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BOUNDARY ASYMPTOTICS OF THE RELATIVE BERGMAN KERNEL METRIC FOR ELLIPTIC CURVES III: 1 & ∞

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Abstract. Explicit asymptotic formulas of the relative Bergman kernel metric for a Legendre family of elliptic curves near the moduli space boundary points 1 and ∞ are obtained respectively. These behaviors also characterize the Poincaré hyperbolic metric and its Kähler potential on $\mathbb{C}\setminus\{0,1\}$.

1. Introduction

The Bergman kernel is a canonical $(1,1)$-form on a complex manifold, and its plurisubharmonic variations were initially studied by Maitani & Yamaguchi [12] and later generalized by Berndtsson [2]. Results on Stein manifolds and complex projective algebraic manifolds (see [3], [15], [5]) indicate semi-positivity properties of relative canonical bundles, and recently turn out to have close relations with the Ohsawa-Takegoshi L^2 extension theorems (cf. [10], [7], [6], [4], [13]). For simplicity, let us consider the one-dimensional case, namely a family of Riemann surfaces parametrized by a complex variable λ, and the Bergman kernel on each fiber X_λ can thus be written as $B_\lambda = k_\lambda(z)dz\wedge d\bar{z}$ in some local coordinate z. Then, due to the variation results of the Bergman kernel, the inequality

$$L_{\lambda,z} := \sqrt{-1}\partial_\lambda\bar{\partial}_\lambda \log k_\lambda(z) \geqslant 0$$

holds whenever the fiber X_λ is smooth.

Assuming that some X_{λ_0} is singular, then a natural question is to characterize the asymptotic behaviors of $L_{\lambda,z}$ near λ_0. The so-called Legendre family of elliptic curves $X_\lambda := \{y^2 = x(x-1)(x-\lambda)\}$ gives a general description of genus one compact Riemann surfaces whose moduli space is $\mathbb{C}\setminus\{0,1\}$, and it degenerates to a singular algebraic curve with a node when λ tends to $0,1$ or ∞. In [8] the author observed that $L_{\lambda,z}$ blows up and has hyperbolic growth as $\lambda\to 0$, in comparison to the Poincaré hyperbolic metric $\omega_{\mathbb{D}^*}$ on $\mathbb{D}^*$, the unit disk removing the origin. However, an explicit four-term asymptotic expansion formula as $\lambda\to 0$ showed that $L_{\lambda,z}$ and $\omega_{\mathbb{D}^*}$ are not the same [9].

The aim of this paper is to investigate $L_{\lambda,z}$ for the above Legendre family of elliptic curves near the other two moduli space boundary points 1 and ∞ respectively. Explicit asymptotic formulas of the relative Bergman kernel metric are as follows.

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THEOREM 1. *Let B_λ be the Bergman kernel of the elliptic curve X_λ, $\lambda \in \mathbb{C} \setminus \{0,1\}$. In a local coordinate z, write $B_\lambda = k_\lambda(z) dz \wedge d\bar{z}$. Then, it follows that*

(i) *as* $\lambda \to 1$,

$$\log k_\lambda(z) \sim \log(-\log|\lambda - 1|),$$

(ii) *and as* $\lambda \to \infty$,

$$\log k_\lambda(z) \sim \log(\log|\lambda|).$$

Here "$\sim$" means that the ratio of both sides tends to "1", as λ tends to each limit point. For example, as $\lambda \to 1$, X_λ degenerates to a singular curve $X_1 := \{y^2 = x(x-1)^2\}$. In particular, both the right hand sides of (i) and (ii) tend to $+\infty$. Rather than taking immediate second-order partial derivatives, we make more careful computations on the curvature forms and derive the following theorem.

THEOREM 2. *Under the same assumptions as in Theorem 1, it follows that*

(i) *as* $\lambda \to 1$,

$$L_{\lambda,z} \sim \frac{\sqrt{-1}}{4|\lambda - 1|^2(\log|\lambda - 1|)^2} d\lambda \wedge d\bar{\lambda},$$

(ii) *and as* $\lambda \to \infty$,

$$L_{\lambda,z} \sim \frac{\sqrt{-1}}{4|\lambda|^2(\log|\lambda|)^2} d\lambda \wedge d\bar{\lambda}.$$

Notice that the right hand sides of (i) and (ii) tend to $+\infty$ and 0^+, respectively. And this is different from the potentials in Theorem 1 which have the same limit. The proofs of the above Theorems 1 and 2 are mainly due to the elliptic modular lambda function's special properties (in particular its behavior under the composition with inverse or translation mappings), which are also used in [8]. On the other hand, same asymptotic behaviors as in Theorems 1 and 2 also characterize the Poincaré hyperbolic metric and its Kähler potential on $\mathbb{C} \setminus \{0,1\}$, since this $L_{\lambda,z}$ indeed has constant Gaussian curvature "-4" (see [9]). Therefore, we get the following corollary.

COROLLARY 1. *Let $\omega_{0,1}$ denote the Poincaré hyperbolic metric on $\lambda \in \mathbb{C} \setminus \{0,1\}$ with a Kähler potential $p(\lambda) := -\log(\operatorname{Im} \tau(\lambda))$, where $\tau(\cdot)$ is the inverse of the elliptic modular lambda function. Then, it follows that*

(i) *as* $\lambda \to 1$,

$$p(\lambda) \sim \log(-\log|\lambda - 1|),$$

$$\omega_{0,1} \sim \frac{\sqrt{-1}}{4|\lambda - 1|^2(\log|\lambda - 1|)^2} d\lambda \otimes d\bar{\lambda},$$

(ii) *and as* $\lambda \to \infty$,

$$p(\lambda) \sim \log(\log|\lambda|),$$

$$\omega_{0,1} \sim \frac{\sqrt{-1}}{4|\lambda|^2(\log|\lambda|)^2} d\lambda \otimes d\bar{\lambda}.$$

We remark that our result agrees in the limiting case $\lambda \to 1$ inside $\mathbb{D}^*$ with the fact that $\omega_{0,1} \leqslant \omega_{\mathbb{D}^*}$ (see e.g. [14, 11]).

2. Preliminaries

We will first recall the definition and basic properties of the elliptic modular lambda function. From [1, p.264], one knows that for any $z \in T_\tau := \mathbb{C}/\big(\mathbb{Z}+\tau\mathbb{Z}\big)$, the Weierstrass-$\wp$ function with respect to the lattice $(1,\tau)$ ($\tau \in \mathbb{C}$, $\operatorname{Im}\tau > 0$) is defined to be

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \neq 0} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right),$$

where the sum ranges over all $\omega = n_1 + n_2\tau$ except 0, and $n_1, n_2 \in \mathbb{Z}$. Denote $e_1 := \wp(\frac{1}{2})$, $e_2 := \wp(\frac{\tau}{2})$ and $e_3 := \wp(\frac{1+\tau}{2})$. Then the elliptic modular lambda function

$$\lambda(\tau) := \frac{e_3 - e_2}{e_1 - e_2}$$

can identify a complex torus $T_\tau = \mathbb{C}/\big(\mathbb{Z}+\tau\mathbb{Z}\big)$ with an elliptic curve $X_\lambda := \{y^2 = x(x-1)(x-\lambda)\}$, since $\lambda(\tau)$ is conformal. In the local coordinate z, the Bergman kernel B_τ of the canonical bundle on X_τ can be simply written as $\frac{1}{\operatorname{Im}\tau} dz \wedge d\bar{z}$, which means that $k_\lambda(z) = \frac{1}{\operatorname{Im}\tau}$. Taking derivatives, one gets that

$$l_{\lambda(\tau),z} := \frac{\partial^2(\log k_{\lambda(\tau)}(z))}{\partial\lambda\partial\bar{\lambda}} = \frac{\partial^2(-\log \operatorname{Im}\tau)}{\partial\lambda\partial\bar{\lambda}} = \frac{|\tau'(\lambda)|^2}{4(\operatorname{Im}\tau)^2}.$$

Since $\tau := \lambda^{-1}$ is holomorphic then $\frac{\partial\tau}{\partial\bar{\lambda}} = 0$, and by the inverse function theorem $\tau'(b) = (\lambda^{-1})'(b) = \frac{1}{\lambda'(a)}$ for any $b = \lambda(a)$ (here λ' being the derivative of λ with respect to τ). Therefore, we have

$$l_{\lambda(\tau),z} = \frac{1}{4(\operatorname{Im}\tau \cdot |\lambda'(\tau)|)^2} > 0. \tag{1}$$

Notice that the inequality above holds due to the fact that the derivative of the elliptic modular lambda function is nowhere vanishing in the domain of definition. Thus $L_{\lambda,z} = \sqrt{-1}\, l_{\lambda,z} d\lambda \wedge d\bar{\lambda}$ is a true metric on the moduli space, i.e. $L_{\lambda,z} > 0$, $\forall \lambda \in \mathbb{C} \setminus \{0,1\}$. Moreover, since

$$-4\frac{\partial^2}{\partial\lambda\partial\bar{\lambda}} \log\left(\frac{|\tau'|}{\operatorname{Im}\tau}\right) = -\frac{|\tau'|^2}{(\operatorname{Im}\tau)^2},$$

it follows that $L_{\lambda,z}$ is the Poincaré hyperbolic metric of $\mathbb{C}\setminus\{0,1\}$ with constant Gaussian curvature "-4".

Next, we introduce two more parameters $\alpha := -\frac{1}{\tau}$ and $\beta := \tau - 1$, both of which have positive imaginary parts as long as $\operatorname{Im}\tau > 0$. As $\tau \to 0$ or equivalently as $\lambda(\tau) \to 1$, it follows that $\operatorname{Im}\alpha \to +\infty$ and $\lambda(\alpha) \to 0$. By the definition of α, it has

$$\operatorname{Im}\tau = \operatorname{Im}\left(\frac{-1}{\operatorname{Re}\alpha + \sqrt{-1}\operatorname{Im}\alpha}\right) = \operatorname{Im}\left(\frac{\sqrt{-1}\operatorname{Im}\alpha - \operatorname{Re}\alpha}{|\alpha|^2}\right) = \operatorname{Im}\alpha \cdot |\tau|^2. \tag{2}$$

Similarly, as $\tau \to 1$ or equivalently as $\lambda(\tau) \to \infty$, it follows that $\beta \to 0$. Since $\operatorname{Im}\beta = \operatorname{Im}\tau$, we know that $\operatorname{Im}\beta \to +\infty$ implies $\lambda(\tau) \to 0$.

3. Proof of Theorem 1

In this section, combining our results in [8] and introducing two new parameters $\alpha := -\frac{1}{\tau}$ and $\beta := \tau - 1$, we will prove new results. We shall use the following well-known properties of the elliptic modular lambda function (see [1, p.279–280]):

(A) As $\operatorname{Im}\alpha \to +\infty$, it holds that

$$\lambda(\alpha) \sim 16 e^{\pi\sqrt{-1}\alpha} \to 0,$$

which means that $\log\lambda(\alpha) \sim \pi\sqrt{-1}\alpha$.

(B) $\lambda(-\frac{1}{\tau}) = 1 - \lambda(\tau)$.

(C) $\lambda(\beta+1) = \frac{\lambda(\beta)}{\lambda(\beta)-1} = 1 + \frac{1}{\lambda(\beta)-1}$ $(\Longrightarrow \lambda(\beta) - 1 = \frac{1}{\lambda(\beta+1)-1})$.

Proof. [Proof of Theorem 1] Claim (i). As $\tau \to 0$ $(\Longleftrightarrow \operatorname{Im}\alpha \to +\infty)$, since $\log k_{\lambda(\tau)}(z) = -\log\operatorname{Im}\tau$, we know by (2) that $\log k_{\lambda(\tau)}(z) \sim -\log\operatorname{Im}\alpha + \log|\alpha|^2$. Theorem 1.3 (i) in [8] says that as $\operatorname{Im}\alpha \to +\infty$, one has

$$-\log\operatorname{Im}\alpha \sim -\log(-\log|\lambda(\alpha)|),$$

which yields as $\tau \to 0$ that

$$\log k_{\lambda(\tau)}(z) \sim -\log(-\log|\lambda(\alpha)|) + 2\log|\alpha|.$$

On the other hand by Property (A) we know that

$$\begin{aligned} \pi|\alpha| &\sim |\log\lambda(\alpha)| = |\log|\lambda(\alpha)| + \sqrt{-1}\arg(\alpha)| \\ &\sim |\log|\lambda(\alpha)|| = -\log|\lambda(\alpha)|, \end{aligned}$$

as $\operatorname{Im}\alpha \to +\infty$ $(\Longleftrightarrow \lambda(\alpha) \to 0)$, which gives that $\log|\alpha| \sim \log(-\log|\lambda(\alpha)|)$.

Therefore, by Property (B) for the Bergman kernel we have proved that

$$\begin{aligned} \log k_{\lambda(\tau)}(z) &\sim -\log(-\log|\lambda(\alpha)|) + 2\log(-\log|\lambda(\alpha)|) \\ &= \log(-\log|\lambda(\alpha)|) = \log(-\log|\lambda(\tau)-1|) \to +\infty, \end{aligned}$$

as $\lambda(\alpha) \to 0$ $(\Longleftrightarrow \lambda(\tau) \to 1)$.

Claim (ii). It follows from Claim (i) that as $\beta \to 0 (\Longleftrightarrow \tau \to 1)$,

$$-\log\operatorname{Im}\beta \sim \log(-\log|\lambda(\beta)-1|),$$

which implies by Property (C) that

$$\begin{aligned} \log k_{\lambda(\tau)}(z) &= -\log\operatorname{Im}\tau = -\log\operatorname{Im}\beta \\ &\sim \log(-\log|\lambda(\beta)-1|) = \log(\log|\lambda(\beta+1)-1|) \\ &= \log(\log|\lambda(\tau)-1|) \sim \log(\log|\lambda(\tau)|) \to +\infty, \end{aligned}$$

as $\lambda(\tau) \to \infty$. $\square$

4. Proof of Theorem 2

Proof. Claim (i). From Property (B), one knows that $\lambda'(\alpha)\cdot\frac{\partial\alpha}{\partial\tau}=-\lambda'(\tau)$, which implies

$$|\lambda'(\tau)|=\frac{|\lambda'(\alpha)|}{|\tau|^2}.$$

By equalities (1) and (2), as $\operatorname{Im}\alpha\to+\infty$, it holds that

$$\begin{aligned}\frac{\partial^2(\log k_{\lambda(\tau)}(z))}{\partial\lambda\partial\bar\lambda}&=\frac{1}{4(\operatorname{Im}\tau\cdot|\lambda'(\tau)|)^2}=\frac{1}{4(\operatorname{Im}\alpha\cdot|\tau|^2\cdot\frac{|\lambda'(\alpha)|}{|\tau|^2})^2}\\&=\frac{1}{4(\operatorname{Im}\alpha\cdot|\lambda'(\alpha)|)^2}=\frac{\partial^2(\log k_{\lambda(\alpha)}(z))}{\partial\lambda\partial\bar\lambda}.\end{aligned}$$

Theorem 1.3 (ii) in [8] says that

$$\frac{\partial^2(\log k_{\lambda(\alpha)}(z))}{\partial\lambda\partial\bar\lambda}\sim\frac{1}{4|\lambda(\alpha)|^2(\log|\lambda(\alpha)|)^2},$$

as $\operatorname{Im}\alpha\to+\infty(\Longleftrightarrow\tau\to0)$, which yields that

$$\frac{\partial^2(\log k_{\lambda(\tau)}(z))}{\partial\lambda\partial\bar\lambda}\sim\frac{1}{4|\lambda(\tau)-1|^2(\log|\lambda(\tau)-1|)^2}\to+\infty,$$

as $\lambda(\tau)\to1$.

Claim (ii). By Property (B) we get that

$$\lambda'(\beta)\cdot\frac{\partial\beta}{\partial\tau}\cdot(\lambda(\tau)-1)+(\lambda(\beta)-1)\cdot\lambda'(\tau)=0.$$

This means $\lambda'(\tau)=\frac{\lambda'(\beta)\cdot(\lambda(\tau)-1)}{-\lambda(\beta)+1}$ and therefore

$$|\lambda'(\tau)|=\frac{|\lambda'(\beta)|\cdot|(\lambda(\tau)-1)|}{|\lambda(\beta)-1|}.$$

By (1) again, it follows that

$$\begin{aligned}\frac{\partial^2(\log k_{\lambda(\tau)}(z))}{\partial\lambda\partial\bar\lambda}&=\frac{1}{4\,(\operatorname{Im}\tau\cdot|\lambda'(\tau)|)^2}\\&=\frac{|\lambda(\beta)-1|^2}{4\,(\operatorname{Im}\beta\cdot|\lambda'(\beta)|\cdot|\lambda(\tau)-1|)^2}=\frac{\partial^2(\log k_{\lambda(\beta)}(z))}{\partial\lambda\partial\bar\lambda}\cdot\frac{|\lambda(\beta)-1|^2}{|\lambda(\tau)-1|^2}.\end{aligned}$$

By Claim (i), as $\lambda(\beta)\to1$ it holds that

$$\frac{\partial^2(\log k_{\lambda(\beta)}(z))}{\partial\lambda\partial\bar\lambda}\sim\frac{1}{4\,(|\lambda(\beta)-1|\cdot\log|\lambda(\beta)-1|)^2},$$

which means that

$$\begin{aligned}\frac{\partial^2(\log k_{\lambda(\tau)}(z))}{\partial\lambda\partial\bar{\lambda}} &\sim \frac{1}{4\left(|\lambda(\beta)-1|\cdot\log|\lambda(\beta)-1|\right)^2}\cdot\frac{|\lambda(\beta)-1|^2}{|\lambda(\tau)-1|^2}\\ &= \frac{1}{4(\log|\lambda(\beta)-1|)^2\cdot|\lambda(\tau)-1|^2}\\ &= \frac{1}{4(-\log|\lambda(\tau)-1|)^2\cdot|\lambda(\tau)-1|^2}\\ &\sim \frac{1}{4(\log|\lambda(\tau)|)^2\cdot|\lambda(\tau)|^2}\to 0^+,\end{aligned}$$

as $\lambda(\tau)\to\infty$. The proof is thus finished. $\square$

At last, we summarize this paper by making the following table indicating how the relative Bergman kernel and its curvature form change as the parameter varies. Here τ is the inverse function of the elliptic modular lambda function. As we can see, all the three cases have different asymptotic behaviors.

As the parameter τ tends to	As the parameter λ tends to	relative Bergman kernel $\log k_\lambda(z)$	the curvature form $\sqrt{-1}\partial_\lambda\bar{\partial}_\lambda\log k_\lambda(z)$
∞	0	$\to-\infty$	$\to+\infty$
0	1	$\to+\infty$	$\to+\infty$
1	∞	$\to+\infty$	$\to 0^+$

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