^3}\right) &= \sum_{n=1}^{\infty}\frac{H_n^2}{n^3} + \sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^3} = \sum_{n=1}^{\infty}\int_0^1 \frac{x^{n-1}}{n^2}\log^2(1-x)\mathrm{d}x \\ &= \int_0^1 \sum_{n=1}^{\infty}\frac{x^{n-1}}{n^2}\log^2(1-x)\mathrm{d}x.\end{aligned} \tag{2}$$

Now, recall the generating function of the harmonic numbers is $-\frac{\log(1-t)}{1-t} = \sum_{n=1}^{\infty} t^n H_n$, $|t| < 1$. If integrating both sides from $t = 0$ to $t = x$, we get that

$$\frac{1}{2}\log^2(1-x) = \int_0^x \sum_{n=1}^{\infty} t^n H_n \mathrm{d}t = \sum_{n=1}^{\infty}\int_0^x t^n H_n \mathrm{d}t = \sum_{n=1}^{\infty}\frac{x^{n+1}}{n+1}H_n. \tag{3}$$

Using (3) in (2), we get that

$$\begin{aligned}\int_0^1 \sum_{k=1}^{\infty}\frac{x^{k-1}}{k^2}\log^2(1-x)\mathrm{d}x &= 2\int_0^1 \sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{x^{k+n}}{k^2(n+1)}H_n\mathrm{d}x \\ &= 2\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\int_0^1 \frac{x^{k+n}}{k^2(n+1)}H_n\mathrm{d}x \\ &= 2\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{H_n}{k^2(n+1)(k+n+1)} \\ &= 2\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{H_n}{k^2(n+1)(k+n+1)}.\end{aligned} \tag{4}$$

Since we have, by partial fraction decomposition, that

$$\begin{aligned}\frac{1}{k^2(k+n+1)} &= \frac{1}{k(n+1)}\left(\frac{1}{k} - \frac{1}{k+n+1}\right) \\ &= \frac{1}{k^2(n+1)} - \frac{1}{(n+1)^2}\left(\frac{1}{k} - \frac{1}{k+n+1}\right),\end{aligned}$$

then

$$\sum_{k=1}^{\infty} \frac{1}{k^2(k+n+1)} = \sum_{k=1}^{\infty} \left(\frac{1}{k^2(n+1)} - \frac{1}{(n+1)^2} \left(\frac{1}{k} - \frac{1}{k+n+1} \right) \right)$$
$$= \frac{\zeta(2)}{n+1} - \frac{H_{n+1}}{(n+1)^2}. \tag{5}$$

Using (5) in (4), we obtain

$$\begin{aligned} 2\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \frac{H_n}{k^2(n+1)(k+n+1)} &= 2\sum_{n=1}^{\infty} \frac{H_n}{n+1} \left(\frac{\zeta(2)}{n+1} - \frac{H_{n+1}}{(n+1)^2} \right) \\ &= 2\zeta(2)\sum_{n=1}^{\infty} \frac{H_n}{(n+1)^2} - 2\sum_{n=1}^{\infty} \frac{H_n H_{n+1}}{(n+1)^3} \\ &= 2\zeta(2)\sum_{n=1}^{\infty} \frac{H_{n+1} - \frac{1}{n+1}}{(n+1)^2} - 2\sum_{n=1}^{\infty} \frac{(H_{n+1} - \frac{1}{n+1})H_{n+1}}{(n+1)^3} \\ &= 2\zeta(2)\sum_{n=1}^{\infty} \frac{H_{n+1}}{(n+1)^2} - 2\zeta(2)\sum_{n=1}^{\infty} \frac{1}{(n+1)^3} \\ &\quad - 2\sum_{n=1}^{\infty} \frac{H_{n+1}^2}{(n+1)^3} + 2\sum_{n=1}^{\infty} \frac{H_{n+1}}{(n+1)^4} \\ &= 2\zeta(2)\sum_{n=1}^{\infty} \frac{H_n}{n^2} - 2\zeta(2)\sum_{n=1}^{\infty} \frac{1}{n^3} - 2\sum_{n=1}^{\infty} \frac{H_n^2}{n^3} + 2\sum_{n=1}^{\infty} \frac{H_n}{n^4} \\ &= 6\zeta(5) - 2\sum_{n=1}^{\infty} \frac{H_n^2}{n^3}, \end{aligned} \tag{6}$$

where we used that $\sum_{n=1}^{\infty} \frac{H_n}{n^2} = 2\zeta(3)$ and $\sum_{n=1}^{\infty} \frac{H_n}{n^4} = 3\zeta(5) - \zeta(2)\zeta(3)$ that are both obtained from Euler's identity in (1).

Thus, combining (6), (4) and (2), we get that

$$\sum_{n=1}^{\infty} \frac{H_n^2}{n^3} = 2\zeta(5) - \frac{1}{3}\sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n^3}. \tag{7}$$

Combining (7) and Lemma 3, we obtain that

$$\sum_{n=1}^{\infty} \frac{H_n^2}{n^3} = \frac{7}{2}\zeta(5) - \zeta(2)\zeta(3),$$

and the theorem is proved. □

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REFERENCES

[1] C. I. VĂLEAN AND O. FURDUI, *Reviving the quadratic series of Au–Yeung*, JCA **6**, (2015), no. 2, 113–118.

[2] P. FLAJOLET AND B. SALVY, *Euler sums and contour integral representations*, Experiment. Math,. **7** (1998), 15–35.

[3] H. M. SRIVASTAVA, J. CHOI, *Zeta and q-Zeta Functions And Associated Series And Integrals*, Elsevier, Amsterdam (2012).

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