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LEGENDRE–TYPE RELATIONS FOR GENERALIZED COMPLETE ELLIPTIC INTEGRALS

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Dedicated to Professor Shigeru Sakaguchi for the occasion of his 60th birthday

Abstract. Legendre's relation for the complete elliptic integrals of the first and second kinds is generalized. The proof depends on an application of the generalized trigonometric functions and is alternative to the proof for Elliott's identity.

1. Introduction

Let $k \in [0,1)$. The complete elliptic integrals of the first kind

$$K(k) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}$$

and of the second kind

$$E(k) = \int_0^1 \sqrt{\frac{1-k^2t^2}{1-t^2}}\,dt$$

play important roles in classical analysis (see for instance [11]).

In this paper, we consider generalizations of $K(k)$ and $E(k)$ as

$$K_{p,q,r}(k) := \int_0^1 \frac{dt}{(1-t^q)^{1/p}(1-k^qt^q)^{1/r}}$$

and

$$E_{p,q,r}(k) := \int_0^1 \frac{(1-k^qt^q)^{1/r}}{(1-t^q)^{1/p}}\,dt,$$

where $p \in \mathbb{P}^* := (-\infty,0) \cup (1,\infty]$ and $q,\ r \in (1,\infty)$. In case $p=q=r=2$, $K_{p,q,r}(k)$ and $E_{p,q,r}(k)$ are reduced to the classical $K(k)$ and $E(k)$, respectively. For $p=\infty$ we regard $K_{p,q,r}$ and $E_{p,q,r}$ as

$$K_{\infty,q,r}(k) := \int_0^1 \frac{dt}{(1-k^qt^q)^{1/r}}$$

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and

$$E_{\infty,q,r}(k) := \int_0^1 (1-k^q t^q)^{1/r}\,dt.$$

Let s^* be the number such that $1/s+1/s^*=1$ for s. Under the convention that $1/\infty=0$ and $1/0=\infty$, we should note that $s\in\mathbb{P}^*$ if and only if $s^*\in(0,\infty)$, particularly, $\infty^*=1$.

There is a lot of literature about the generalized complete elliptic integrals. $K_{p,q,p}$ is introduced in [12] with a generalization of the Jacobian elliptic function with a period of $4K_{p,q,p}$ to study a bifurcation problem of a bistable reaction-diffusion equation involving p-Laplacian. Relationship between $K_{p,q,p}$ and $E_{p,q,p}$ has been observed in [3, 16]. Regarding K_{p,q,p^*}, another generalization of Jacobian elliptic function with a period of K_{p,q,p^*} is given and the basis properties for the family of these functions are shown in [13]. Moreover, K_{p,q,p^*} is also applied to a problem on Bhatia-Li's mean and a curious relation between K_{p,q,p^*} and $E_{p,q,p}$ is given in [9].

It is well known that $K(k)$ and $E(k)$ satisfy the famous *Legendre's relation* (see, for example, [2, 4, 6]):

$$E(k)K(k')+K(k)E(k')-K(k)K(k')=\frac{\pi}{2}, \tag{1}$$

where $k'=\sqrt{1-k^2}$. Our purpose in the present paper is to generalize Legendre's relation (1) to the generalized complete elliptic integrals above.

To state the results, we will give some notations. For $p\in\mathbb{P}^*$ and $q\in(1,\infty)$, let

$$\pi_{p,q} := 2\int_0^1 \frac{dt}{(1-t^q)^{1/p}} = \frac{2}{q}B\left(\frac{1}{q},\frac{1}{p^*}\right),$$

where B denotes the beta function. In particular, $\pi_{\infty,q}=2$ for any $q\in(1,\infty)$. We write $K_{p,q}:=K_{p,q,q^*}$, $E_{p,q}:=E_{p,q,q}$ for $p\in\mathbb{P}^*$ and $q\in(1,\infty)$; $K_p:=K_{p,p,p^*}$, $E_p:=E_{p,p,p}$, $\pi_p:=\pi_{p,p}$ for $p\in(1,\infty)$.

THEOREM 1. *Let $p\in\mathbb{P}^*$, q, $r\in(1,\infty)$ and $k\in(0,1)$. Then*

$$E_{p,q,r}(k)K_{p,r,q^*}(k') + K_{p,q,r^*}(k)E_{p,r,q}(k') - K_{p,q,r^*}(k)K_{p,r,q^*}(k') = \frac{\pi_{p,q}\pi_{s,r}}{4}, \tag{2}$$

where $k':=(1-k^q)^{1/r}$ and $1/s=1/p-1/q$.

COROLLARY 1. (Case $q=r$) *Let $p\in\mathbb{P}^*$, $q\in(1,\infty)$ and $k\in(0,1)$. Then*

$$E_{p,q}(k)K_{p,q}(k')+K_{p,q}(k)E_{p,q}(k')-K_{p,q}(k)K_{p,q}(k')=\frac{\pi_{p,q}\pi_{s,q}}{4}, \tag{3}$$

where $k':=(1-k^q)^{1/q}$ and $1/s=1/p-1/q$.

COROLLARY 2. ([14], Case $p=q=r$) *Let $p\in(1,\infty)$ and $k\in(0,1)$. Then*

$$E_p(k)K_p(k')+K_p(k)E_p(k')-K_p(k)K_p(k')=\frac{\pi_p}{2}, \tag{4}$$

where $k':=(1-k^p)^{1/p}$.

REMARK 1. Using (4), the author establishes computation formulas of π_p for $p=3$ in [14]; for $p=4$ in [15].

In fact, (2) is equivalent to *Elliott's identity* (5) below. The advantage of our result lies in the facts that it is understandable without acknowledge of hypergeometric functions and that its proof gives an alternative proof for Elliott's identity with straightforward calculations.

2. Proof of Theorem 1

The following property immediately follows from the definitions of $K_{p,q,r}$ and $E_{p,q,r}$.

PROPOSITION 1. *Let $p \in \mathbb{P}^*$, q, $r \in (1,\infty)$. Then, $K_{p,q,r}(k)$ is increasing on $[0,1)$ and*

$$K_{p,q,r}(0) = \frac{\pi_{p,q}}{2},$$
$$\lim_{k\to 1-0} K_{p,q,r}(k) = \begin{cases} \infty & \text{if } 1/p+1/r \geqslant 1, \\ \pi_{u,q}/2 \ (1/u = 1/p+1/r) & \text{if } 1/p+1/r < 1; \end{cases}$$

and $E_{p,q,r}(k)$ is decreasing on $[0,1]$ and

$$E_{p,q,r}(0) = \frac{\pi_{p,q}}{2}, \quad E_{p,q,r}(1) = \frac{\pi_{v,q}}{2} \ (1/v = 1/p-1/r).$$

For $p \in \mathbb{P}^*$ and $q \in (1,\infty)$, the *generalized trigonometric function* $\sin_{p,q} x$ is the inverse function of

$$\sin_{p,q}^{-1} x := \begin{cases} \displaystyle\int_0^x \frac{dt}{(1-t^q)^{1/p}} & \text{if } p \neq \infty, \\ x & \text{if } p = \infty. \end{cases}$$

Clearly, $\sin_{p,q} x$ is increasing function from $[0,\pi_{p,q}/2]$ onto $[0,1]$.

For $p=q=2$, $\sin_{p,q}\theta$ and $\pi_{p,q} = 2\sin_{p,q}^{-1} 1$ are identical to the classical $\sin\theta$ and π, respectively. Moreover, $\sin_{p,q}\theta$ and $\pi_{p,q}$ play important roles to express the solutions (λ,u) of inhomogeneous eigenvalue problem of p-Laplacian $-(|u'|^{p-2}u')' = \lambda|u|^{q-2}u$, p, $q \in (1,\infty)$, with a boundary condition (see [5, 10, 12] and the references given there).

For $p \neq \infty$ and $x \in (0,\pi_{p,q}/2)$, we also define $\cos_{p,q} x := (\sin_{p,q} x)'$. It is easy to check that for $x \in (0,\pi_{p,q}/2)$,

$$\cos_{p,q}^p x + \sin_{p,q}^q x = 1, \quad (\cos_{p,q} x)' = -\frac{q}{p} \sin_{p,q}^{q-1} x \cos_{p,q}^{2-p} x.$$

Now, we apply the generalized trigonometric function to the generalized complete elliptic integrals. For $p \in \mathbb{P}^*$ and q, $r \in (1,\infty)$, using $\sin_{p,q}\theta$ and $\pi_{p,q}$, we can express

$K_{p,q,r}(k)$ and $E_{p,q,r}(k)$ as follows.

$$K_{p,q,r}(k)=\int_0^{\pi_{p,q}/2}\frac{d\theta}{(1-k^q\sin_{p,q}^q\theta)^{1/r}},$$
$$E_{p,q,r}(k)=\int_0^{\pi_{p,q}/2}(1-k^q\sin_{p,q}^q\theta)^{1/r}\,d\theta.$$

Then, we see that the functions $K_{p,q,r^*}(k)$ and $E_{p,q,r}(k)$ satisfy a system of linear differential equations.

PROPOSITION 2. *Let $p\in\mathbb{P}^*$, q, $r\in(1,\infty)$. Then,*

$$\frac{dE_{p,q,r}}{dk}=\frac{q(E_{p,q,r}-K_{p,q,r^*})}{rk},$$
$$\frac{dK_{p,q,r^*}}{dk}=\frac{aE_{p,q,r}-(a-k^q)K_{p,q,r^*}}{k(1-k^q)},$$

where $a:=1+q/r-q/p$.

Proof. We consider the case $p\neq\infty$. Differentiating $E_{p,q,r}(k)$ we have

$$\frac{dE_{p,q,r}}{dk}=\frac{q}{r}\int_0^{\pi_{p,q}/2}\frac{-k^{q-1}\sin_{p,q}^q\theta}{(1-k^q\sin_{p,q}^q\theta)^{1/r^*}}\,d\theta=\frac{q}{rk}(E_{p,q,r}-K_{p,q,r^*}).$$

Next, for $K_{p,q,r^*}(k)$

$$\frac{dK_{p,q,r^*}}{dk}=\frac{q}{r^*}\int_0^{\pi_{p,q}/2}\frac{k^{q-1}\sin_{p,q}^q\theta}{(1-k^q\sin_{p,q}^q\theta)^{1+1/r^*}}\,d\theta.$$

Here we see that

$$\frac{d}{d\theta}\left(\frac{-\cos_{p,q}^{p/r^*}\theta}{(1-k^q\sin_{p,q}^q\theta)^{1/r^*}}\right)=\frac{q(1-k^q)\sin_{p,q}^{q-1}\theta\cos_{p,q}^{1-p/r}\theta}{r^*(1-k^q\sin_{p,q}^q\theta)^{1+1/r^*}},$$
$$\lim_{\theta\to\pi_{p,q}/2}\cos_{p,q}^{p-1}\theta=\lim_{\theta\to\pi_{p,q}/2}(1-\sin_{p,q}^q\theta)^{1/p^*}=0;$$

so that we use integration by parts as

$$\begin{aligned}
\frac{dK_{p,q,r^*}}{dk} &= \frac{k^{q-1}}{1-k^q}\int_0^{\pi_{p,q}/2} \frac{d}{d\theta}\left(\frac{-\cos_{p,q}^{p/r^*}\theta}{(1-k^q\sin_{p,q}^q\theta)^{1/r^*}}\right)\sin_{p,q}\theta\cos_{p,q}^{p/r-1}\theta\,d\theta \\
&= \frac{k^{q-1}}{1-k^q}\left[\frac{-\sin_{p,q}\theta\cos_{p,q}^{p-1}\theta}{(1-k^q\sin_{p,q}^q\theta)^{1/r^*}}\right]_0^{\pi_{p,q}/2} \\
&\quad + \frac{k^{q-1}}{1-k^q}\int_0^{\pi_{p,q}/2}\frac{\cos_{p,q}^{p/r^*}\theta}{(1-k^q\sin_{p,q}^q\theta)^{1/r^*}}\left(\cos_{p,q}^{p/r}\theta-\frac{(q/r-q/p)\sin_{p,q}^q\theta}{\cos_{p,q}^{p/r^*}\theta}\right)d\theta \\
&= \frac{k^{q-1}}{1-k^q}\int_0^{\pi_{p,q}/2}\frac{\cos_{p,q}^p\theta-(q/r-q/p)\sin_{p,q}^q\theta}{(1-k^q\sin_{p,q}^q\theta)^{1/r^*}}\,d\theta \\
&= \frac{k^{q-1}}{1-k^q}\int_0^{\pi_{p,q}/2}\frac{(1+q/r-q/p)(1-k^q\sin_{p,q}^q\theta)-(1+q/r-q/p-k^q)}{k^q(1-k^q\sin_{p,q}^q\theta)^{1/r^*}}\,d\theta \\
&= \frac{(1+q/r-q/p)E_{p,q,r}-(1+q/r-q/p-k^q)K_{p,q,r^*}}{k(1-k^q)}.
\end{aligned}$$

The case $p=\infty$ is proved similarly. Indeed,

$$\frac{dE_{\infty,q,r}}{dk} = \frac{q}{r}\int_0^1\frac{-k^{q-1}\theta^q}{(1-k^q\theta^q)^{1/r^*}}\,d\theta = \frac{q}{rk}(E_{\infty,q,r}-K_{\infty,q,r^*})$$

and

$$\begin{aligned}
\frac{dK_{\infty,q,r^*}}{dk} &= \frac{q}{r^*}\int_0^1\frac{k^{q-1}\theta^q}{(1-k^q\theta^q)^{1+1/r^*}}\,d\theta \\
&= \frac{k^{q-1}}{1-k^q}\int_0^1\frac{d}{d\theta}\left(-\left(\frac{1-\theta^q}{1-k^q\theta^q}\right)^{1/r^*}\right)\theta(1-\theta^q)^{1/r}\,d\theta \\
&= \frac{k^{q-1}}{1-k^q}\left[\frac{-\theta(1-\theta^q)}{(1-k^q\theta^q)^{1/r^*}}\right]_0^1 \\
&\quad + \frac{k^{q-1}}{1-k^q}\int_0^1\left(\frac{1-\theta^q}{1-k^q\theta^q}\right)^{1/r^*}\left((1-\theta^q)^{1/r}-\frac{(q/r)\theta^q}{(1-\theta^q)^{1/r^*}}\right)d\theta \\
&= \frac{k^{q-1}}{1-k^q}\int_0^1\frac{1-\theta^q-(q/r)\theta^q}{(1-k^q\theta^q)^{1/r^*}}\,d\theta \\
&= \frac{k^{q-1}}{1-k^q}\int_0^1\frac{(1+q/r)(1-k^q\theta^q)-(1+q/r-k^q)}{k^q(1-k^q\theta^q)^{1/r^*}}\,d\theta \\
&= \frac{(1+q/r)E_{\infty,q,r}-(1+q/r-k^q)K_{\infty,q,r^*}}{k(1-k^q)}.
\end{aligned}$$

This completes the proof.

Proposition 2 now yields Theorem 1.

Proof. [Proof of Theorem 1] Let $k' := (1-k^q)^{1/r}$, $E'_{p,r,q}(k) := E_{p,r,q}(k')$ and $K'_{p,r,q^*}(k) := K_{p,r,q^*}(k')$. As $dk'/dk = -(q/r)k^{q-1}/(k')^{r-1}$, Proposition 2 gives

$$\begin{aligned}
\frac{dE_{p,q,r}}{dk} &= \frac{q(E_{p,q,r}-K_{p,q,r^*})}{rk},\\
\frac{dK_{p,q,r^*}}{dk} &= \frac{aE_{p,q,r}-(a-k^q)K_{p,q,r^*}}{k(k')^r},\\
\frac{dE'_{p,r,q}}{dk} &= \frac{k^{q-1}(-E'_{p,r,q}+K'_{p,r,q^*})}{(k')^r},\\
\frac{dK'_{p,r,q^*}}{dk} &= \frac{q(-bE'_{p,r,q}+(b-(k')^r)K'_{p,r,q^*})}{rk(k')^r},
\end{aligned}$$

where $a := 1+q/r-q/p$ and $b := 1+r/q-r/p$.

We denote the left-hand side of (2) by $L(k)$. A direct computation shows that

$$\begin{aligned}
&\frac{d}{dk}L(k)\\
&= \frac{q(E_{p,q,r}-K_{p,q,r^*})}{rk}\cdot K'_{p,r,q^*} + E_{p,q,r}\cdot\frac{q(-bE'_{p,r,q}+(b-(k')^r)K'_{p,r,q^*})}{rk(k')^r}\\
&\quad + \frac{aE_{p,q,r}-(a-k^q)K_{p,q,r^*}}{k(k')^r}\cdot E'_{p,r,q} + K_{p,q,r^*}\cdot\frac{k^{q-1}(-E'_{p,r,q}+K'_{p,r,q^*})}{(k')^r}\\
&\quad - \frac{aE_{p,q,r}-(a-k^q)K_{p,q,r^*}}{k(k')^r}\cdot K'_{p,r,q^*} - K_{p,q,r^*}\cdot\frac{q(-bE'_{p,r,q}+(b-(k')^r)K'_{p,r,q^*})}{rk(k')^r}\\
&= \left(\frac{q}{rk}+\frac{q(b-(k')^r)}{rk(k')^r}-\frac{a}{k(k')^r}\right)E_{p,q,r}K'_{p,r,q^*}\\
&\quad + \left(-\frac{q}{rk}+\frac{k^{q-1}}{(k')^r}+\frac{a-k^q}{k(k')^r}-\frac{q(b-(k')^r)}{rk(k')^r}\right)K_{p,q,r^*}K'_{p,r,q^*}\\
&\quad + \left(-\frac{qb}{rk(k')^r}+\frac{a}{k(k')^r}\right)E_{p,q,r}E'_{p,r,q}\\
&\quad + \left(-\frac{a-k^q}{k(k')^r}-\frac{k^{q-1}}{(k')^r}+\frac{qb}{rk(k')^r}\right)K_{p,q,r^*}E'_{p,r,q}\\
&= \frac{qb-ra}{rk(k')^r}(E_{p,q,r}K'_{p,r,q^*}-K_{p,q,r^*}K'_{p,r,q^*}-E_{p,q,r}E'_{p,r,q}+K_{p,q,r^*}E'_{p,r,q}).
\end{aligned}$$

Since $qb-ra=0$, we see that $dL/dk=0$. Thus $L(k)$ is a constant C.

We will evaluate C as follows. Since

$$
\begin{aligned}
&|(K_{p,q,r^*}-E_{p,q,r})K'_{p,r,q^*}| \\
&\quad = \int_0^{\pi_{p,q}/2} \left(\frac{1}{(1-k^q \sin_{p,q}^q \theta)^{1/r^*}} - (1-k^q \sin_{p,q}^q \theta)^{1/r} \right) d\theta \\
&\qquad \times \int_0^{\pi_{p,r}/2} \frac{d\theta}{(1-(k')^r \sin_{p,r}^r \theta)^{1/q^*}} \\
&\quad = \int_0^{\pi_{p,q}/2} \frac{k^q \sin_{p,q}^q \theta}{(1-k^q \sin_{p,q}^q \theta)^{1/r^*}}\, d\theta \cdot \int_0^{\pi_{p,r}/2} \frac{d\theta}{(\cos_{p,r}^p \theta + k^q \sin_{p,r}^r \theta)^{1/q^*}} \\
&\quad \leqslant k^q K_{p,q,r^*}(k) \cdot \frac{1}{k^{q-1}} \frac{\pi_{p,r}}{2} \\
&\quad = \frac{\pi_{p,r}}{2} k K_{p,q,r^*}(k),
\end{aligned}
$$

we obtain $\lim_{k\to+0}(K_{p,q,r^*}-E_{p,q,r})K'_{p,r,q^*}=0$. Therefore, from Proposition 1

$$
C = \lim_{k\to+0} K_{p,q,r^*} E'_{p,r,q} = K_{p,q,r^*}(0) E_{p,r,q}(1) = \frac{\pi_{p,q}\pi_{s,r}}{4},
$$

where $1/s = 1/p - 1/q$. Thus, we conclude the assertion.

Finally, we will give a remark for Theorem 1. From the series expansion and the termwise integration, it is possible to express the generalized complete elliptic integrals by Gaussian hypergeometric functions

$$
\begin{aligned}
K_{p,q,r}(k) &= \frac{\pi_{p,q}}{2} F\left(\frac{1}{q}, \frac{1}{r}; \frac{1}{p^*} + \frac{1}{q}; k^q\right), \\
E_{p,q,r}(k) &= \frac{\pi_{p,q}}{2} F\left(\frac{1}{q}, -\frac{1}{r}; \frac{1}{p^*} + \frac{1}{q}; k^q\right).
\end{aligned}
$$

By these expressions and letting $1/p = 1/2 - b,\ 1/q = 1/2 + a,\ 1/r = 1/2 - c$ and $k^q = x$ in (2), we obtain *Elliott's identity* (see Elliott [7]; see also [1], [2, Theorem 3.2.8] and [8, (13) p. 85]):

$$
\begin{aligned}
&F\left(\begin{matrix} 1/2+a, -1/2-c \\ a+b+1 \end{matrix}; x\right) F\left(\begin{matrix} 1/2-a, 1/2+c \\ b+c+1 \end{matrix}; 1-x\right) \\
&\qquad + F\left(\begin{matrix} 1/2+a, 1/2-c \\ a+b+1 \end{matrix}; x\right) F\left(\begin{matrix} -1/2-a, 1/2+c \\ b+c+1 \end{matrix}; 1-x\right) \\
&\qquad - F\left(\begin{matrix} 1/2+a, 1/2-c \\ a+b+1 \end{matrix}; x\right) F\left(\begin{matrix} 1/2-a, 1/2+c \\ b+c+1 \end{matrix}; 1-x\right) \\
&\qquad\qquad = \frac{\Gamma(a+b+1)\Gamma(b+c+1)}{\Gamma(a+b+c+3/2)\Gamma(b+1/2)}
\end{aligned} \tag{5}
$$

for $|a|, |c| < 1/2$ and $b \in (-1/2, \infty)$, where Γ denotes the gamma function. Also, letting $1/p = 2 - c - a$ and $1/q = 1 - a$ in (3) of Corollary 1, we have the identity of

[1, Corollary 3.13 (5)] for $a \in (0,1)$ and $c \in (1-a,\infty)$. A series of Vuorinen's works on Elliott's identity with his coauthors starting from [1] deals with the concavity/convexity properties of certain related functions to the left-hand side of (5).

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