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ON THE RATIO OF TWO SETS IN REAL LINE

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Abstract. In this paper, assuming Martin's axiom we show that there exists a Lebesgue measurable subset A of the real line $\mathbb{R}$ such that the set $\{c \in \mathbb{R} : R(A, c+A)$ contains an interval$\}$ is non-measurable. Here the set $R(A, c+A) = \{\frac{a}{c+a'} : a, a' \in A, c+a' \neq 0\}$. Also other two results on the ratio set of linear sets are presented.

1. Introduction

In 1920, H. Steinhaus [14] established that the sum set $A+B = \{a+b : a \in A,\ b \in B\}$ of two linear sets A and B each of which having positive Lebesgue measure contains an interval.

We recall the definition of the property of Baire of a linear set as follows:

DEFINITION 1. ([12]) A set A is said to have the property of Baire if it can be expressed as symmetric difference of an open set and a set of first category .

The category analogue of Steinhaus's result was established by Piccard [13] in the following way:

If both A and B are second category subsets of $\mathbb{R}$, each having the property of Baire, then the set $A+B = \{a+b : a \in A, b \in B\}$ contains an interval.

Several authors ([3], [4], [5], [7], [8], [10], [11]) have generalized the results of Steinhaus and Piccard in many directions. In 1962, N. C. Bose Majumder [2] introduced the concept of Ratio set of A in $\mathbb{R}$ with non zero abscissa which is stated as follows:

DEFINITION 2. ([2]) The Ratio set of a linear set A of non zero abscissa denoted by $R(A,A)$ is defined by $R(A,A) = \{\frac{a}{b}$ or $\frac{b}{a} : a, b \in A\}$.
Also we define $R(A, c+A) = \{\frac{a}{c+a'} : a, a' \in A, c+a' \neq 0\}$.

Bose Majumder [2] established that the Ratio set $R(A,A)$ of a linear set A with positive abscissa having positive Lebesgue measure contains an interval with left hand end point 1. The category analogue of Bose Majumder result was established by Ganguly and Basu [6] in the following way:

If A is a second category subset of non zero reals with the property of Baire, then the set $R(A,A)$ contains an interval of the form $[1, \xi)$, $(\xi > 1)$.

Again we recall a definition of Bernstein set in $\mathbb{R}$.

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DEFINITION 3. ([12]) A set B $(\subseteq \mathbb{R})$ is said to be Bernstein set if both B and B^c $(B^c = \mathbb{R} \setminus B)$ have non empty intersection with every uncountable closed set in $\mathbb{R}$.

If B is Bernstein set then it follows that B^c is as well a Bernstein set. Such sets are neither Lebesgue measurable nor have the property of Baire and as such are of interest to analysists, topologists and descriptive set theorists alike. It follows from the definition that a Bernstein set is everywhere full measure (outer) and everywhere second category.

In 1970, D.A.Martin and R.M.Solovay [9] introduced the notion of *Martin's axiom* which is independent of the usual axioms of ZFC set theory. The consequences of Martin's axiom are as follows:

The union of any cardinal k or fewer null sets on a Polish space (Complete and Separable space) is null. In particular, the union of k or fewer subsets of $\mathbb{R}$ of Lebesgue measure zero also has Lebesgue measure zero and union of k or fewer first category subsets of $\mathbb{R}$ is also a first category subset of $\mathbb{R}$.

In this paper, at first we show that for any given distinct real sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ with distinct members each, there exist subsets A and B of $\mathbb{R}$ such that

$$R(A, x_n + B) = \mathbb{R}^* = \mathbb{R} \setminus \{0\} \text{ for all } n \in \mathbb{N} \text{ and}$$
$$R(A, y_n + B) \text{ contains no interval for all } n \in \mathbb{N},$$
$$\text{where } \mathbb{N} \text{ is the set of natural numbers.}$$

Next we show there exists a G_δ-set A which is contained in a set of positive measure but not contained in any F_σ-set of measure zero such that the ratio set of A is empty interior.

Finally with the help of Martin's axiom, an attempt has been made to established that there exists a measurable subset A of $\mathbb{R}$ such that the set $\{c \in \mathbb{R} : R(A, c + A)$ contains an interval $\}$ is nonmeasurable.

2. Results

THEOREM 1. *If $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are any two sequences of real numbers such that $x_n \neq x_m$ and $y_n \neq y_m$ for all $n \neq m$ and $x_n \neq y_m$ for all $n, m \in \mathbb{N}$, where $\mathbb{N}$ is the set of natural numbers, then there exist subsets A and B of $\mathbb{R}$ such that*

(a) $R(A, x_n + B) = \mathbb{R}^* = \mathbb{R} \setminus \{0\}$ for all $n \in \mathbb{N}$ and

(b) $R(A, y_n + B)$ contains no interval for all $n \in \mathbb{N}$.

Proof. Let $\{z_\alpha : \alpha < \omega_c\}$ be well ordering of all non zero real numbers $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$, where ω_c is the first ordinal number having cardinal c of the continuum. We will use transfinite induction to construct two sequences $\{u_{\alpha n}\}_{n=1}^{\infty}$ and $\{v_{\alpha n}\}_{n=1}^{\infty}$ for each $\alpha < \omega_c$ such that the sets $A = \{u_{\alpha n} : \alpha < \omega_c, n \in \mathbb{N}\}$ and $B = \{v_{\alpha n} : \alpha < \omega_c, n \in \mathbb{N}\}$ satisfy the conditions of the theorem. At first we will construct the sequences $\{u_{1n}\}_{n=1}^{\infty}$ and $\{v_{1n}\}_{n=1}^{\infty}$ by ordinary induction.

For x_1, y_n for all $n \in \mathbb{N}$ and z_1 we will show that there exist $u_{11} \in \mathbb{R}^*$ and $v_{11} \in \mathbb{R}$ such that $\frac{u_{11}}{x_1 + v_{11}} = z_1$ and $\frac{u_{11}}{y_n + v_{11}}$ is an irrational number for all $n \in \mathbb{N}$. If we consider

$u_{11} = u$ an arbitrary nonzero real number, then in order to establish $\frac{u_{11}}{x_1+v_{11}} = z_1$ we have $v_{11} = z_1^{-1}u - x_1$. For this value of v_{11}, we have $\frac{u_{11}}{y_n+v_{11}} = \frac{u}{y_n+z_1^{-1}u-x_1}$. From hypothesis of the sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ the last expression is irrational for all fixed $n \in \mathbb{N}$ and for all $u \in \mathbb{R}^* \setminus D$, where D is a denumerable subset of $\mathbb{R}$. Thus, if $u_{11} = u$ and $v_{11} = z_1^{-1}u - x_1$ then $\frac{u_{11}}{x_1+v_{11}} = z_1$ and $\frac{u_{11}}{y_n+v_{11}}$ is irrational for all $n \in \mathbb{N}$ provided $u \in \mathbb{R}^* \setminus D$, where D is a denumerable subset of $\mathbb{R}$.

Suppose there exist nonzero real numbers $\{u_{1n}\}_{n=1}^{k}$ and real numbers $\{v_{1n}\}_{n=1}^{k}$ such that $\frac{u_{1n}}{x_n+v_{1n}} = z_1$ for all $n = 1,2,3,......,k$ and $\frac{u_{1i}}{y_n+v_{1j}}$ is irrational for all $i, j \in \{1,2,3,.......,k\}$ and $n \in \mathbb{N}$.

Let $u_{1,k+1} = u$ be an arbitrary nonzero real number and $v_{1,k+1} = z_1^{-1}u - x_{k+1}$, then by same argument as first step, we have $\frac{u_{1n}}{x_n+v_{1n}} = z_1$ for all $n = 1,2,3,......,k+1$ and $\frac{u_{1i}}{y_n+v_{1j}}$ is irrational for all $i, j \in \{1,2,3,.......,k+1\}$ and for all $n \in \mathbb{N}$ provided $u \in \mathbb{R}^* \setminus D_1$, where D_1 is a denumerable subset of $\mathbb{R}$.

So, by finite induction there exist sequence of nonzero real numbers $\{u_{1n}\}_{n=1}^{\infty}$ and sequence of real numbers $\{v_{1n}\}_{n=1}^{\infty}$ such that $\frac{u_{1n}}{x_n+v_{1n}} = z_1$ for all $n \in \mathbb{N}$ and $\frac{u_{1i}}{y_n+v_{1j}}$ is irrational for each $i, j, n \in \mathbb{N}$.

Now we apply transfinite induction. Let $\alpha < \omega_c$. Suppose for each $\beta < \alpha$ the sequence of nonzero real numbers $\{u_{\beta n}\}_{n=1}^{\infty}$ and sequence of real numbers $\{v_{\beta n}\}_{n=1}^{\infty}$ have been defined in such a way that $\frac{u_{\beta n}}{x_n+v_{\beta n}} = z_\beta$ for every $\beta < \alpha$, for every $n \in \mathbb{N}$ and $\frac{u_{\beta i}}{y_n+v_{\gamma j}}$ is irrational for every $\beta, \gamma < \alpha$ and $i, j, n \in \mathbb{N}$. So, by transfinite induction and same argument as before, for all $\alpha < \omega_c$ there exist sequence of nonzero real numbers $\{u_{\alpha n}\}_{n=1}^{\infty}$ and sequence of real numbers $\{v_{\alpha n}\}_{n=1}^{\infty}$ such that $\frac{u_{\alpha n}}{x_n+v_{\alpha n}} = z_\alpha$ for every $n \in \mathbb{N}$ and $\frac{u_{\beta i}}{y_n+v_{\gamma j}}$ is irrational for every $\beta, \gamma \leqslant \alpha$ and $i, j, n \in \mathbb{N}$. If we consider

$$A = \{u_{\alpha n} : \alpha < \omega_c, n \in \mathbb{N}\} \text{ and } B = \{v_{\alpha n} : \alpha < \omega_c, n \in \mathbb{N}\}$$

then $R(A, x_n + B) = \mathbb{R}^* = \mathbb{R} \setminus \{0\}$ for all $n \in \mathbb{N}$ and $\frac{a}{y_n+b}$ is irrational for each $n \in \mathbb{N}$, $a \in A$ and $b \in B$. Hence for each $n \in \mathbb{N}$ the set $R(A, y_n + B)$ contains no interval.

COROLLARY 1. *There exist two subsets A and B of $\mathbb{R}$ and real numbers r_1, r_2 and r with $r_1 < r < r_2$ such that $R(A, r_1 + B)$ and $R(A, r_2 + B)$ both contain intervals, but $R(A, r + B)$ contains no interval.*

Proof of the Corollary is immediate.

THEOREM 2. *If $M \subset \mathbb{R} \setminus \{0\} (= \mathbb{R}^*)$ be a set of positive measure there exists a G_δ-set $A \subseteq M$ not contained in any F_σ-set of measure zero such that $Int R(A, A) = \emptyset$.*

Proof. Without loss of generality we assume that M is a compact nowhere dense set. As M is compact we consider a sequence $\{x_i\}_{i \in \mathbb{N}} \subseteq M$ which is dense in M. Then for each $i \in \mathbb{N}$ the set $x_i M$ is nowhere dense and so the set $P = \bigcup_{i=1}^{\infty} x_i M$ is of first category in $\mathbb{R}^*$. Therefore the set $\mathbb{R}^* \setminus P$ is residual set. Then we can find a sequence

$\{y_i\} \subseteq \mathbb{R}^* \setminus P$ which is dense in $\mathbb{R}^*$. For each $i \in \mathbb{N}$, we have $\{y_i\} \cap \{x_i M\} = \emptyset$. Thus the closed set $A_i = M \cap x_i M$ is a nowhere dense in M for each i. Let $A = M \setminus \bigcup_{i=1}^{\infty} A_i$. But $\bigcup_{i=1}^{\infty} A_i$ is an F_σ-set and hence A is a G_δ set, residual in M. Since $\mathbb{R}$ is a complete metric space with usual metric, hence M is a complete metric subspace of $\mathbb{R}^*$. Therefore A is dense in M. It now follows from the category argument that A is not contained in any F_σ-set of measure zero (cf. Lemma 1.1 of [1]). Suppose $Int R(A,A) \neq \emptyset$. Then for some $i \in \mathbb{N}$, we can find $x_i \in R(A,A)$. Therefore $x_i = \frac{z}{y}$ for $y \in A$ and $z \in A$. Then $z \in M$ and $y \in M$; $z = x_i y \in x_i M$ and therefore $z \in A \cap A_i = \emptyset$, a contradiction. Hence $Int R(A,A) = \emptyset$ i.e. $R(A,A)$ has empty interior.

3. Nonmeasurable Set

THEOREM 3. *Assuming the Martin's axiom, there exists a measurable subset A of $\mathbb{R}$ such that the set $\{c \in \mathbb{R} : R(A, c + A)$ contains an interval $\}$ is a nonmeasurable set.*

Proof. Let $\{z_\alpha : \alpha < \omega_c\}$ be well ordering of $\mathbb{R}^*$. Assume $\{F_\alpha : \alpha < \omega_c\}$ and $\{G_\alpha : \alpha < \omega_c\}$ are well ordering of all uncountable closed subsets of $\mathbb{R}^*$ and collection of all dense open sets in $\mathbb{R}^*$ respectively. We will use transfinite induction to construct two single indexed transfinite sequences of real numbers

$$\{x_\alpha : \alpha < \omega_c\} \text{ and } \{y_\alpha : \alpha < \omega_c\}$$

and two double indexed transfinite sequences of real numbers

$$\{u_{\alpha\beta} : \alpha, \beta < \omega_c\} \text{ and } \{v_{\alpha\beta} : \alpha, \beta < \omega_c\}.$$

We will now proceed to construct four sequences with the desired properties. Pick $x_1 \in F_1, y_1 \in F_1 \setminus \{x_1\}$. Since G_1 is a dense open set in $\mathbb{R}$, by similar argument of existence of u_{11}, v_{11} in the proof of the Theorem 1, there exist

$$u_{11}, v_{11} \in G_1 \setminus \{\pm x_1, \pm y_1\}$$

such that

$$\tfrac{u_{11}}{x_1 + v_{11}} = z_1 \text{ and } R(K_1, y_1 + K_1) \cap \mathbb{Q} = \emptyset,$$

where $K_1 = \{u_{11}, v_{11}\}$ and $\mathbb{Q}$ is the set of rational numbers. Next pick $x_2 \in F_2 \setminus \{x_1, y_1\}$, $y_2 \in F_2 \setminus \{x_1, y_1, x_2\}$. Since the character of G_2 is same as G_1, there exist distinct pairs

$$u_{12}, v_{12}; u_{21}, v_{21}; u_{22}, v_{22} \in G_1 \cap G_2 \setminus \{\pm x_1, \pm x_2, \pm y_1, \pm y_2\}$$

such that

$$\tfrac{u_{12}}{x_1 + v_{12}} = z_2,\ \tfrac{u_{21}}{x_2 + v_{21}} = z_1,\ \tfrac{u_{22}}{x_2 + v_{22}} = z_2 \text{ and } R(K_2, y_i + K_2) \cap \mathbb{Q} = \emptyset, i = 1, 2,$$

where

$$K_2 = \{u_{11}, v_{11}, u_{12}, v_{12}, u_{21}, v_{21}, u_{22}, v_{22}\}.$$

Suppose that $\alpha < \omega_c$ and that $\{x_\beta : \beta < \alpha\}$, $\{y_\beta : \beta < \alpha\}$, $\{u_{\gamma\delta} : \gamma, \delta \leqslant \beta < \alpha\}$ and $\{v_{\gamma\delta} : \gamma, \delta \leqslant \beta < \alpha\}$ have been selected in such a way that the following conditions are satisfied.

- $x_\beta \in F_\beta \setminus \{x_\gamma, y_\gamma : \gamma < \beta\}$, $y_\beta \in F_\beta \setminus \{x_\beta, y_\gamma : \gamma < \beta\}$, for every $\beta < \alpha$.
- For each $\beta < \alpha$, $u_{i\beta}, v_{i\beta} \in \bigcap\limits_{i \leqslant \beta} G_i \setminus \{\pm x_i, \pm y_i\}$ for each $i < \beta$ and $u_{\beta i}, v_{\beta i} \in \bigcap\limits_{i \leqslant \beta} G_i \setminus \{\pm x_i, \pm y_i\}$ for each $i \leqslant \beta$.
- $\frac{u_{\gamma\delta}}{x_\gamma + v_{\gamma\delta}} = z_\delta$ for $\gamma, \delta \leqslant \beta < \alpha$ and $R(K_\beta, y_i + K_\beta) \cap \mathbb{Q} = \emptyset$ for each $i \leqslant \beta < \alpha$, where $K_\beta = \{u_{\gamma\delta} : \gamma, \delta \leqslant \beta < \alpha\} \cup \{v_{\gamma\delta} : \gamma, \delta \leqslant \beta < \alpha\}$.

We will now pick, in order,

$u_{1\alpha}, v_{1\alpha}; u_{2\alpha}, v_{2\alpha}; \ldots\ldots; u_{i\alpha}, v_{i\alpha}$; (for all $i < \alpha$), $x_\alpha, y_\alpha; u_{\alpha 1}, v_{\alpha 1}; u_{\alpha 2}, v_{\alpha 2}; \ldots\ldots; u_{\alpha\alpha}, v_{\alpha\alpha}$

with the desired properties. First we show that there exist

$$u_{1\alpha}, v_{1\alpha} \in \bigcap_{i \leqslant \alpha} G_i \setminus \{\pm x_j, \pm y_j : j < \alpha\}$$

such that

$$\frac{u_{1\alpha}}{x_1 + v_{1\alpha}} = z_\alpha \text{ and } R(K_\beta \cup \{u_{1\alpha}, v_{1\alpha}\}, y_j + (K_\beta \cup \{u_{1\alpha}, v_{1\alpha}\})) \cap \mathbb{Q} = \emptyset$$

for each $j < \alpha$.

Since $\alpha < \omega_c$, therefore α has cardinality less than that of the continuum. By Martin's axiom $\bigcap\limits_{i \leqslant \beta} G_i$ is a dense set in $\mathbb{R}^*$. If we consider $u_{1\alpha} = u$ then from the relation $\frac{u_{1\alpha}}{x_1 + v_{1\alpha}} = z_\alpha$, we get $v_{1\alpha} = z_\alpha^{-1} u - x_1$. Therefore $\frac{u_{1\alpha}}{y_j + v_{1\alpha}} = \frac{u}{y_j + z_\alpha^{-1} u - x_1}$ and $\frac{v_{1\alpha}}{y_j + u_{1\alpha}} = \frac{z_\alpha^{-1} - x_1}{y_j + u}$ are irrational for all $u \in \bigcap\limits_{i \leqslant \alpha} G_i \setminus \{\pm x_j, \pm y_j : j < \alpha\}$ (since $y_j \neq x_1$ for each $j < \alpha$). So, there exist u such that

$$u, (u z_\alpha^{-1} - x_1) \in \bigcap_{i \leqslant \alpha} G_i \setminus \{\pm x_j, \pm y_j : j < \alpha\} \text{ with } u_{1\alpha} = u \text{ and } v_{1\alpha} = u z_\alpha^{-1} - x_1.$$

Clearly $\frac{u_{1\alpha}}{x_1 + v_{1\alpha}} = z_\alpha$ and $R(K_\beta \cup \{u_{1\alpha}, v_{1\alpha}\}, y_j + (K_\beta \cup \{u_{1\alpha}, v_{1\alpha}\})) \cap \mathbb{Q} = \emptyset$ for each $j < \alpha$. By same argument, there exist

$$u_{2\alpha}, v_{2\alpha} \in \bigcap_{i \leqslant \alpha} G_i \setminus \{\pm x_j, \pm y_j : j < \alpha\}$$

such that $\frac{u_{2\alpha}}{x_2 + v_{2\alpha}} = z_\alpha$ and

$$R(K_\beta \cup \{u_{1\alpha}, v_{1\alpha}, u_{2\alpha}, v_{2\alpha}\}, y_j + (K_\beta \cup \{u_{1\alpha}, v_{1\alpha}, u_{2\alpha}, v_{2\alpha}\})) \cap \mathbb{Q} = \emptyset$$

for each $j < \alpha$.

By above argument we pick, in order, the elements $u_{3\alpha}, v_{3\alpha}; u_{4\alpha}, v_{4\alpha}; \ldots\ldots; u_{j\alpha}, v_{j\alpha}$ (for all $j < \alpha$) with the desired properties. That is there exist

$$u_{j\alpha}, v_{j\alpha} \in \bigcap_{i \leqslant \alpha} G_i \setminus \{\pm x_j, \pm y_j : j < \alpha\} \text{ such that } \tfrac{u_{j\alpha}}{x_j + v_{j\alpha}} = z_\alpha \text{ for all } j < \alpha$$

and

$$R(K_\beta \bigcup_{j<\alpha} \{u_{j\alpha}, v_{j\alpha}\}, y_j + (K_\beta \bigcup_{j<\alpha} \{u_{j\alpha}, v_{j\alpha}\})) \cap \mathbb{Q} = \emptyset \text{ for each } j < \alpha.$$

Next we have to pick x_α and y_α. Since α has cardinality less than that of continuum, $R(H_\alpha, y_i + H_\alpha) \cap \mathbb{Q} = \emptyset$ for every $i < \alpha$ (here $H_\alpha = K_\beta \cup \{u_{i\alpha}, v_{i\alpha} : i < \alpha\}$), F_α has cardinality of the continuum and $\mathbb{Q}$ is countable, there exist

$$x_\alpha \in F_\alpha \setminus \{x_\beta, y_\beta : \beta < \alpha\}$$

and

$$y_\alpha \in F_\alpha \setminus \{x_\alpha, y_\beta : \beta < \alpha\}$$

such that

$$R(H_\alpha, y_i + H_\alpha) \cap \mathbb{Q} = \emptyset$$

for every $i \leqslant \alpha$.

Now we proceed to pick, in order the elements $u_{\alpha 1}, v_{\alpha 1}; u_{\alpha 2}, v_{\alpha 2}; \ldots\ldots; u_{\alpha\alpha}, v_{\alpha\alpha}$ with the desired properties. We first pick

$$u_{\alpha 1}, v_{\alpha 1} \in \bigcap_{i \leqslant \alpha} G_i \setminus \{\pm x_i, \pm y_i\}$$

such that

$$\tfrac{u_{\alpha 1}}{x_\alpha + v_{\alpha 1}} = z_1$$

and

$$R(H_\alpha \cup \{u_{\alpha 1}, v_{\alpha 1}\}, y_i + (H_\alpha \cup \{u_{\alpha 1}, v_{\alpha 1}\})) \cap \mathbb{Q} = \emptyset$$

for each $i \leqslant \alpha$.

Such pair of elements exist by the similar argument of existence of $u_{1\alpha}$ and $v_{1\alpha}$ of the proof of this theorem. Now using the above argument, the elements $u_{\alpha 2}, v_{\alpha 2}; \ldots\ldots; u_{\alpha\alpha}, v_{\alpha\alpha}$ with the desired properties are selected. Therefore, by transfinite induction, there exist four transfinite sequences

$$\{x_\alpha : \alpha < \omega_c\},\ \{y_\alpha : \alpha < \omega_c\},\ \{u_{\alpha\beta} : \alpha, \beta < \omega_c\} \text{ and } \{v_{\alpha\beta} : \alpha, \beta < \omega_c\}$$

satisfying the following properties:

- $x_\alpha \in F_\alpha \setminus \{x_\beta, y_\beta : \beta < \alpha\}$ and $y_\alpha \in F_\alpha \setminus \{x_\alpha, y_\beta : \beta < \alpha\}$ for each $\alpha < \omega_c$.
- For each $\alpha < \omega_c$, we have $u_{i\alpha}, v_{i\alpha} \in \bigcap_{i \leqslant \alpha} G_i \setminus \{\pm x_i, \pm y_i\}$ for each $i < \alpha$ and $u_{\alpha i}, v_{\alpha i} \in \bigcap_{i \leqslant \alpha} G_i \setminus \{\pm x_i, \pm y_i\}$ for each $i \leqslant \alpha$.
- $\frac{u_{\alpha\beta}}{x_\alpha + v_{\alpha\beta}} = z_\beta$ for $\alpha, \beta < \omega_c$ and $R(A, y_i + A) \cap \mathbb{Q} = \emptyset$ for every $i < \omega_c$, where $A = \{u_{\alpha\beta} : \alpha, \beta < \omega_c\} \cup \{v_{\alpha\beta} : \alpha, \beta < \omega_c\}$.

Let $T = \{c \in \mathbb{R} : R(A, c + A)$ contains an interval$\}$, where $R(A, c + A)$ is defined. By above, $x_\alpha \in T$ for each $\alpha < \omega_c$ and $y_\alpha \notin T$ for each $\alpha < \omega_c$. Since $x_\alpha, y_\alpha \in F_\alpha$ for each $\alpha < \omega_c$, it follows that T is a Bernstein set and hence T is nonmeasurable [12]. By the above properties of the sequences $\{u_{\alpha\beta} : \alpha, \beta < \omega_c\}$ and $\{v_{\alpha\beta} : \alpha, \beta < \omega_c\}$ and by Martin's axioms it follows that, if $\varepsilon > 0$ and $G = G_\alpha$ (for some $\alpha < \omega_c$) is a dense open set in $\mathbb{R}^*$ and having measure less than ε, then $A = (A \cap G_\alpha) \cup (A \setminus G_\alpha)$ has measure less than ε, since $A \setminus G_\alpha$ has cardinality less than that of the continuum and hence has measure zero. Therefore $m(A) < \varepsilon$ for each $\varepsilon > 0$ and $m(A) = 0$. Hence the result.

4. Question

QUESTION 1. It is unknown whether formulation of Theorem 3 is possible or not without *Martin's Axiom* (MA).

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