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APPROXIMATION OF PERIODIC FUNCTIONS BY ZYGMUND MEANS IN ORLICZ SPACES

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Abstract. In the present work we investigate the approximation of the functions by Zygmund means in the Orlicz spaces $L_M(\mathbb{T})$ in the terms of the best approximation $E_n(f)_M$ and modulus of smoothness $\omega_k(\cdot,f)_M$.

1. Introduction and Main Results

Let $M(u)$ be a continuous increasing convex function on $[0,\infty)$ such that $M(u)/u \to 0$ if $u \to 0$, and $M(u)/u \to \infty$ if $u \to \infty$. We denote by N *the complementary of* M *in Young's sense,* i.e. $N(u) = \max\{uv - M(v) : v \geqslant 0\}$ if $u \geqslant 0$. We will say that M satisfies the Δ_2-condition if $M(2u) \leqslant cM(u)$ for any $u \geqslant u_0 \geqslant 0$ with some constant c, independent of u.

Let $\mathbb{T}$ denote the interval $[-\pi,\pi]$, $\mathbb{C}$ the complex plane, and $L_p(\mathbb{T})$, $1 \leqslant p \leqslant \infty$, the Lebesgue space of measurable complex-valued functions on $\mathbb{T}$.

For a given Young function M, let $\widetilde{L}_M(\mathbb{T})$ denote the set of all Lebesgue measurable functions $f : \mathbb{T} \to \mathbb{C}$ for which

$$\int_{\mathbb{T}} M(|f(x)|)\,dx < \infty.$$

Let N be the complementary Young function of M. It is well-known [21, p. 69], [34, pp. 52-68] that the linear span of $\widetilde{L}_M(\mathbb{T})$ equipped with the *Orlicz norm*

$$\|f\|_{L_M(\mathbb{T})} := \sup\left\{ \int_{\mathbb{T}} |f(x)g(x)|\,dx : g \in \widetilde{L}_N(\mathbb{T}),\ \int_{\mathbb{T}} N(|g(x)|)\,dx \leqslant 1 \right\},$$

or with the *Luxemburg norm*

$$\|f\|^*_{L_M(\mathbb{T})} := \inf\left\{ k > 0 : \int_{\mathbb{T}} M\left(\frac{|f(x)|}{k}\right) dx \leqslant 1 \right\}$$

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becomes a Banach space. This space is denoted by $L_M(\mathbb{T})$ and is called an *Orlicz space* [21, p. 26]. The Orlicz spaces are known as the generalizations of the Lebesgue spaces $L_p(\mathbb{T}),\ 1<p<\infty$. The Luxemburg and Orlicz norms are equivalent since [21, p. 80]

$$\|f\|^*_{L_M(\mathbb{T})} \leqslant \|f\|_{L_M(\mathbb{T})} \leqslant 2\,\|f\|^*_{L_M(\mathbb{T})}, \quad f\in L_M(\mathbb{T}).$$

If we choose $M(u)=u^p/p \ \ (1<p<\infty)$ then the complementary function is $N(u)=u^q/q$ with $\frac{1}{p}+\frac{1}{q}=1$ and we have the relation

$$p^{-1/p}\,\|u\|_{L_p(\mathbb{T})} = \|u\|^*_{L_M(\mathbb{T})} \leqslant \|u\|_{L_M(\mathbb{T})} \leqslant q^{1/q}\,\|u\|_{L_p(\mathbb{T})},$$

where $\|u\|_{L_p(\mathbb{T})} = \left(\int\limits_{\mathbb{T}} |u(x)|^p\,dx\right)^{1/p}$ denotes the usual norm of the $L_p(\mathbb{T})$ space.

The Orlicz space $L_M(\mathbb{T})$ is *reflexive* if and only if the $N-$function M and its complementary function N both satisfy the Δ_2-condition [34, p. 113].

Note that the detailed information about properties of the Orlicz spaces can be found in [6], [7], [21], [27], [28] and [29].

Let $L_M(\mathbb{T})$ be an Orlicz space. Suppose that x,h are real, and let us take into account the sum

$$\Delta_h^k f(x) = \sum_{\nu=0}^{k} (-1)^{k-\nu} \binom{k}{\nu} f(x+\nu h),\ \ f\in L_M(\mathbb{T}),\ k\in\mathbb{N},$$

where

$$\binom{k}{\nu} := \frac{k(k-1)...(k-\nu+1)}{\nu!}.$$

The function

$$\omega_k(f,\delta)_M := \sup_{0<h\leqslant\delta} \left\|\Delta_h^k f(x)\right\|_{L_M(\mathbb{T})},\ \delta>0$$

is called $k-th$ *modulus of smoothness* of $f\in L_M(\mathbb{T})$.

It can easily be shown that $\omega_k(f,\delta)_M$ is a continuous, non-negative and non-decreasing function satisfying the conditions

$$\lim_{\delta\to 0^+} \omega_k(f,\delta)_M = 0,\ \omega_k(f+g,\cdot)_M \leqslant \omega_k(f,\cdot)_M + \omega_k(g,\cdot)_M,$$

for $f,g\in L_M(\mathbb{T})$.

Let

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} A_k(x,f) \tag{1}$$

be the Fourier series of the function $f\in L_1(\mathbb{T})$, where $A_k(x,f) := (a_k(f)\cos kx + b_k(f)\sin kx)$, $a_k(f)$ and $b_k(f)$ are Fourier coefficients of the function $f\in L_1(\mathbb{T})$.

The *$n-th$ partial sums*, *Zygmund means of order k* $(k \in \mathbb{N})$ of the series (1) are defined, respectively as [10], [40]

$$S_n(x,f) = \frac{a_0}{2} + \sum_{\nu=1}^{n} A_\nu(x,f),$$

$$Z_{n,k}(x,f) = \frac{a_0}{2} + \sum_{\nu=1}^{n} (1 - \frac{\nu^k}{(n+1)^k}) A_\nu(x,f), \ k = 1,2,..., \ n = 1,2,...$$

It is clear that

$$S_0(x,f) = Z_{0,k}(x,f) = \frac{a_0}{2}.$$

We denote by $E_n(f)_M$ the best approximation of $f \in L_M(\mathbb{T})$ by trigonometric polynomials of degree not exceeding n, i.e.,

$$E_n(f)_M = \inf\{\| f - T_n \|_{L_M(\mathbb{T})} : T_n \in \Pi_n\}$$

where Π_n denotes the class of trigonometric polynomials of degree at most n.

The approximation problems by trigonometric polynomials in nonweighted and weighted Orlicz spaces have been investigated by several authors (see, for example, [1]-[4], [9], [10], [13], [15]-18], [23], [31], [32], [35], [42]). The approximation of the functions by the means of Fourier trigonometric series in different spaces were studied in [5], [11], [12], [19], [24]-[26], [36]-[41] and [43]. In the present paper, we investigate the deviation of functions from their Zygmund means in the terms of the best approximation $E_n(f)_M$ and modulus of smoothness $\omega_k(f,\cdot)_M$ of these functions in the Orlicz spaces $L_M(\mathbb{T})$. Note that in the proof of the main results we use the methot in the [40] and [11].

Our main results are the following.

THEOREM 1. *Let $L_M(\mathbb{T})$ be a reflexive Orlicz space and $k \in \mathbb{N}$. Then for every $f \in L_M(\mathbb{T})$ the inequality*

$$\| f - Z_{n,k}(\cdot,f) \|_{L_M(\mathbb{T})} \leqslant \frac{c_1(M,k)}{(n+1)^k} \sum_{\nu=0}^{n} (\nu+1)^{k-1} E_\nu(f)_M \tag{2}$$

holds.

THEOREM 2. *Let $L_M(\mathbb{T})$ be a reflexive Orlicz space and $k \in \mathbb{N}$. Then for every $f \in L_M(\mathbb{T})$ the inequality*

$$\| f - Z_{n,k}(\cdot,f) \|_{L_M(\mathbb{T})} \leqslant c_2(M,k)\, \omega_k(f, \frac{\pi}{n})_M \tag{3}$$

holds.

Note that Theorems 1 and 2 in the Lebesgue spaces $L_p(\mathbb{T})$, $p \geqslant 1$ were obtained in [40] and [11] respectively.

2. Proofs of the Results

Proof of Theorem 1. We consider the trigonometric polynomial

$$T_n(x)=\sum_{\nu=0}^{n}(\alpha_\nu\cos\nu x+\beta_\nu\sin\nu x).$$

The following inequality holds:

$$\begin{aligned}
\|f-Z_{n,k}(\cdot,f)\|_{L_M(\mathbb{T})}&=\left\|f-\sum_{\nu=0}^{n}(1-\frac{\nu^k}{(n+1)^k})A_\nu(\cdot,f)\right\|_{L_M(\mathbb{T})}\\
&\leqslant\|f-T_n\|_{L_M(\mathbb{T})}+\left\|T_n-\sum_{\nu=0}^{n}\left(1-\frac{\nu^k}{(n+1)^k}\right)(\alpha_\nu\cos\nu x+\beta_\nu\sin\nu x)\right\|_{L_M(\mathbb{T})}\\
&+\left\|\sum_{\nu=0}^{n}\left(1-\frac{\nu^k}{(n+1)^k}\right)A_\nu(\cdot,\ f)-\sum_{\nu=0}^{n}(\alpha_\nu\cos\nu x+\beta_\nu\sin\nu x)\left(1-\frac{\nu^k}{(n+1)^k}\right)\right\|_{L_M(\mathbb{T})}\\
&=\|f-T_n\|_{L_M(\mathbb{T})}+\left\|T_n-\sum_{\nu=0}^{n}\left(1-\frac{\nu^k}{(n+1)^k}\right)(\alpha_\nu\cos\nu x+\beta_\nu\sin\nu x)\right\|_{L_M(\mathbb{T})}\\
&+\left\|\frac{1}{\pi}\int_0^{2\pi}\{f(x+\theta)-T_n(x+\theta)\}\left\{\frac{1}{2}+\sum_{\nu=1}^{n}\left(1-\frac{\nu^k}{(n+1)^k}\right)\cos\nu\theta\right\}d\theta\right\|_{L_M(\mathbb{T})}\\
&\leqslant\|f-T_n\|_{L_M(\mathbb{T})}+K_n\|f(\cdot+h)-T_n(\cdot+h)\|_{L_M(\mathbb{T})}\\
&+\left\|T_n-\sum_{\nu=0}^{n}\left(1-\frac{\nu^k}{(n+1)^k}\right)(\alpha_\nu\cos\nu x+\beta_\nu\sin\nu x)\right\|_{L_M(\mathbb{T})}\\
&\leqslant(1+K_n)\|f-T_n\|_{L_M(\mathbb{T})}+R_n(T_n)_M,
\end{aligned}\tag{4}$$

where

$$\begin{aligned}
K_n&=\frac{2}{\pi}\int_0^{\pi}\left|\frac{1}{2}+\sum_{\nu=1}^{n}\lambda_\nu(n)\cos\nu\theta\right|d\theta,\\
\lambda_\nu(n)&=1-\frac{\nu^k}{(n+1)^k},\ k=1,2,...\\
R_n(T_n)_M&=\left\|T_n-\sum_{\nu=0}^{n}\left(1-\frac{\nu^k}{(n+1)^k}\right)(\alpha_\nu\cos\nu x+\beta_\nu\sin\nu x)\right\|_{L_M(\mathbb{T})}.
\end{aligned}$$

Let $f\in L_M(\mathbb{T})$ and let $T_n\in\Pi_n$ $(n=0,1,2,...)$ be the polynomial of best approximation to f, i. e.

$$E_n(f)_M=\|f-T_n\|_{L_M(\mathbb{T})}.$$

Then using (4) we obtain

$$\begin{aligned} &\left\|f-Z_{n,k}(\cdot,f)\right\|_{L_M(\mathbb{T})} \\ &\leqslant (1+K_n)E_n(f)_M+\frac{1}{(n+1)^k}\left\|\sum_{\nu=1}^{n}\nu^k(\alpha_\nu\cos\nu x+\beta_\nu\sin\nu x)\right\|_{L_M(\mathbb{T})} \end{aligned} \tag{5}$$

Note that according to [20] $K_n \leqslant c_3$. Then the inequality (5) we write the following form:

$$\begin{aligned} &\left\|f-Z_{n,k}(\cdot,f)\right\|_{L_M(\mathbb{T})} \\ &\leqslant c_3E_n(f)_M+\frac{1}{(n+1)^k}\left\|\sum_{\nu=1}^{n}\nu^k(\alpha_\nu\cos\nu x+\beta_\nu\sin\nu x)\right\|_{L_M(\mathbb{T})} \end{aligned} \tag{6}$$

We suppose that k is even and the number $m\in N$ satisfies condition $2^m\leqslant n<2^{m+1}$. Then we have

$$\begin{aligned} &\left\|f-Z_{n,k}(\cdot,f)\right\|_{L_M(\mathbb{T})} \\ &\leqslant c_4E_n(f)_M+\frac{1}{(n+1)^k}\left\|T_n^{(k)}\right\|_{L_M(T)} \\ &\leqslant c_4E_n(f)_M+\frac{1}{(n+1)^k}\{\left\|T_2^{(k)}-T_0^{(k)}\right\|_{L_M(\mathbb{T})} \\ &\quad+\sum_{\nu=1}^{m}\left\|T_{2^{\nu+1}}^{(k)}-T_{2^\nu}^{(k)}\right\|_{L_M(\mathbb{T})}+\left\|T_n^{(k)}-T_{2^{m+1}}^{(k)}\right\|_{L_M(\mathbb{T})}\}. \end{aligned} \tag{7}$$

Since T_n is the polynomial best approximation we obtain

$$\begin{aligned} &\|T_{2^{\nu+1}}-T_{2^\nu}\|_{L_M(\mathbb{T})} \\ &\leqslant \|T_{2^{\nu+1}}-f\|_{L_M(\mathbb{T})}+\|f-T_{2^\nu}\|_{L_M(\mathbb{T})} \\ &\leqslant E_{2^{\nu+1}}(f)_M+E_{2^\nu}(f)_M\leqslant 2E_{2^\nu}(f)_M \end{aligned} \tag{8}$$

Using (8) and Bernstein inequality for trigonometric polynomial in the Orlicz spaces [22], [14] we have

$$\begin{aligned} \left\|T_{2^{\nu+1}}^{(k)}-T_{2^\nu}^{(k)}\right\|_{L_M(\mathbb{T})} &\leqslant c_5 2^{(\nu+1)k}\|T_{2^{\nu+1}}-T_{2^\nu}\|_{L_M(\mathbb{T})} \\ &\leqslant c_6 2^{(\nu+1)k}E_{2^\nu}(f)_M. \end{aligned} \tag{9}$$

Consideration of (7)and (9) gives us

$$\begin{aligned} &\left\|f-Z_{n,k}(\cdot,f)\right\|_{L_M(\mathbb{T})} \\ &\leqslant c_5E_n(f)_M+\frac{c_7}{(n+1)^k}\{\|T_2-T_0\|_{L_M(\mathbb{T})} \\ &\quad+\sum_{\nu=1}^{m}2^{(\nu+1)k}E_{2^\nu}(f)_M+\|T_n-T_{2^{m+1}}\|_{L_M(\mathbb{T})}\} \end{aligned} \tag{10}$$

The inequality

$$2^{(\nu+1)k}E_{2^\nu}(f)_M \leqslant 2^{2k}\sum_{m=2^{\nu-1}+1}^{2^\nu} m^{k-1}E_m(f)_M \tag{11}$$

holds. Really,

$$\sum_{m=2^{\nu-1}+1}^{2^\nu} m^{k-1} \geqslant \left(2^{\nu-1}\right)^{k-1}2^{\nu-1} = 2^{k(\nu-1)}.$$

Since $E_m(f)_M$ is monotonically decreasing, we conclude that

$$2^{(\nu+1)k}E_{2^\nu}(f)_M \leqslant 2^{2k}\sum_{m=2^{\nu-1}+1}^{2^\nu} m^{k-1}E_m(f)_M.$$

Now, as done in [14], we use the inequality (11) in (10) to obtain

$$\begin{aligned}
&\left\|f - Z_{n,k}(\cdot,f)\right\|_{L_M(\mathbb{T})}\\
&\leqslant c_5E_n(f)_M + \frac{c_8}{(n+1)^k}\{E_0(f)_M + 2^{2k}\sum_{\nu=1}^{m}\left(\sum_{m=2^{\nu-1}+1}^{2^\nu} m^{k-1}E_m(f)_M\right)\}\\
&\leqslant c_9E_n(f)_M + \frac{c_{10}}{(n+1)^k}\{E_0(f)_M + 2^{2k}\sum_{m=2}^{2^m} m^{k-1}E_m(f)_M\}\\
&\leqslant \frac{c_{11}}{(n+1)^k}\sum_{\nu=0}^{n}(\nu+1)^{k-1}E_\nu(f)_M.
\end{aligned}$$

Consequently, if k is even the inequality (2) is proved. Now let $k \geqslant 3$ be a odd. Then

$$\begin{aligned}
&R_n(T_n)_M\\
&= \frac{1}{(n+1)^{k-1}}\left\|T_n^{(k-1)} - \sum_{\nu=0}^{n}\left(1-\frac{\nu}{n+1}\right)\nu^{k-1}(\alpha_\nu\cos\nu x + \beta_\nu\sin\nu x)\right\|_{L_M(\mathbb{T})}.
\end{aligned} \tag{12}$$

According [18] we have

$$R_n(T_n)_M \leqslant \frac{c_{12}}{(n+1)^k}\sum_{\nu=0}^{n-1}E_\nu(T_n^{(k-1)})_M. \tag{13}$$

Note that by [23] and [3] the inequality

$$E_n(f^{(k)})_M \leqslant c_{13}\left\{n^kE_n(f)_M + \sum_{\nu=n+1}^{\infty}\nu^{k-1}E_\nu(f)_M.\right\} \tag{14}$$

holds. Using properties of sequence $\{E_n(f)_M\}$ and (14) we find that

$$\begin{aligned}
&\sum_{\nu=0}^{n-1} E_\nu(T_n^{(k-1)})_M \\
&\leqslant c_{14}\sum_{\nu=0}^{n-1}\left\{(\nu+1)^{k-1}E_\nu(T_n)_M+\sum_{s=\nu}^{n-1}(s+1)^{k-2}E_s(T_n)_M\right\} \\
&\leqslant c_{15}\sum_{\nu=0}^{n-1}(\nu+1)^{k-1}E_\nu(T_n)_M \leqslant c_{16}\sum_{\nu=0}^{n-1}(\nu+1)^{k-1}E_\nu(f)_M. \qquad (15)
\end{aligned}$$

Use of (15), (13) and (6) gives us inequality (2). Theorem 1 is proved.

Proof of Theorem 2. Let $f\in L_M(\mathbb{T})$. Then the following inequality holds:

$$\begin{aligned}
\left\|f-Z_{n,k}(\cdot,f)\right\|_{L_M(\mathbb{T})} &\leqslant \|f-S_n(\cdot,f)\|_{L_M(\mathbb{T})}+(n+1)^{-k}\left\|\nu^k A_\nu(\cdot,f)\right\|_{L_M(\mathbb{T})} \\
&= U_1+U_2^{(k)}. \qquad (16)
\end{aligned}$$

It is well known from [33], [14] that

$$U_1=\|f-S_n(\cdot,f)\|_{L_M(\mathbb{T})}\leqslant c_{17}(M)E_n(f)_M. \qquad (17)$$

By [32] and [2] we have

$$E_n(f)_M\leqslant c_{18}(k,M)\omega_k(f,\frac{\pi}{n})_M. \qquad (18)$$

Then by (17) and (18) we get

$$U_1=\|f-S_n(\cdot,f)\|_{L_M(\mathbb{T})}\leqslant c_{19}(k,M)\omega_k(f,\frac{\pi}{n})_M. \qquad (19)$$

We note that if k is even

$$\sum_{\nu=1}^{n}\nu^k A_\nu(x,f)=(-1)^{k/2}S_n^{(k)}(x,f),$$

if k is odd

$$\sum_{\nu=1}^{n}\nu^k A_\nu(x,f)=(-1)^{(k+3)/2}\widetilde{S_n}^{(k)}(x,f),$$

where $\widetilde{g(x)}$ is the function that is trigonometricaly conjugate to $g(x)$. Then

$$U_2^{(k)}=\begin{cases}(n+1)^{-k}\left\|S_n^{(k)}(\cdot,f)\right\|_{L_M(\mathbb{T})}, & k-\text{even}\\ (n+1)^{-k}\left\|\widetilde{S_n}^{(k)}(\cdot,f)\right\|_{L_M(\mathbb{T})}, & k-\text{odd}.\end{cases} \qquad (20)$$

If k is even, by inequalities (2.11), (3.1) of [4] and (20) we have

$$\begin{aligned} U_2^{(k)} &= (n+1)^{-k} \left\| S_n^{(k)}(\cdot,f) \right\|_{L_M(\mathbb{T})} \\ &\leqslant c_{20}(n+1)^{-k}2^{-k}n^{-k} \left\| \Delta_{\pi/n}^k S_n(\cdot,f) \right\|_{L_M(\mathbb{T})} \\ &\leqslant 2^{-k}c_{21} \left\| \Delta_{\pi/n}^k S_n(\cdot,f) \right\|_{L_M(\mathbb{T})} = 2^{-k}c_{21} \left\| \Delta_{\pi/n}^k (S_n(\cdot,f) - f + f) \right\|_{L_M(\mathbb{T})} \\ &\leqslant c_{22}(M,k) \left\{ \|f - S_n(\cdot,f)\|_{L_M(\mathbb{T})} + \left\| \Delta_{\pi/n}^k (f) \right\|_{L_M(\mathbb{T})} \right\} \\ &\leqslant c_{23}(M,k)\omega_k(f,\frac{\pi}{n})_M. \end{aligned} \tag{21}$$

Considering [33], [14] we have

$$\left\| \widetilde{S_n}^{(k)}(\cdot,f) \right\|_{L_M(\mathbb{T})} \leqslant c_{24} \left\| S_n^{(k)}(\cdot,f) \right\|_{L_M(\mathbb{T})}. \tag{22}$$

If k is odd, consideration of (20), (22) and (21) gives us

$$\begin{aligned} U_2^{(k)} &= (n+1)^{-k} \left\| \widetilde{S_n}^{(k)}(\cdot,f) \right\|_{L_M(\mathbb{T})} \\ &\leqslant c_{25}(n+1)^{-k} \left\| S_n^{(k)}(\cdot,f) \right\|_{L_M(\mathbb{T})} \leqslant c_{26}(M,k)\omega_k(f,\frac{\pi}{n})_M. \end{aligned} \tag{23}$$

Taking into account the realizations (6), (19), (21) and (23) we obtain the inequality (3). Theorem 2 is completely proved.

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