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# FURTHER RESULTS ON GENERALISED ITERATION OF ENTIRE FUNCTIONS WITH FINITE ITERATED ORDER

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*Abstract.* We consider generalised iteration of two entire functions and study further growth properties of generalised iterated entire functions of finite iterated order.

## 1. Introduction and definitions

If $f(z)$ and $g(z)$ be two transcendental entire functions, Clunie [4] showed that $\lim_{r\to\infty}\frac{T(r,f\circ g)}{T(r,f)}=\infty$ and $\lim_{r\to\infty}\frac{T(r,f\circ g)}{T(r,g)}=\infty$. After this several authors [7, 8, 9, 10, 11, 14] made close investigations on growth properties of composition of two entire functions with finite order. In 1987, Bernal [3] introduced the notions of finite iterated order and finiteness degree of the order as follows.

DEFINITION 1.1. [3, 6] The iterated $i$ order $\rho_i(f)$ of an entire function $f$ is defined by

$$\rho_i(f)=\limsup_{r\to\infty}\frac{\log^{[i+1]}M(r,f)}{\log r}=\limsup_{r\to\infty}\frac{\log^{[i]}T(r,f)}{\log r},\quad (i\in\mathbb{N}).$$

Similarly, the iterated $i$ lower order $\mu_i(f)$ of an entire function $f$ is defined by

$$\mu_i(f)=\liminf_{r\to\infty}\frac{\log^{[i+1]}M(r,f)}{\log r}=\liminf_{r\to\infty}\frac{\log^{[i]}T(r,f)}{\log r},\quad (i\in\mathbb{N}),$$

where following Sato [12] we write $\log^{[0]}x=x$, $\exp^{[0]}x=x$ and for positive integer $m$, $\log^{[m]}x=\log(\log^{[m-1]}x)$, $\exp^{[m]}x=\exp(\exp^{[m-1]}x)$.

DEFINITION 1.2. [3, 6] The finiteness degree of the order of an entire function $f$ is defined by

$$i(f)=\begin{cases}0 & \text{if } f(z) \text{ is a polynomial;}\\ \min\{k\in\{1,2,\ldots\},\ \rho_k(f)<\infty\} & \text{if } f(z) \text{ is transcendental;}\\ \infty & \text{when } \rho_k(f)=\infty \text{ for all k.}\end{cases}$$

Recently, Jin Tu et. al [13] investigated the growth of two composite entire functions of finite iterated order. In 2012, Banerjee and Mondal [2] introduced a new type of iteration called generalised iteration defined as follows .

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DEFINITION 1.3. [2] Let $f(z)$ and $g(z)$ be entire functions and $\alpha \in (0,1]$ be any real number. Then the generalised iteration of $f(z)$ with respect to $g(z)$ is defined as follows :

$$
\begin{aligned}
f_{1,g}(z) &= (1-\alpha)g_{0,f}(z) + \alpha f(z), \text{ where } g_{0,f}(z) = z,\\
f_{2,g}(z) &= (1-\alpha)g_{1,f}(z) + \alpha f(g_{1,f}(z)),\\
f_{3,g}(z) &= (1-\alpha)g_{2,f}(z) + \alpha f(g_{2,f}(z)),\\
&\dots \quad \dots \quad \dots\\
f_{n,g}(z) &= (1-\alpha)g_{n-1,f}(z) + \alpha f(g_{n-1,f}(z))
\end{aligned}
$$

and so are

$$
\begin{aligned}
g_{1,f}(z) &= (1-\alpha)f_{0,g}(z) + \alpha g(z), \text{ where } f_{0,g}(z) = z,\\
g_{2,f}(z) &= (1-\alpha)f_{1,g}(z) + \alpha g(f_{1,g}(z)),\\
g_{3,f}(z) &= (1-\alpha)f_{2,g}(z) + \alpha g(f_{2,g}(z)),\\
&\dots \quad \dots \quad \dots\\
g_{n,f}(z) &= (1-\alpha)f_{n-1,g}(z) + \alpha g(f_{n-1,g}(z)).
\end{aligned}
$$

Clearly all $f_{n,g}(z)$ and $g_{n,f}(z)$ are entire functions.

With this iteration process introduced by Banerjee and Mondal [2], in a recent paper Banerjee and Mandal [1] proved some results of Jin Tu et. al [13] for generalised iterated entire functions with finite iterated order. In this paper we investigate further growth properties for generalised iterated entire functions with finite iterated order.

Throughout we assume $f(z)$ and $g(z)$ are entire functions having finite iterated order if $\rho_p(f) < \infty$, $\rho_q(g) < \infty$ and positive iterated lower order if $\mu_p(f) > 0$, $\mu_q(g) > 0$. Clearly a function having positive iterated lower order must be transcendental. Also we use the standard notations and definitions of the theory of meromorphic functions which are available in [5].

## 2. Known lemmas

The following known lemmas will be required in the sequel.

LEMMA 2.1. [9] *Let $f(z)$ and $g(z)$ be entire functions. If $M(r,g) > \frac{2+\varepsilon}{\varepsilon}|g(0)|$ for any $\varepsilon > 0$, then*

$$T(r, f(g)) < (1+\varepsilon)T(M(r,g), f).$$

*In particular if $g(0) = 0$, then $T(r, f(g)) \leqslant T(M(r,g), f)$ for all $r > 0$.*

LEMMA 2.2. [4] *Let $f(z)$ and $g(z)$ be entire functions with $g(0) = 0$. Let $\beta$ satisfy $0 < \beta < 1$ and let $c(\beta) = \frac{(1-\beta)^2}{4\beta}$. Then for $r > 0$,*

$$M(r, f(g)) \geqslant M(c(\beta)M(\beta r, g), f).$$

*Furthermore if $\beta = \frac{1}{2}$, for sufficiently large $r$*

$$M(r, f(g)) \geqslant M(\frac{1}{8}M(\frac{r}{2}, g), f).$$

LEMMA 2.3. [5] *Let $f(z)$ and $g(z)$ be transcendental entire functions. Then*

$$\frac{T(r,f)}{T(r,g(f))} \to 0 \text{ as } r \to \infty.$$

## 3. Preliminary theorem

THEOREM 3.1. [1] *Let $f(z)$ and $g(z)$ be entire functions of finite iterated order and positive iterated lower order with $i(f) = p$, $i(g) = q$.*

*(i) If $n$ is odd, then $i(f_{n,g}) = \frac{n+1}{2}p + \frac{n-1}{2}q$ and $\rho_{\frac{n+1}{2}p+\frac{n-1}{2}q}(f_{n,g}) = \rho_p(f)$.*

*(ii) If $n$ is even, then $i(f_{n,g}) = \frac{n}{2}(p+q)$ and $\rho_{\frac{n}{2}(p+q)}(f_{n,g}) = \rho_q(g)$.*

COROLLARY 3.1. *If $f(z)$ be an entire function of finite iterated order and positive iterated lower order, then $\rho_{np}(f_{n,f}) = \rho_p(f)$, where $f_{n,f}$ denotes the $n$-th generalised iteration of $f(z)$ with respect to itself.*

This follows from Theorem 3.1 by putting $g(z) = f(z)$. □

COROLLARY 3.2. *Let $f(z)$ and $g(z)$ be entire functions of finite iterated order and positive iterated lower order with $i(f) = p$, $i(g) = q$.*

*(i) If $n$ is odd, then $\mu_p(f) \leqslant \rho_{\frac{n+1}{2}p+\frac{n-1}{2}q}(f_{n,g}) \leqslant \rho_p(f)$,*

*(ii) if $n$ is even, then $\mu_q(g) \leqslant \rho_{\frac{n}{2}(p+q)}(f_{n,g}) \leqslant \rho_q(g)$.*

Since $\mu_p(f) \leqslant \rho_p(f)$, this follows from Theorem 3.1. □

## 4. Main results

THEOREM 4.1. *Let $f(z)$ and $g(z)$ be entire functions of finite iterated order and positive iterated lower order.*

*(i) If $n$ is odd and $p \leqslant i(f) \leqslant l$, $i(g) = q$ then*

$$\frac{n+1}{2}p + \frac{n-1}{2}q \leqslant i(f_{n,g}) \leqslant \frac{n+1}{2}l + \frac{n-1}{2}q$$

*and*

$$\rho_{\frac{n+1}{2}p+\frac{n-1}{2}q}(f_{n,g}) \geqslant \rho_l(f), \quad \rho_{\frac{n+1}{2}l+\frac{n-1}{2}q}(f_{n,g}) \leqslant \rho_p(f).$$

*(ii) If $n$ is even and $i(f) = p$, $q \leqslant i(g) \leqslant l$, then*

$$\frac{n}{2}(p+q) \leqslant i(f_{n,g}) \leqslant \frac{n}{2}(p+l)$$

*and*

$$\rho_{\frac{n}{2}(p+q)}(f_{n,g}) \geqslant \rho_q(g), \quad \rho_{\frac{n}{2}(p+l)}(f_{n,g}) \leqslant \rho_q(g).$$

*Proof.* Case (i). Suppose $n$ is odd.

Let $i(f) = k \ (\in \mathbb{N})$. So from Theorem 3.1, we have

$$i(f_{n,g}) = \frac{n+1}{2}k + \frac{n-1}{2}q \quad \text{and} \quad \rho_{\frac{n+1}{2}k+\frac{n-1}{2}q}(f_{n,g}) = \rho_k(f).$$

Now $p \leqslant k \leqslant l$ i.e.,

$$\frac{n+1}{2}p + \frac{n-1}{2}q \leqslant \frac{n+1}{2}k + \frac{n-1}{2}q \leqslant \frac{n+1}{2}l + \frac{n-1}{2}q$$

i.e.,

$$\frac{n+1}{2}p + \frac{n-1}{2}q \leqslant i(f_{n,g}) \leqslant \frac{n+1}{2}l + \frac{n-1}{2}q.$$

Also, $\rho_l(f) \leqslant \rho_k(f) \ \leqslant \rho_p(f)$ and consequently we have $\rho_{\frac{n+1}{2}p+\frac{n-1}{2}q}(f_{n,g}) \geqslant \rho_l(f)$ and $\rho_{\frac{n+1}{2}l+\frac{n-1}{2}q}(f_{n,g}) \leqslant \rho_p(f)$.

Case (ii). Suppose $n$ is even. Let $i(g) = k \ (\in \mathbb{N})$. So from Theorem 3.1, we have $i(f_{n,g}) = \frac{n}{2}(p+k)$ and $\rho_{\frac{n}{2}(p+k)}(f_{n,g}) = \rho_q(g)$. Again since $q \leqslant k \leqslant l$, so

$$\frac{n}{2}(p+q) \leqslant \frac{n}{2}(p+k) \leqslant \frac{n}{2}(p+l)$$

i.e.,

$$\frac{n}{2}(p+q) \leqslant i(f_{n,g}) \leqslant \frac{n}{2}(p+l).$$

Also $\rho_l(g) \leqslant \rho_k(g) \ \leqslant \rho_q(g)$ and consequently $\rho_{\frac{n}{2}(p+q)}(f_{n,g}) \geqslant \rho_q(g)$ and $\rho_{\frac{n}{2}(p+l)}(f_{n,g}) \leqslant \rho_q(g)$. □

THEOREM 4.2. *Let $f(z)$, $g(z)$ be entire functions of finite iterated order and positive iterated lower order.*

*(i) If $n$ is odd, then*

$$\frac{\mu_p(f)}{\rho_q(g)} \leqslant \liminf_{r\to\infty} \frac{\log^{[\frac{n+1}{2}p+\frac{n-1}{2}q]} T(r, f_{n,g})}{\log^{[q]} T(r,g)} \leqslant \min\{\frac{\mu_p(f)}{\mu_q(g)}, \frac{\rho_p(f)}{\rho_q(g)}\}$$
$$\leqslant \max\{\frac{\mu_p(f)}{\mu_q(g)}, \frac{\rho_p(f)}{\rho_q(g)}\} \leqslant \limsup_{r\to\infty} \frac{\log^{[\frac{n+1}{2}p+\frac{n-1}{2}q]} T(r, f_{n,g})}{\log^{[q]} T(r,g)} \leqslant \frac{\rho_p(f)}{\mu_q(g)}.$$

*(ii) If $n$ is even, then*

$$\frac{\mu_q(g)}{\rho_p(f)} \leqslant \liminf_{r\to\infty} \frac{\log^{[\frac{n}{2}(p+q)]} T(r, f_{n,g})}{\log^{[p]} T(r,f)} \leqslant \min\{\frac{\mu_q(g)}{\mu_p(f)}, \frac{\rho_q(g)}{\rho_p(f)}\}$$
$$\leqslant \max\{\frac{\mu_q(g)}{\mu_p(f)}, \frac{\rho_q(g)}{\rho_p(f)}\} \leqslant \limsup_{r\to\infty} \frac{\log^{[\frac{n}{2}(p+q)]} T(r, f_{n,g})}{\log^{[p]} T(r,f)} \leqslant \frac{\rho_q(g)}{\mu_p(f)}.$$

*Proof.* By definition, for sufficiently large $r$ and for given $\varepsilon \ (>0)$, we have

$$(\mu_q(g) - \varepsilon)\log r \leqslant \log^{[q]} T(r,g) \leqslant (\rho_q(g) + \varepsilon)\log r. \tag{4.1}$$

Case (i). Suppose $n$ is odd. For sufficiently large $r$, we have

$$\begin{aligned}
T(r,f_{n,g}) &\leqslant T(r,g_{n-1,f})+T(r,f(g_{n-1,f}))+O(1)\\
&=(1+o(1))T(r,f(g_{n-1,f})),\ \text{using Lemma 2.3}\\
&\leqslant 2T(M(r,g_{n-1,f}),f),\ \text{using Lemma 2.1}\\
&\leqslant \exp^{[p-1]}\{M(r,g_{n-1,f})\}^{\rho_p(f)+2\varepsilon}\\
&\leqslant \exp^{[p]}[(\rho_p(f)+2\varepsilon)\{\log M(r,f_{n-2,g})+\log M(r,g(f_{n-2,g}))+O(1)\}]\\
&\leqslant \exp^{[p]}[(\rho_p(f)+2\varepsilon)\{\log M(M(r,f_{n-2,g}),g)\\
&\qquad\qquad +\log M(M(r,f_{n-2,g}),g)+O(1)\}]\\
&\leqslant \exp^{[p]}\{3(\rho_p(f)+2\varepsilon)\log M(M(r,f_{n-2,g}),g)\}\\
&\leqslant \exp^{[p]}[3(\rho_p(f)+2\varepsilon)\log\{\exp^{[q]}\{M(r,f_{n-2,g})\}^{\rho_q(g)+\varepsilon}\}]\\
&\leqslant \exp^{[p+q]}[3(\rho_q(g)+2\varepsilon)\log\{\exp^{[p]}\{M(r,g_{n-3,f})\}^{\rho_p(f)+\varepsilon}\}]\\
&\leqslant \exp^{[2p+q]}[3(\rho_p(f)+2\varepsilon)\log\{\exp^{[q]}\{M(r,f_{n-4,g})\}^{\rho_q(g)+\varepsilon}\}]\\
&\leqslant \exp^{[2p+2q]}\{(\rho_q(g)+2\varepsilon)\log M(r,f_{n-4,g})\} \qquad (4.2)\\
&\quad\vdots\\
&\leqslant \exp^{[\frac{n-1}{2}p+\frac{n-1}{2}q]}\{(\rho_q(g)+2\varepsilon)\log M(r,f_{1,g})\}\\
&\leqslant \exp^{[\frac{n-1}{2}(p+q)]}[(\rho_q(g)+2\varepsilon)\{\log M(r,z)+\log M(r,f)+O(1)\}]\\
&\leqslant \exp^{[\frac{n-1}{2}(p+q)]}\{(\rho_q(g)+2\varepsilon)(1+o(1))\log M(r,f)\} \qquad (4.3)\\
&\leqslant \exp^{[\frac{n+1}{2}p+\frac{n-1}{2}q]}\{\log(r^{\rho_p(f)+2\varepsilon})\}. \qquad (4.4)
\end{aligned}$$

On the other hand for sufficiently large $r$ and chosen $\varepsilon$ $(0<2\varepsilon<\min\{\mu_p(f),\mu_q(g)\})$, we have

$$\begin{aligned}
T(r,f_{n,g}) &\geqslant T(r,f(g_{n-1,f}))-T(r,g_{n-1,f})+O(1)\\
&=(1+o(1))T(r,f(g_{n-1,f})),\ \text{using Lemma 2.3}\\
&\geqslant \frac{1}{3}(1+o(1))\log M(\frac{1}{9}M(\frac{r}{4},g_{n-1,f}),f),\ \text{using Lemma 2.2}\\
&\geqslant \exp^{[p]}[\log\{M(\frac{r}{4},g_{n-1,f})\}^{\mu_p(f)-2\varepsilon}]\\
&\geqslant \exp^{[p]}[(\mu_p(f)-2\varepsilon)T(\frac{r}{4},g_{n-1,f})]\\
&\geqslant \exp^{[p]}[(\mu_p(f)-2\varepsilon)\{T(\frac{r}{4},g(f_{n-2,g}))-T(\frac{r}{4},f_{n-2,g})+O(1)\}]\\
&=\exp^{[p]}\{(\mu_p(f)-2\varepsilon)(1+o(1))T(\frac{r}{4},g(f_{n-2,g}))\},\ \text{using Lemma 2.3}\\
&\geqslant \exp^{[p]}\{\frac{1}{3}(\mu_p(f)-2\varepsilon)(1+o(1))\log M(\frac{1}{9}M(\frac{r}{4^2},f_{n-2,g}),g)\}\\
&\geqslant \exp^{[p]}[\exp^{[q]}\{\log\{M(\frac{r}{4^2},f_{n-2,g})\}^{\mu_q(g)-2\varepsilon}\}]
\end{aligned}$$

$$= \exp^{[p+q]}\{(\mu_q(g)-2\varepsilon)\log M(\frac{r}{4^2}, f_{n-2,g})\} \tag{4.5}$$

$$\vdots$$

$$\geqslant \exp^{[\frac{n-1}{2}(p+q)]}\{(\mu_q(g)-2\varepsilon)\log M(\frac{r}{4^{n-1}}, f_{1,g})\}$$

$$\geqslant \exp^{[\frac{n-1}{2}(p+q)]}\{(\mu_q(g)-2\varepsilon)(1+o(1))\log M(\frac{r}{4^{n-1}}, f)\} \tag{4.6}$$

$$\geqslant \exp^{[\frac{n+1}{2}p+\frac{n-1}{2}q]}\{\log(r^{\mu_p(f)-2\varepsilon})\}. \tag{4.7}$$

From (4.1), (4.4) and (4.7), for sufficiently large $r$ and for chosen $\varepsilon$ such that $0 < 2\varepsilon < \min\{\mu_p(f), \mu_q(g)\}$, we have

$$\frac{\mu_p(f)-2\varepsilon}{\rho_q(g)+\varepsilon} \leqslant \frac{\log^{[\frac{n+1}{2}p+\frac{n-1}{2}q]} T(r, f_{n,g})}{\log^{[q]} T(r,g)} \leqslant \frac{\rho_p(f)+2\varepsilon}{\mu_q(g)-\varepsilon}.$$

Since $\varepsilon$ $(>0)$ is arbitrary, we get

$$\begin{cases} \liminf\limits_{r\to\infty} \dfrac{\log^{[\frac{n+1}{2}p+\frac{n-1}{2}q]} T(r, f_{n,g})}{\log^{[q]} T(r,g)} \geqslant \dfrac{\mu_p(f)}{\rho_q(g)} \\ \limsup\limits_{r\to\infty} \dfrac{\log^{[\frac{n+1}{2}p+\frac{n-1}{2}q]} T(r, f_{n,g})}{\log^{[q]} T(r,g)} \leqslant \dfrac{\rho_p(f)}{\mu_q(g)} \end{cases}. \tag{4.8}$$

By definition, there exist two sequences $\{r_l\}$ and $\{r_m\}$ tending to infinity such that

$$\begin{cases} \log^{[q]} T(r_l, g) \geqslant (\rho_q(g)-\varepsilon)\log r_l \\ \log^{[q]} T(r_m, g) \leqslant (\mu_q(g)+\varepsilon)\log r_m. \end{cases} \tag{4.9}$$

Also, there exists a sequence $\{r_l'\}$ tending to infinity such that we have from (4.6)

$$\begin{aligned} T(r_l', f_{n,g}) &\geqslant \exp^{[\frac{n-1}{2}(p+q)]}\{(\mu_q(g)-2\varepsilon)(1+o(1))\log M(\frac{r_l'}{4^{n-1}}, f)\} \\ &\geqslant \exp^{[\frac{n+1}{2}p+\frac{n-1}{2}q]}\{\log(r_l')^{\rho_p(f)-2\varepsilon}\}. \end{aligned} \tag{4.10}$$

Further there exists a sequence $\{r_m'\}$ tending to infinity such that we have from (4.3)

$$\begin{aligned} T(r_m', f_{n,g}) &\leqslant \exp^{[\frac{n-1}{2}(p+q)]}\{(\rho_q(g)+2\varepsilon)(1+o(1))\log M(r_m', f)\} \\ &\leqslant \exp^{[\frac{n+1}{2}p+\frac{n-1}{2}q]}\{\log(r_m')^{\mu_p(f)+2\varepsilon}\}. \end{aligned} \tag{4.11}$$

From (4.1), (4.11) and (4.4), (4.9) we have

$$\frac{\log^{[\frac{n+1}{2}p+\frac{n-1}{2}q]} T(r_m', f_{n,g})}{\log^{[q]} T(r_m', g)} \leqslant \frac{\mu_p(f)+2\varepsilon}{\mu_q(g)-\varepsilon}$$

and

$$\frac{\log^{[\frac{n+1}{2}p+\frac{n-1}{2}q]} T(r_l, f_{n,g})}{\log^{[q]} T(r_l, g)} \leqslant \frac{\rho_p(f)+2\varepsilon}{\rho_q(g)-\varepsilon}.$$

Since $\varepsilon\ (>0)$ is arbitrary,

$$\liminf_{r\to\infty} \frac{\log^{[\frac{n+1}{2}p+\frac{n-1}{2}q]} T(r, f_{n,g})}{\log^{[q]} T(r, g)} \leqslant \min\{\frac{\mu_p(f)}{\mu_q(g)}, \frac{\rho_p(f)}{\rho_q(g)}\}. \tag{4.12}$$

Also from (4.7), (4.9) and (4.1), (4.10) we have

$$\frac{\log^{[\frac{n+1}{2}p+\frac{n-1}{2}q]} T(r_m, f_{n,g})}{\log^{[q]} T(r_m, g)} \geqslant \frac{\mu_p(f)-2\varepsilon}{\mu_q(g)+\varepsilon}$$

and

$$\frac{\log^{[\frac{n+1}{2}p+\frac{n-1}{2}q]} T(r'_l, f_{n,g})}{\log^{[q]} T(r'_l, g)} \geqslant \frac{\rho_p(f)-2\varepsilon}{\rho_q(g)+\varepsilon}.$$

Therefore,

$$\limsup_{r\to\infty} \frac{\log^{[\frac{n+1}{2}p+\frac{n-1}{2}q]} T(r, f_{n,g})}{\log^{[q]} T(r, g)} \geqslant \max\{\frac{\mu_p(f)}{\mu_q(g)}, \frac{\rho_p(f)}{\rho_q(g)}\}. \tag{4.13}$$

Combining (4.8), (4.12) and ( 4.13), we obtain

$$\begin{aligned}\frac{\mu_p(f)}{\rho_q(g)} &\leqslant \liminf_{r\to\infty} \frac{\log^{[\frac{n+1}{2}p+\frac{n-1}{2}q]} T(r, f_{n,g})}{\log^{[q]} T(r, g)} \\ &\leqslant \min\{\frac{\mu_p(f)}{\mu_q(g)}, \frac{\rho_p(f)}{\rho_q(g)}\} \\ &\leqslant \max\{\frac{\mu_p(f)}{\mu_q(g)}, \frac{\rho_p(f)}{\rho_q(g)}\} \\ &\leqslant \limsup_{r\to\infty} \frac{\log^{[\frac{n+1}{2}p+\frac{n-1}{2}q]} T(r, f_{n,g})}{\log^{[q]} T(r, g)} \leqslant \frac{\rho_p(f)}{\mu_q(g)}.\end{aligned}$$

Case (ii). Suppose $n$ is even. By definition, for sufficiently large $r$ and for given $\varepsilon\ (>0)$, we have

$$(\mu_p(f)-\varepsilon)\log r \leqslant \log^{[p]} T(r, f) \leqslant (\rho_p(f)+\varepsilon)\log r. \tag{4.14}$$

In this case, for sufficiently large $r$, from (4.2) we get

$$\begin{aligned} T(r, f_{n,g}) &\leqslant \exp^{[\frac{n}{2}p+\frac{n-2}{2}q]}\{(\rho_p(f)+2\varepsilon)\log M(r, g_{1.f})\} \\ &\leqslant \exp^{[\frac{n}{2}p+\frac{n-2}{2}q]}[(\rho_p(f)+2\varepsilon)\{\log M(r, z)+\log M(r, g)+O(1)\}] \\ &\leqslant \exp^{[\frac{n}{2}p+\frac{n-2}{2}q]}\{(\rho_p(f)+2\varepsilon)(1+o(1))\log M(r, g)\} \end{aligned} \tag{4.15}$$

$$\leqslant \exp^{[\frac{n}{2}(p+q)]}\{\log(r^{\rho_q(g)+2\varepsilon})\}. \tag{4.16}$$

On the other hand for sufficiently large $r$, from (4.5) we have

$$\begin{aligned} T(r,f_{n,g}) &\geqslant \exp^{[\frac{np}{2}+\frac{n-2}{2}q]}\{(\mu_p(f)-2\varepsilon)\log M(\frac{r}{4^{n-1}},g_{1,f})\} \\ &\geqslant \exp^{[\frac{np}{2}+\frac{n-2}{2}q]}\{(\mu_p(f)-2\varepsilon)(1+o(1))\log M(\frac{r}{4^{n-1}},g)\} \qquad (4.17) \\ &\geqslant \exp^{[\frac{n}{2}(p+q)]}\{\log(r^{\mu_q(g)-2\varepsilon})\}. \qquad (4.18) \end{aligned}$$

From (4.14), (4.16) and (4.18), for sufficiently large $r$ and for chosen $\varepsilon$ such that $0<2\varepsilon<\min\{\mu_p(f),\mu_q(g)\}$, we have

$$\frac{\mu_q(g)-2\varepsilon}{\rho_p(f)+\varepsilon} \leqslant \frac{\log^{[\frac{n}{2}(p+q)]}T(r,f_{n,g})}{\log^{[p]}T(r,f)} \leqslant \frac{\rho_q(g)+2\varepsilon}{\mu_p(f)-\varepsilon}.$$

Since $\varepsilon\ (>0)$ is arbitrary, we get

$$\begin{cases} \liminf\limits_{r\to\infty}\dfrac{\log^{[\frac{n}{2}(p+q)]}T(r,f_{n,g})}{\log^{[p]}T(r,f)} \geqslant \dfrac{\mu_q(g)}{\rho_p(f)} \\ \limsup\limits_{r\to\infty}\dfrac{\log^{[\frac{n}{2}(p+q)]}T(r,f_{n,g})}{\log^{[p]}T(r,f)} \leqslant \dfrac{\rho_q(g)}{\mu_p(f)} \end{cases} \tag{4.19}$$

By definition, there exist two sequences $\{r_l\}$ and $\{r_m\}$ tending to infinity such that

$$\log^{[p]}T(r_l,f) \geqslant (\rho_p(f)-\varepsilon)\log r_l \quad\text{and}\quad \log^{[p]}T(r_m,f) \leqslant (\mu_p(f)+\varepsilon)\log r_m \tag{4.20}$$

Also, there exists a sequence $\{r_l'\}$ tending to infinity such that we have from (4.17)

$$\begin{aligned} T(r_l',f_{n,g}) &\geqslant \exp^{[\frac{np}{2}+\frac{n-2}{2}q]}\{(\mu_p(f)-2\varepsilon)(1+o(1))\log M(\frac{r_l'}{4^{n-1}},g)\} \\ &\geqslant \exp^{[\frac{n}{2}(p+q)]}\{\log(r_l')^{\rho_q(g)-2\varepsilon}\}. \qquad (4.21) \end{aligned}$$

Further, there exists a sequence $\{r_m'\}$ tending to infinity such that we have from (4.15)

$$\begin{aligned} T(r_m',f_{n,g}) &\leqslant \exp^{[\frac{n}{2}p+\frac{n-2}{2}q]}\{(\rho_p(f)+2\varepsilon)(1+o(1))\log M(r_m',g)\} \\ &\leqslant \exp^{[\frac{n}{2}(p+q)]}\{\log(r_m')^{\mu_q(g)+2\varepsilon}\}. \qquad (4.22) \end{aligned}$$

From (4.14), (4.22) and (4.16), (4.20) we have

$$\frac{\log^{[\frac{n}{2}(p+q)]}T(r_m',f_{n,g})}{\log^{[p]}T(r_m',f)} \leqslant \frac{\mu_q(g)+2\varepsilon}{\mu_p(f)-\varepsilon} \quad\text{and}\quad \frac{\log^{[\frac{n}{2}(p+q)]}T(r_l,f_{n,g})}{\log^{[p]}T(r_l,f)} \leqslant \frac{\rho_q(g)+2\varepsilon}{\rho_p(f)-\varepsilon}.$$

Since $\varepsilon\ (>0)$ is arbitrary,

$$\liminf_{r\to\infty}\frac{\log^{[\frac{n}{2}(p+q)]}T(r,f_{n,g})}{\log^{[p]}T(r,f)} \leqslant \min\{\frac{\mu_q(g)}{\mu_p(f)},\frac{\rho_q(g)}{\rho_p(f)}\}. \tag{4.23}$$

Also from (4.18), (4.20) and (4.14), (4.21) we have

$$\frac{\log^{[\frac{n}{2}(p+q)]} T(r_m, f_{n,g})}{\log^{[p]} T(r_m, f)} \geqslant \frac{\mu_q(g) - 2\varepsilon}{\mu_p(f) + \varepsilon} \text{ and } \frac{\log^{[\frac{n}{2}(p+q)]} T(r'_l, f_{n,g})}{\log^{[p]} T(r'_l, f)} \geqslant \frac{\rho_q(g) - 2\varepsilon}{\rho_p(f) + \varepsilon}.$$

Therefore,

$$\limsup_{r\to\infty} \frac{\log^{[\frac{n}{2}(p+q)]} T(r, f_{n,g})}{\log^{[p]} T(r, f)} \geqslant \max\{\frac{\mu_q(g)}{\mu_p(f)}, \frac{\rho_q(g)}{\rho_p(f)}\}. \tag{4.24}$$

Combining (4.19), (4.23) and (4.24) we get the required result for case (ii). □

COROLLARY 4.1. *Let $f(z)$, $g(z)$ satisfy the hypothesis of Theorem 4.2.*
*(i) If $n$ is odd, then*

$$\frac{\mu_p(f)}{\rho_q(g)} \leqslant \liminf_{r\to\infty} \frac{\log^{[\frac{n+1}{2}p+\frac{n-1}{2}q]} T(r, f_{n,g})}{\log^{[q]} T(r, g^{(k)})} \leqslant \min\{\frac{\mu_p(f)}{\mu_q(g)}, \frac{\rho_p(f)}{\rho_q(g)}\}$$
$$\leqslant \max\{\frac{\mu_p(f)}{\mu_q(g)}, \frac{\rho_p(f)}{\rho_q(g)}\} \leqslant \limsup_{r\to\infty} \frac{\log^{[\frac{n+1}{2}p+\frac{n-1}{2}q]} T(r, f_{n,g})}{\log^{[q]} T(r, g^{(k)})} \leqslant \frac{\rho_p(f)}{\mu_q(g)}.$$

*(ii) If $n$ is even, then*

$$\frac{\mu_q(g)}{\rho_p(f)} \leqslant \liminf_{r\to\infty} \frac{\log^{[\frac{n}{2}(p+q)]} T(r, f_{n,g})}{\log^{[p]} T(r, f^{(k)})} \leqslant \min\{\frac{\mu_q(g)}{\mu_p(f)}, \frac{\rho_q(g)}{\rho_p(f)}\}$$
$$\leqslant \max\{\frac{\mu_q(g)}{\mu_p(f)}, \frac{\rho_q(g)}{\rho_p(f)}\} \leqslant \limsup_{r\to\infty} \frac{\log^{[\frac{n}{2}(p+q)]} T(r, f_{n,g})}{\log^{[p]} T(r, f^{(k)})} \leqslant \frac{\rho_q(g)}{\mu_p(f)},$$

*for $k = 0, 1, 2, \dots$.*

*Proof.* If $f(z)$ is meromorphic in the complex plane, then $f(z)$ and its $k$th order derivatives $f^{(k)}(z)$ have the same order and lower order, where $k$ is any positive integer. By this fact and from Theorem 4.2, we have the conclusion of Corollary 4.1. □

THEOREM 4.3. *Let $f(z)$, $g(z)$ be entire functions of finite iterated order and positive iterated lower order.*
*(i) If $n$ is odd, then*

$$\frac{\mu_p(f)}{\rho_p(f)} \leqslant \liminf_{r\to\infty} \frac{\log^{[\frac{n+1}{2}p+\frac{n-1}{2}q]} T(r, f_{n,g})}{\log^{[p]} T(r, f)} \leqslant 1 \leqslant \limsup_{r\to\infty} \frac{\log^{[\frac{n+1}{2}p+\frac{n-1}{2}q]} T(r, f_{n,g})}{\log^{[p]} T(r, f)} \leqslant \frac{\rho_p(f)}{\mu_p(f)}.$$

*If $n$ is even, then*

$$\frac{\mu_q(g)}{\rho_q(g)} \leqslant \liminf_{r\to\infty} \frac{\log^{[\frac{n}{2}(p+q)]} T(r, f_{n,g})}{\log^{[q]} T(r, g)} \leqslant 1 \leqslant \limsup_{r\to\infty} \frac{\log^{[\frac{n}{2}(p+q)]} T(r, f_{n,g})}{\log^{[q]} T(r, g)} \leqslant \frac{\rho_q(g)}{\mu_q(g)}.$$

*Proof.* Case (i). Suppose $n$ is odd. From (4.7) and (4.14), for sufficiently large $r$, we have

$$\frac{\log^{[\frac{n+1}{2}p+\frac{n-1}{2}q]} T(r, f_{n,g})}{\log^{[p]} T(r, f)} \geqslant \frac{\mu_p(f) - 2\varepsilon}{\rho_p(f) + \varepsilon}.$$

Since $\varepsilon\ (>0)$ is arbitrary, we get

$$\liminf_{r\to\infty}\frac{\log^{[\frac{n+1}{2}p+\frac{n-1}{2}q]}T(r,f_{n,g})}{\log^{[p]}T(r,f)}\geqslant\frac{\mu_p(f)}{\rho_p(f)}. \tag{4.25}$$

Also from (4.4) and (4.20), we have

$$\frac{\log^{[\frac{n+1}{2}p+\frac{n-1}{2}q]}T(r_l,f_{n,g})}{\log^{[p]}T(r_l,f)}\leqslant\frac{\rho_p(f)+2\varepsilon}{\rho_p(f)-\varepsilon}.$$

Since $\varepsilon\ (>0)$ is arbitrary,

$$\liminf_{r\to\infty}\frac{\log^{[\frac{n+1}{2}p+\frac{n-1}{2}q]}T(r,f_{n,g})}{\log^{[p]}T(r,f)}\leqslant 1. \tag{4.26}$$

Similarly from (4.10) and (4.14), we have

$$\frac{\log^{[\frac{n+1}{2}p+\frac{n-1}{2}q]}T(r'_l,f_{n,g})}{\log^{[p]}T(r'_l,f)}\geqslant\frac{\rho_p(f)-2\varepsilon}{\rho_p(f)+\varepsilon}.$$

Therefore,

$$\limsup_{r\to\infty}\frac{\log^{[\frac{n+1}{2}p+\frac{n-1}{2}q]}T(r,f_{n,g})}{\log^{[p]}T(r,f)}\geqslant 1. \tag{4.27}$$

Again by (4.4) and (4.14), we have

$$\limsup_{r\to\infty}\frac{\log^{[\frac{n+1}{2}p+\frac{n-1}{2}q]}T(r,f_{n,g})}{\log^{[p]}T(r,f)}\leqslant\frac{\rho_p(f)}{\mu_p(f)}. \tag{4.28}$$

Combining (4.25), (4.26), (4.27) and (4.28) we get the required result for the case (i).

Case (ii). Suppose $n$ is even. From (4.1) and (4.18), we have

$$\liminf_{r\to\infty}\frac{\log^{[\frac{n}{2}(p+q)]}T(r,f_{n,g})}{\log^{[q]}T(r,g)}\geqslant\frac{\mu_q(g)}{\rho_q(g)}. \tag{4.29}$$

Also from (4.9) and (4.16), we have

$$\frac{\log^{[\frac{n}{2}(p+q)]}T(r_l,f_{n,g})}{\log^{[q]}T(r_l,g)}\leqslant\frac{\rho_q(g)+2\varepsilon}{\rho_q(g)-\varepsilon}.$$

Since $\varepsilon\ (>0)$ is arbitrary,

$$\liminf_{r\to\infty}\frac{\log^{[\frac{n}{2}(p+q)]}T(r,f_{n,g})}{\log^{[q]}T(r,g)}\leqslant 1. \tag{4.30}$$

Similarly from (4.1) and (4.21), we have

$$\frac{\log^{[\frac{n}{2}(p+q)]}T(r'_l,f_{n,g})}{\log^{[q]}T(r'_l,g)}\geqslant\frac{\rho_q(g)-2\varepsilon}{\rho_q(g)+\varepsilon}.$$

Therefore,

$$\limsup_{r\to\infty}\frac{\log^{[\frac{n}{2}(p+q)]}T(r,f_{n,g})}{\log^{[q]}T(r,g)}\geqslant 1. \tag{4.31}$$

Again by (4.1) and (4.16), we have

$$\limsup_{r\to\infty}\frac{\log^{[\frac{n}{2}(p+q)]}T(r,f_{n,g})}{\log^{[q]}T(r,g)}\leqslant\frac{\rho_q(g)}{\mu_q(g)}. \tag{4.32}$$

Combining (4.29), (4.30), (4.31) and (4.32) we get the required result for the case (ii).

COROLLARY 4.2. *Under the hypothesis of Theorem 4.3*
*(i) if $n$ is odd, then*

$$\frac{\mu_p(f)}{\rho_p(f)}\leqslant\liminf_{r\to\infty}\frac{\log^{[\frac{n+1}{2}p+\frac{n-1}{2}q+1]}M(r,f_{n,g})}{\log^{[p+1]}M(r,f)}\leqslant 1$$
$$\leqslant\limsup_{r\to\infty}\frac{\log^{[\frac{n+1}{2}p+\frac{n-1}{2}q+1]}M(r,f_{n,g})}{\log^{[p+1]}M(r,f)}\leqslant\frac{\rho_p(f)}{\mu_p(f)},$$

*(ii) if $n$ is even, then*

$$\frac{\mu_q(g)}{\rho_q(g)}\leqslant\liminf_{r\to\infty}\frac{\log^{[\frac{n}{2}(p+q)+1]}M(r,f_{n,g})}{\log^{[q+1]}M(r,g)}\leqslant 1$$
$$\leqslant\limsup_{r\to\infty}\frac{\log^{[\frac{n}{2}(p+q)+1]}M(r,f_{n,g})}{\log^{[q+1]}M(r,g)}\leqslant\frac{\rho_q(g)}{\mu_q(g)}.$$

COROLLARY 4.3. *Let $f(z)$, $g(z)$ satisfy the hypothesis of Theorem 4.3 with $\mu_p(f)=\rho_p(f)$ and $\mu_q(g)=\rho_q(g)$.*
*(i) If $n$ is odd, then*

$$\lim_{r\to\infty}\frac{\log^{[\frac{n+1}{2}p+\frac{n-1}{2}q]}T(r,f_{n,g})}{\log^{[p]}T(r,f)}=1.$$

*(ii) If $n$ is even, then*

$$\lim_{r\to\infty}\frac{\log^{[\frac{n}{2}(p+q)]}T(r,f_{n,g})}{\log^{[q]}T(r,g)}=1.$$

COROLLARY 4.4. *Let $f(z)$, $g(z)$ satisfy the hypothesis of Theorem 4.3 with $\mu_p(f)=\rho_p(f)$ and $\mu_q(g)=\rho_q(g)$.*
*(i) If $n$ is odd, then*

$$\lim_{r\to\infty}\frac{\log^{[\frac{n+1}{2}p+\frac{n-1}{2}q+1]}M(r,f_{n,g})}{\log^{[p+1]}M(r,f)}=1.$$

*(ii) If $n$ is even, then*

$$\lim_{r\to\infty}\frac{\log^{[\frac{n}{2}(p+q)+1]}M(r,f_{n,g})}{\log^{[q+1]}M(r,g)}=1.$$

COROLLARY 4.5. *Let $f(z)$, $g(z)$ satisfy the hypothesis of Theorem 4.3.*
*(i) If $n$ is odd, then*

$$\frac{\mu_p(f)}{\rho_p(f)}\leqslant\liminf_{r\to\infty}\frac{\log^{[\frac{n+1}{2}p+\frac{n-1}{2}q]}T(r,f_{n,g})}{\log^{[p]}T(r,f^{(k)})}\leqslant 1$$
$$\leqslant\limsup_{r\to\infty}\frac{\log^{[\frac{n+1}{2}p+\frac{n-1}{2}q]}T(r,f_{n,g})}{\log^{[p]}T(r,f^{(k)})}\leqslant\frac{\rho_p(f)}{\mu_p(f)}.$$

*(ii) If $n$ is even, then*

$$\frac{\mu_q(g)}{\rho_q(g)}\leqslant\liminf_{r\to\infty}\frac{\log^{[\frac{n}{2}(p+q)]}T(r,f_{n,g})}{\log^{[q]}T(r,g^{(k)})}\leqslant 1$$
$$\leqslant\limsup_{r\to\infty}\frac{\log^{[\frac{n}{2}(p+q)]}T(r,f_{n,g})}{\log^{[q]}T(r,g^{(k)})}\leqslant\frac{\rho_q(g)}{\mu_q(g)},$$

*for $k=0,1,2,\ldots$.*

COROLLARY 4.6.. *Let $f(z)$, $g(z)$ satisfy the hypothesis of Theorem 4.3.*
*(i) If $n$ is odd, then*

$$\frac{\mu_p(f)}{\rho_p(f)}\leqslant\liminf_{r\to\infty}\frac{\log^{[\frac{n+1}{2}p+\frac{n-1}{2}q+1]}M(r,f_{n,g})}{\log^{[p+1]}M(r,f^{(k)})}\leqslant 1$$
$$\leqslant\limsup_{r\to\infty}\frac{\log^{[\frac{n+1}{2}p+\frac{n-1}{2}q+1]}M(r,f_{n,g})}{\log^{[p+1]}M(r,f^{(k)})}\leqslant\frac{\rho_p(f)}{\mu_p(f)}.$$

*(ii) If $n$ is even, then*

$$\frac{\mu_q(g)}{\rho_q(g)}\leqslant\liminf_{r\to\infty}\frac{\log^{[\frac{n}{2}(p+q)+1]}M(r,f_{n,g})}{\log^{[q+1]}M(r,g^{(k)})}\leqslant 1$$
$$\leqslant\limsup_{r\to\infty}\frac{\log^{[\frac{n}{2}(p+q)+1]}M(r,f_{n,g})}{\log^{[q+1]}M(r,g^{(k)})}\leqslant\frac{\rho_q(g)}{\mu_q(g)},$$

*for $k=0,1,2,\ldots$.*

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