APPROXIMATION OF PERIODIC FUNCTIONS
BY ZYGMUND MEANS IN ORLICZ SPACES

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Abstract. In the present work we investigate the approximation of the functions by Zygmund means in the Orlicz spaces $L_M(T)$ in the terms of the best approximation $E_n(f)_M$ and modulus of smoothness $\omega_k(\cdot, f)_M$.

1. Introduction and Main Results

Let $M(u)$ be a continuous increasing convex function on $[0, \infty)$ such that $M(u)/u \to 0$ if $u \to 0$, and $M(u)/u \to \infty$ if $u \to \infty$. We denote by $N$ the complementary of $M$ in Young’s sense, i.e. $N(u) = \max \{uv - M(v) : v \geq 0\}$ if $u \geq 0$. We will say that $M$ satisfies the $\Delta_2$–condition if $M(2u) \leq cM(u)$ for any $u \geq u_0 \geq 0$ with some constant $c$, independent of $u$.

Let $\mathbb{T}$ denote the interval $[-\pi, \pi]$, $\mathbb{C}$ the complex plane, and $L_p(\mathbb{T})$, $1 \leq p \leq \infty$, the Lebesgue space of measurable complex-valued functions on $\mathbb{T}$.

For a given Young function $M$, let $\tilde{L}_M(\mathbb{T})$ denote the set of all Lebesgue measurable functions $f : \mathbb{T} \to \mathbb{C}$ for which

$$\int_{\mathbb{T}} M(|f(x)|) \, dx < \infty.$$ 

Let $N$ be the complementary Young function of $M$. It is well-known [21, p. 69], [34, pp. 52-68] that the linear span of $\tilde{L}_M(\mathbb{T})$ equipped with the Orlicz norm

$$\|f\|_{L_M(\mathbb{T})} := \sup \left\{ \int_{\mathbb{T}} |f(x)| g(x) \, dx : g \in \tilde{L}_N(\mathbb{T}), \int_{\mathbb{T}} N(|g(x)|) \, dx \leq 1 \right\},$$

or with the Luxemburg norm

$$\|f\|_{L_M(\mathbb{T})}^* := \inf \left\{ k > 0 : \int_{\mathbb{T}} M\left(\frac{|f(x)|}{k}\right) \, dx \leq 1 \right\}.$$


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becomes a Banach space. This space is denoted by \( L_M(\mathbb{T}) \) and is called an Orlicz space [21, p. 26]. The Orlicz spaces are known as the generalizations of the Lebesgue spaces \( L_p(\mathbb{T}) \), \( 1 < p < \infty \). The Luxemburg and Orlicz norms are equivalent since [21, p. 80]

\[
\|f\|_{L^*_M(\mathbb{T})} \leq \|f\|_{L^*_M(\mathbb{T})} \leq 2 \|f\|_{L^*_M(\mathbb{T})}, \quad f \in L_M(\mathbb{T}).
\]

If we choose \( M(u) = u^p / p \) \( (1 < p < \infty) \) then the complementary function is \( N(u) = u^q / q \) with \( \frac{1}{p} + \frac{1}{q} = 1 \) and we have the relation

\[
p^{-1/p} \|u\|_{L^p(\mathbb{T})} = \|u\|_{L^*_M(\mathbb{T})} \leq \|u\|_{L^*_M(\mathbb{T})} \leq q^{1/q} \|u\|_{L^p(\mathbb{T})},
\]

where \( \|u\|_{L^p(\mathbb{T})} = \left( \int_{\mathbb{T}} |u(x)|^p \, dx \right)^{1/p} \) denotes the usual norm of the \( L_p(\mathbb{T}) \) space.

The Orlicz space \( L_M(\mathbb{T}) \) is reflexive if and only if the \( N \) function \( M \) and its complementary function \( N \) both satisfy the \( \Delta_2 \) condition [34, p. 113].

Note that the detailed information about properties of the Orlicz spaces can be found in [6], [7], [21], [27], [28] and [29].

Let \( L_M(\mathbb{T}) \) be an Orlicz space. Suppose that \( x, h \) are real, and let us take into account the sum

\[
\Delta^k_h f(x) = \sum_{\nu=0}^k (-1)^{k-\nu} \binom{k}{\nu} f(x + \nu h), \quad f \in L_M(\mathbb{T}), \ k \in \mathbb{N},
\]

where

\[
\binom{k}{\nu} := \frac{k(k-1)\ldots(k-\nu+1)}{\nu!}.
\]

The function

\[
\omega_k(f, \delta)_M := \sup_{0<h<\delta} \left\| \Delta^k_h f(x) \right\|_{L^*_M(\mathbb{T})}, \quad \delta > 0
\]

is called \( k \)th modulus of smoothness of \( f \in L_M(\mathbb{T}) \).

It can easily be shown that \( \omega_k(f, \delta)_M \) is a continuous, non-negative and non-decreasing function satisfying the conditions

\[
\lim_{\delta \to 0^+} \omega_k(f, \delta)_M = 0, \quad \omega_k(f + g, \cdot)_M \leq \omega_k(f, \cdot)_M + \omega_k(g, \cdot)_M,
\]

for \( f, g \in L_M(\mathbb{T}) \).

Let

\[
a_0 + \sum_{k=1}^{\infty} A_k(x, f)
\]

be the Fourier series of the function \( f \in L_1(\mathbb{T}) \), where \( A_k(x, f) := (a_k(f) \cos kx + b_k(f) \sin kx) \), \( a_k(f) \) and \( b_k(f) \) are Fourier coefficients of the function \( f \in L_1(\mathbb{T}) \).
The $n$–th partial sums, Zygmund means of order $k$ ($k \in \mathbb{N}$) of the series (1) are defined, respectively as [10], [40]

\[
S_n(x, f) = \frac{a_0}{2} + \sum_{\nu=1}^{n} A_\nu(x, f), \\
Z_{n,k}(x, f) = \frac{a_0}{2} + \sum_{\nu=1}^{n} \left(1 - \frac{\nu^k}{(n+1)^k}\right) A_\nu(x, f), \quad k = 1, 2, \ldots, \ n = 1, 2, \ldots
\]

It is clear that

\[
S_0(x, f) = Z_{0,k}(x, f) = \frac{a_0}{2}.
\]

We denote by $E_n(f)_M$ the best approximation of $f \in L_M(\mathbb{T})$ by trigonometric polynomials of degree not exceeding $n$, i.e.,

\[
E_n(f)_M = \inf\{\|f - T_n\|_{L_M(\mathbb{T})} : T_n \in \Pi_n\}
\]

where $\Pi_n$ denotes the class of trigonometric polynomials of degree at most $n$.

The approximation problems by trigonometric polynomials in nonweighted and weighted Orlicz spaces have been investigated by several authors (see, for example, [1]-[4], [9], [10], [13], [15]-[18], [23], [31], [32], [35], [42]). The approximation of the functions by the means of Fourier trigonometric series in different spaces were studied in [5], [11], [12], [19], [24]-[26], [36]-[41] and [43]. In the present paper, we investigate the deviation of functions from their Zygmund means in the terms of the best approximation $E_n(f)_M$ and modulus of smoothness $\omega_k(f, \cdot)_M$ of these functions in the Orlicz spaces $L_M(\mathbb{T})$. Note that in the proof of the main results we use the method in the [40] and [11].

Our main results are the following.

**THEOREM 1.** Let $L_M(\mathbb{T})$ be a reflexive Orlicz space and $k \in \mathbb{N}$. Then for every $f \in L_M(\mathbb{T})$ the inequality

\[
\|f - Z_{n,k}(\cdot, f)\|_{L_M(\mathbb{T})} \leq \frac{c_1(M,k)}{(n+1)^k} \sum_{\nu=0}^{n} (\nu+1)^{k-1} E_\nu(f)_M
\]

holds.

**THEOREM 2.** Let $L_M(\mathbb{T})$ be a reflexive Orlicz space and $k \in \mathbb{N}$. Then for every $f \in L_M(\mathbb{T})$ the inequality

\[
\|f - Z_{n,k}(\cdot, f)\|_{L_M(\mathbb{T})} \leq c_2(M,k) \omega_k(f, \frac{\pi}{n})_M
\]

holds.

Note that Theorems 1 and 2 in the Lebesgue spaces $L_p(\mathbb{T}), \ p \geq 1$ were obtained in [40] and [11] respectively.
2. Proofs of the Results

Proof of Theorem 1. We consider the trigonometric polynomial

\[ T_n(x) = \sum_{\nu=0}^{n} (\alpha_\nu \cos \nu x + \beta_\nu \sin \nu x). \]

The following inequality holds:

\[
\left\| f - Z_{n,k} \right\|_{L_M(\mathbb{T})} = \left\| f - \sum_{\nu=0}^{n} \left( 1 - \frac{\nu^k}{(n+1)^k} \right) A_\nu(\cdot, f) \right\|_{L_M(\mathbb{T})} \\
\leq \left\| f - T_n \right\|_{L_M(\mathbb{T})} + \left\| T_n - \sum_{\nu=0}^{n} \left( 1 - \frac{\nu^k}{(n+1)^k} \right) (\alpha_\nu \cos \nu x + \beta_\nu \sin \nu x) \right\|_{L_M(\mathbb{T})} \\
+ \left\| \sum_{\nu=0}^{n} \left( 1 - \frac{\nu^k}{(n+1)^k} \right) A_\nu(\cdot, f) - \sum_{\nu=0}^{n} \left( 1 - \frac{\nu^k}{(n+1)^k} \right) (\alpha_\nu \cos \nu x + \beta_\nu \sin \nu x) \right\|_{L_M(\mathbb{T})} \\
= \left\| f - T_n \right\|_{L_M(\mathbb{T})} + \left\| T_n - \sum_{\nu=0}^{n} \left( 1 - \frac{\nu^k}{(n+1)^k} \right) (\alpha_\nu \cos \nu x + \beta_\nu \sin \nu x) \right\|_{L_M(\mathbb{T})} \\
+ \left\| \frac{1}{\pi} \int_{0}^{2\pi} \sum_{\nu=0}^{n} \left( 1 - \frac{\nu^k}{(n+1)^k} \right) (\alpha_\nu \cos \nu x + \beta_\nu \sin \nu x) \right\|_{L_M(\mathbb{T})} \\
\leq (1 + K_n) \left\| f - T_n \right\|_{L_M(\mathbb{T})} + R_n(T_n),
\]

where

\[ K_n = \frac{2}{\pi} \int_{0}^{\pi} \frac{1}{2} + \sum_{\nu=1}^{n} \lambda_\nu(n) \cos \nu \theta \, d\theta; \]

\[ \lambda_\nu(n) = 1 - \frac{\nu^k}{(n+1)^k}, \quad k = 1, 2, \ldots \]

\[ R_n(T_n) = \left\| T_n - \sum_{\nu=0}^{n} \left( 1 - \frac{\nu^k}{(n+1)^k} \right) (\alpha_\nu \cos \nu x + \beta_\nu \sin \nu x) \right\|_{L_M(\mathbb{T})}. \]

Let \( f \in L_M(\mathbb{T}) \) and let \( T_n \in \Pi_n \ (n = 0, 1, 2, \ldots) \) be the polynomial of best approximation to \( f \), i. e.

\[ E_n(f)_M = \left\| f - T_n \right\|_{L_M(\mathbb{T})}. \]
Then using (4) we obtain
\[
\| f - Z_{n,k}(\cdot, f) \|_{L_M(\T)} \\
\leq (1 + K_n) E_n(f)_M + \frac{1}{(n+1)^k} \left| \sum_{1 \leq \nu \leq n} \nu^k (\alpha\nu \cos \nu x + \beta\nu \sin \nu x) \right|_{L_M(\T)}
\] (5)

Note that according to [20] \( K_n \leq c_3 \). Then the inequality (5) we write the following form:
\[
\| f - Z_{n,k}(\cdot, f) \|_{L_M(\T)} \\
\leq c_3 E_n(f)_M + \frac{1}{(n+1)^k} \left| \sum_{1 \leq \nu \leq n} \nu^k (\alpha\nu \cos \nu x + \beta\nu \sin \nu x) \right|_{L_M(\T)}
\] (6)

We suppose that \( k \) is even and the number \( m \in \mathbb{N} \) satisfies condition \( 2^m \leq n < 2^{m+1} \). Then we have
\[
\| f - Z_{n,k}(\cdot, f) \|_{L_M(\T)} \\
\leq c_4 E_n(f)_M + \frac{1}{(n+1)^k} \left\{ \| T_n^{(k)} - T_0^{(k)} \|_{L_M(\T)} + \| T_{2^{m+1}}^{(k)} - T_{2^{m+1}}^{(k)} \|_{L_M(\T)} \right\}
\] (7)

Since \( T_n \) is the polynomial best approximation we obtain
\[
\| T_{2^{m+1}}^{(k)} - T_{2^{m+1}}^{(k)} \|_{L_M(\T)} \\
\leq \| T_{2^{m+1}} - f \|_{L_M(\T)} + \| f - T_{2^{m+1}} \|_{L_M(\T)} \\
\leq E_{2^{m+1}}(f)_M + E_{2^{m+1}}(f)_M \leq 2E_{2^{m+1}}(f)_M
\] (8)

Using (8) and Bernstein inequality for trigonometric polynomial in the Orlicz spaces [22], [14] we have
\[
\| T_{2^{m+1}}^{(k)} - T_{2^{m+1}}^{(k)} \|_{L_M(\T)} \\
\leq c_5 2^{(v+1)k} \| T_{2^{m+1}} - T_{2^{m+1}} \|_{L_M(\T)} \\
\leq c_6 2^{(v+1)k} E_{2^{m+1}}(f)_M.
\] (9)

Consideration of (7)and (9) gives us
\[
\| f - Z_{n,k}(\cdot, f) \|_{L_M(\T)} \\
\leq c_7 E_n(f)_M + \frac{c_7}{(n+1)^k} \left\{ \| T_2 - T_0 \|_{L_M(\T)} + \| T_n - T_{2^{m+1}} \|_{L_M(\T)} \right\}
\] (10)
The inequality
\[ 2^{(v+1)k} E_{2^v}(f)_M \leq 2^{2k} \sum_{m=2^{v-1}+1}^{2^v} m^{k-1} E_m(f)_M \] (11)
holds. Really,
\[ \sum_{m=2^{v-1}+1}^{2^v} m^{k-1} \geq (2^{v-1})^{k-1} 2^{v-1} = 2^{k(v-1)}. \]
Since \( E_m(f)_M \) is monotonically decreasing, we conclude that
\[ 2^{(v+1)k} E_{2^v}(f)_M \leq 2^{2k} \sum_{m=2^{v-1}+1}^{2^v} m^{k-1} E_m(f)_M. \]

Now, as done in [14], we use the inequality (11) in (10) to obtain
\[
\left\| f - Z_{n,k}(\cdot, f) \right\|_{LM(\mathbb{T})} \\
\leq c_5 E_n(f)_M + \frac{c_8}{(n+1)^k} \left\{ E_0(f)_M + 2^{2k} \sum_{v=1}^{m} \left( \sum_{m=2^{v-1}+1}^{2^v} m^{k-1} E_m(f)_M \right) \right\} \\
\leq c_9 E_n(f)_M + \frac{c_{10}}{(n+1)^k} \left\{ E_0(f)_M + 2^{2k} \sum_{m=2}^{2m} m^{k-1} E_m(f)_M \right\} \\
\leq \frac{c_{11}}{(n+1)^k} \sum_{v=0}^{n} (v+1)^{k-1} E_v(f)_M. \]

Consequently, if \( k \) is even the inequality (2) is proved. Now let \( k \geq 3 \) be a odd.

Then
\[
R_{n}(T_n)M = \frac{1}{(n+1)^{k-1}} \left\| T_n^{(k-1)} - \sum_{v=0}^{n} \left( 1 - \frac{v}{n+1} \right) v^{k-1}(\alpha_v \cos \nu x + \beta_v \sin \nu x) \right\|_{LM(\mathbb{T})} \] (12)
According [18] we have
\[
R_{n}(T_n)M \leq \frac{c_{12}}{(n+1)^k} \sum_{v=0}^{n-1} E_v(T_n^{(k-1)})_M. \] (13)

Note that by [23] and [3] the inequality
\[
E_n(f^{(k)})_M \leq c_{13} \left\{ n^k E_n(f)_M + \sum_{v=n+1}^{\infty} v^{k-1} E_v(f)_M \right\} \] (14)
holds. Using properties of sequence \( \{E_n(f)_M\} \) and (14) we find that
\[
\sum_{\nu=0}^{n-1} E_\nu(T_n(k-1))_M \\
\leq c_{14} \sum_{\nu=0}^{n-1} (\nu + 1)^{k-1} E_\nu(T_n)_M + \sum_{s=\nu}^{n-1} (s + 1)^{k-2} E_s(T_n)_M
\]
\[
\leq c_{15} \sum_{\nu=0}^{n-1} (\nu + 1)^{k-1} E_\nu(T_n)_M \leq c_{16} \sum_{\nu=0}^{n-1} (\nu + 1)^{k-1} E_\nu(f)_M.
\]
(15)

Use of (15), (13) and (6) gives us inequality (2). Theorem 1 is proved.

Proof of Theorem 2. Let \( f \in LM(\mathbb{T}) \). Then the following inequality holds:
\[
\|f - Z_{n,k}(\cdot, f)\|_{LM(\mathbb{T})} \leq \|f - S_n(\cdot, f)\|_{LM(\mathbb{T})} + (n + 1)^{-k} \left\| v^k A_\nu(\cdot, f) \right\|_{LM(\mathbb{T})}
\]
\[
= U_1 + U_2^{(k)}.
\]
(16)

It is well known from [33], [14] that
\[
U_1 = \|f - S_n(\cdot, f)\|_{LM(\mathbb{T})} \leq c_{17}(M) E_n(f)_M.
\]
(17)

By [32] and [2] we have
\[
E_n(f)_M \leq c_{18}(k, M) \omega_k(f, \frac{\pi}{n})_M.
\]
(18)

Then by (17) and (18) we get
\[
U_1 = \|f - S_n(\cdot, f)\|_{LM(\mathbb{T})} \leq c_{19}(k, M) \omega_k(f, \frac{\pi}{n})_M.
\]
(19)

We note that if \( k \) is even
\[
\sum_{\nu=1}^{n} v^k A_\nu(x, f) = (-1)^{k/2} S_n^{(k)}(x, f),
\]
if \( k \) is odd
\[
\sum_{\nu=1}^{n} v^k A_\nu(x, f) = (-1)^{(k+3)/2} \tilde{S}_n^{(k)}(x, f),
\]
where \( \tilde{g}(x) \) is the function that is trigonometrically conjugate to \( g(x) \). Then
\[
U_2^{(k)} = \begin{cases} 
(n + 1)^{-k} \left\| S_n^{(k)}(\cdot, f) \right\|_{LM(\mathbb{T})}, & k \text{ even} \\
(n + 1)^{-k} \left\| \tilde{S}_n^{(k)}(\cdot, f) \right\|_{LM(\mathbb{T})}, & k \text{ odd}
\end{cases}
\]
(20)
If $k$ is even, by inequalities (2.11), (3.1) of [4] and (20) we have

$$U_2^{(k)} = (n + 1)^{-k} \left\| \tilde{S}_n^{(k)}(\cdot, f) \right\|_{LM(T)} \leq c_{20}(n + 1)^{-k} 2^{-k} n^{-k} \left\| \Delta^k_{\pi/n} S_n(\cdot, f) \right\|_{LM(T)}$$

$$\leq 2^{-k} c_{21} \left\| \Delta^k_{\pi/n} S_n(\cdot, f) \right\|_{LM(T)} = 2^{-k} c_{21} \left\| \Delta^k_{\pi/n} (S_n(\cdot, f) - f + f) \right\|_{LM(T)}$$

$$\leq c_{22}(M, k) \left\{ \left\| f - S_n(\cdot, f) \right\|_{LM(T)} + \left\| \Delta^k_{\pi/n} (f) \right\|_{LM(T)} \right\}$$

$$\leq c_{23}(M, k) \omega_k(f, \frac{\pi}{n})_M.$$  \hspace{1cm} (21)

Considering [33], [14] we have

$$\left\| \tilde{S}_n^{(k)}(\cdot, f) \right\|_{LM(T)} \leq c_{24} \left\| S_n^{(k)}(\cdot, f) \right\|_{LM(T)}.$$ \hspace{1cm} (22)

If $k$ is odd, consideration of (20), (22) and (21) gives us

$$U_2^{(k)} = (n + 1)^{-k} \left\| S_n^{(k)}(\cdot, f) \right\|_{LM(T)} \leq c_{25}(n + 1)^{-k} \left\| S_n^{(k)}(\cdot, f) \right\|_{LM(T)} \leq c_{26}(M, k) \omega_k(f, \frac{\pi}{n})_M.$$ \hspace{1cm} (23)

Taking into account the realizations (6), (19), (21) and (23) we obtain the inequality (3). Theorem 2 is completely proved.

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