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BOUNDS FOR EXTREME ZEROS OF QUASI–ORTHOGONAL ULTRASPHERICAL POLYNOMIALS

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Abstract. We discuss and compare upper and lower bounds obtained by two different methods for the positive zero of the ultraspherical polynomial $C_n^{(\lambda)}$ that is greater than 1 when $-3/2 < \lambda < -1/2$. Our first approach uses mixed three term recurrence relations and interlacing of zeros while the second approach uses a method going back to Euler and Rayleigh and already applied to Bessel functions and Laguerre and q-Laguerre polynomials. We use the bounds obtained by the second method to simplify the proof of the known interlacing of the zeros of $(1-x^2)C_n^{(\lambda)}$ and $C_{n+1}^{(\lambda)}$, for $-3/2 < \lambda < -1/2$.

1. Introduction

For $\lambda > -1/2$, the sequence of ultraspherical polynomials $\{C_n^{(\lambda)}\}_{n=0}^{\infty}$ is orthogonal on $[-1,1]$ with respect to the positive measure $(1-x^2)^{\lambda-1/2}$ and all the zeros of $C_n^{(\lambda)}$ lie in $(-1,1)$. As the parameter λ decreases, the zeros of $C_n^{(\lambda)}$ depart from the interval $(-1,1)$ in pairs through the endpoints as λ decreases through the values $-1/2, -3/2, \ldots, -\lfloor n/2 \rfloor + 1/2$. Here, we pay particular attention to the case $-3/2 < \lambda < -1/2$, where λ is fixed and it is known [1, Cor. 2] that the positive zeros, listed in decreasing order, satisfy

$$0 < x_{\lfloor n/2 \rfloor,n}(\lambda) < \cdots < x_{2,n}(\lambda) < 1 < x_{1,n}(\lambda). \tag{1}$$

We apply two methods to investigate upper and lower bounds for the extreme zero $x_{1,n}(\lambda)$ of $C_n^{(\lambda)}$, λ fixed, $-3/2 < \lambda < -1/2$. One method emanates from a suitably chosen mixed three-term recurrence relation and the other from the Euler-Rayleigh technique discussed in [13] where it is used to derive bounds for the smallest real zero of a power series or polynomial, with exclusively real zeros, in terms of the coefficients of the series or polynomial.

Since the first approach involves interlacing properties of zeros of polynomials, we recall the definition:

DEFINITION 1. Let $\{p_n\}_{n=0}^{\infty}$ be a sequence of polynomials and suppose the zeros of p_n are real and simple for each $n \in \mathbb{N}$. Denoting the zeros of p_n in decreasing order by $x_{n,n} < \ldots < x_{2,n} < x_{1,n}$, the zeros of p_n and p_{n-1} are interlacing if, for each $n \in \mathbb{N}$,

$$x_{n,n} < x_{n-1,n-1} < \ldots < x_{2,n} < x_{1,n-1} < x_{1,n}. \tag{2}$$

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The interlacing of the zeros of orthogonal polynomials of consecutive degree p_n and p_{n-1} is a well known classical result [19, §3.3].

Quasi-orthogonal polynomials arise in a natural way in the context of classical orthogonal polynomials that depend on one or more parameters. The concept of quasi-orthogonality of order 1 of a sequence of polynomials was introduced by Riesz [17] in connection with the moment problem. Fejér [10] considered quasi-orthogonality of order 2 and the general case was studied by Shohat [18] and many other authors including Chihara [2], Dickinson [3], Draux [5] and Maroni [14, 15, 16]. The definition of quasi-orthogonality of a sequence of polynomials is the following:

DEFINITION 2. Let $\{q_n\}_{n=0}^{\infty}$ be a sequence of polynomials with degree $q_n = n$ for each $n \in \mathbb{N}$. For a positive integer $r < n$, the sequence $\{q_n\}_{n=0}^{\infty}$ is quasi-orthogonal of order r with respect to a positive Borel measure μ if

$$\int x^k q_n(x)\, d\mu(x) = 0 \quad \text{for} \quad k = 0, \ldots, n-1-r. \tag{3}$$

If (3) holds for $r = 0$, the sequence $\{q_n\}_{n=0}^{\infty}$ is orthogonal with respect to the measure μ.

The sequence of ultraspherical polynomials $\{C_n^{(\lambda)}\}_{n=0}^{\infty}$, is orthogonal on $(-1,1)$ with respect to the weight function $(1-x^2)^{\lambda-1/2}$ when $\lambda > -1/2$. As λ decreases below $-1/2$, the zeros of $C_n^{(\lambda)}$ leave the interval of orthogonality $(-1,1)$ in pairs, through the endpoints, as described in our opening paragraph. A full description of the trajectories of the zeros can be found in [7]. For the range $-3/2 < \lambda < -1/2$, it is shown in [1, Theorem 6] that the sequence $\{C_n^{(\lambda)}\}_{n=0}^{\infty}$ is quasi-orthogonal of order 2 with respect to the weight function $(1-x^2)^{\lambda+1/2}$. Also. the polynomial $C_n^{(\lambda)}$ has $n-2$ real, distinct zeros in the interval $(-1,1)$; its smallest zero is < -1, its largest zero is > 1 and, for each $n \in \mathbb{N}$, the zeros of $C_n^{(\lambda)}$ interlace with the zeros of the (orthogonal) polynomial $C_{n-1}^{(\lambda+1)}$ [1, Corollary 2].

In this paper, we derive upper and lower bounds for the two (symmetric about the origin) zeros of $C_n^{(\lambda)}$ that lie outside $(-1,1)$ using two different methods.

We use the notation $x_{k,n}(\lambda)$ for the kth zero, in *decreasing* order, of $C_n^{(\lambda)}(x)$. This differs from the notation $x_{n,k}(\lambda)$ used in [4], and from other papers where the zeros are listed in increasing order. It follows from [19, (4.7.30)] that $C_n^{(\lambda^*)}(x) \equiv 0$, for $\lambda^* = 0, -1, \ldots, -\lfloor (n-1)/2 \rfloor$, and the concept of "$k$th zero" becomes meaningless. In this paper, we define $x_{k,n}(\lambda^*)$ by

$$x_{k,n}(\lambda^*) = \lim_{\lambda \to \lambda^*} x_{k,n}(\lambda).$$

In [8], we considered some properties of quasi-orthogonal Laguerre polynomials $\{L_n^{(\alpha)}(x)\}$, in the case $-2 < \alpha < -1$.

2. Bounds for the largest zero of $C_n^{(\lambda)}$, $-3/2<\lambda<-1/2$. An approach using mixed three-term recurrence relations.

The co-primality of two polynomials is an important concept when considering interlacing properties of their zeros. Conversely, inequalities satisfied by zeros of polynomials that are interlacing may provide information about the co-primality of the polynomials. The interplay between the co-primality of two polynomials and the interlacing of their (real) zeros facilitates information about upper and lower bounds of the extreme zeros of $C_n^{(\lambda)}$, $-3/2<\lambda<-1/2$.

THEOREM 1. *Suppose that $\{C_n^{(\lambda)}\}_{n=0}^{\infty}$ is the sequence of ultraspherical polynomials where λ is fixed and lies in the range $-3/2<\lambda<-1/2$. For each $n=2,3,\ldots$, the largest zero, $x_{1,n}(\lambda)$, of $C_n^{(\lambda)}(x)$, satisfies*

$$1<x_{1,n}(\lambda)<\left(1+\frac{2\lambda+1}{n-1}\right)^{-1/2}=\left(\frac{n-1}{2\lambda+n}\right)^{1/2}. \tag{4}$$

Proof. From [6, (16)], we have

$$\begin{aligned}&4\lambda(\lambda+1)(1-x^2)^2C_{n-2}^{(\lambda+2)}(x)=\\&(2\lambda+n)[x^2(n+2\lambda+1)-n]C_n^{(\lambda)}(x)-(2\lambda+1)(n+1)xC_{n+1}^{(\lambda)}(x).\end{aligned} \tag{5}$$

Evaluating (5) at the largest two zeros $x_{1,n+1}>1>x_{2,n+1}$ of $C_{n+1}^{(\lambda)}$, we have

$$\begin{aligned}&16\lambda^2(\lambda+1)^2(1-x_{1,n+1}^2)^2(1-x_{2,n+1}^2)^2C_{n-2}^{(\lambda+2)}(x_{1,n+1})C_{n-2}^{(\lambda+2)}(x_{2,n+1})=\\&(2\lambda+n)^2(2\lambda+n+1)^2C_n^{(\lambda)}(x_{1,n+1})C_n^{(\lambda)}(x_{2,n+1})\\&\times\left[x_{1,n+1}^2-\frac{n}{n+2\lambda+1}\right]\left[x_{2,n+1}^2-\frac{n}{n+2\lambda+1}\right].\end{aligned} \tag{6}$$

We know from [9, Theorem 3.1(i)], with n replaced by $n+1$, that $C_n^{(\lambda)}$ does not change sign between the two largest zeros of $C_{n+1}^{(\lambda)}$. Further, from (5), since $C_n^{(\lambda)}$ and $C_{n+1}^{(\lambda)}$ are co-prime for each $n\in\mathbb{N}$, $\lambda\in\mathbb{R}$ (also from [9, Theorem 3.1(i)]), the only possible common zeros of $C_{n-2}^{(\lambda+2)}(x)$ and $C_{n+1}^{(\lambda)}(x)$ are at the values $x^2=n/(n+2\lambda+1)=1-(2\lambda+1)/(n+2\lambda+1)$ which have absolute value >1 for each λ satisfying $-3/2<\lambda<-1/2$ and $n\geqslant 3$. Since, because of orthogonality, all the zeros of $C_{n-2}^{(\lambda+2)}$ lie in $(-1,1)$ we deduce that $C_{n-2}^{(\lambda+2)}$ and $C_{n+1}^{(\lambda)}$ are co-prime for each $n\in\mathbb{N}$, and each λ satisfying $-3/2<\lambda<-1/2$.

From [9, Theorem 3.5(ii)] with n replaced by $n+1$, we know that the zeros of $(1-x^2)C_{n-2}^{(\lambda+2)}(x)$ and $C_{n+1}^{(\lambda)}(x)$ are interlacing. Since the point 1 lies between the two zeros $x_{1,n+1}$ and $x_{2,n+1}$, we see that in (6), both the left-hand side and the product of the first four factors on the right are positive and therefore $\sqrt{n/(2\lambda+n+1)}\notin(x_{2,n+1},x_{1,n+1})$. Also, $\sqrt{n/(2\lambda+n+1)}>1$ so that $x_{1,n+1}<\sqrt{n/(2\lambda+n+1)}$. Finally, replacing n by $n-1$, we have the stated result. $\square$

THEOREM 2. *Suppose that* $\{C_n^{(\lambda)}\}_{n=0}^{\infty}$ *is the sequence of ultraspherical polynomials where* λ *is fixed and lies in the range* $-3/2<\lambda<-1/2$. *For each* $n=2,3,\ldots$, *the largest zero,* $x_{1,n}(\lambda)$ *of* $C_n^{(\lambda)}(x)$ *satisfies*

$$x_{1,n}(\lambda) > \left(1+\frac{(2\lambda+1)(2\lambda+3)}{(n-1)(n+2\lambda+1)}\right)^{-1/2} > 1. \tag{7}$$

Proof. From [6, (18)], we have

$$8\lambda(\lambda+1)(\lambda+2)(1-x^2)^3 C_{n-2}^{(\lambda+3)}(x) = \tag{8}$$
$$(2\lambda+n)[x^2[n(n+2\lambda+2)+(2\lambda+1)(2\lambda+3)]-n(n+2\lambda+2)]C_n^{(\lambda)}(x)-g(x)C_{n+1}^{(\lambda)}(x),$$

where $g(x)$ is a polynomial in x. Evaluating (8) at the largest two zeros $x_{2,n+1}<1<x_{1,n+1}$ of $C_{n+1}^{(\lambda)}$, we get

$$\begin{aligned} 64\lambda^2(\lambda+1)^2(\lambda+2)^2(1-x_{1,n+1}^2)^3(1-x_{2,n+1}^2)^3 C_{n-2}^{(\lambda+3)}(x_{1,n+1})C_{n-2}^{(\lambda+3)}(x_{2,n+1}) = \\ f(n,\lambda)\left[x_{1,n+1}^2-\frac{n(n+2\lambda+2)}{n(n+2\lambda+2)+(2\lambda+1)(2\lambda+3)}\right] \\ \times\left[x_{2,n+1}^2-\frac{n(n+2\lambda+2)}{n(n+2\lambda+2)+(2\lambda+1)(2\lambda+3)}\right]C_n^{(\lambda)}(x_{1,n+1})C_n^{(\lambda)}(x_{2,n+1}), \end{aligned} \tag{9}$$

where

$$f(n,\lambda)=(2\lambda+n)^2(n(n+2\lambda+2)+(2\lambda+1)(2\lambda+3))^2.$$

As in the proof of Theorem 1, we know from [9, Theorem 3.1(i)] that $C_n^{(\lambda)}$ does not change sign between the two largest zeros of $C_{n+1}^{(\lambda)}$ and, in addition, $C_n^{(\lambda)}$ and $C_{n+1}^{(\lambda)}$ are co-prime. Therefore, the only possible common zeros of $C_{n-2}^{(\lambda+3)}(x)$ and $C_{n+1}^{(\lambda)}(x)$ are at the values

$$x^2=\left[1+\frac{(2\lambda+1)(2\lambda+3)}{n(n+2\lambda+2)}\right]^{-1}$$

which have absolute value >1 for each λ satisfying $-3/2<\lambda<-1/2$ and $n\geqslant 2$. On the other hand, all the zeros of $C_{n-2}^{(\lambda+3)}(x)$ are in $(-1,1)$ so $C_{n-2}^{(\lambda+3)}(x)$ and $C_{n+1}^{(\lambda)}(x)$ have no common zeros. Thus the left-hand side of (9) is negative and, since the product of the last two terms on the right-hand side is positive, we see that the product of the square-bracketed terms is negative and, in particular,

$$x_{1,n+1}^2 > \frac{n(n+2\lambda+2)}{n(n+2\lambda+2)+(2\lambda+1)(2\lambda+3)}. \tag{10}$$

Finally, replacing n by $n-1$, we have the stated result. □

3. The Euler-Rayleigh method

A method described in [20, pp. 500-501], developed further in [13], and applied to Laguerre and q-Laguerre polynomials in [11], may be applied to prove bounds for the largest zero of $C_n^{(\lambda)}$, λ fixed, $-3/2 < \lambda < -1/2$. It can also be applied to finding bounds for the largest zero of $C_n^{(\lambda)}$ in the orthogonal cases when $\lambda > -1/2$.

The idea is as follows. If a polynomial $f(t) = \sum_{k=0}^{n} a_n t^n$ has all its zeros real at the points

$$t_1 < t_2 < \cdots < t_n, \tag{11}$$

and we use the notation

$$S_j = \sum_{k=1}^{n} t_k^{-j},$$

then the sums S_j can be expressed in terms of the coefficients by $S_1 = -a_1$, $S_2 = -2a_2 + a_1^2$, and, in general

$$S_j = -na_n - \sum_{i=1}^{j-1} a_i S_{j-i}.$$

Two special cases can be considered.

LEMMA 1. *If* $0 < t_1 < t_2 < \cdots < t_n$, *then* $S_m > 0$, $m = 1,2,\ldots$, *and*

$$S_m^{-1/m} < t_1 < S_m/S_{m+1},\ m = 1,2,\ldots$$

where the lower limits increase and the upper limits decrease with increasing m.

This is [13, Lemma 3.2].

LEMMA 2. *If* $t_1 < 0 < t_2 < \cdots < t_n$, *and* $a_1 > 0$, *then* $S_m < 0$ *for odd* m,

$$-\left|S_{2m-1}\right|^{-1/(2m-1)} < t_1 < -S_{2m}^{-1/(2m)} < S_{2m-1}/S_{2m},\ m = 1,2,\ldots, \tag{12}$$

and

$$S_{2m}/S_{2m+1} < t_1,\ m = 1,2,\ldots,. \tag{13}$$

This is [13, Lemma 3.3] with the addition of the the remark at the end of [13, §3]. We apply Lemma 2 in the special case $m = 1$ to the representation [19, (4.7.6)]

$$C_n^{(\lambda)}(x) = \binom{n+2\lambda-1}{n} {}_2F_1\left(-n,\, n+2\lambda;\, \lambda+1/2;\, t\right),\ \ t = \frac{1-x}{2}. \tag{14}$$

Since we are assuming that $-3/2 < \lambda < -1/2$ with λ fixed, it follows from [1, Cor. 2] that $x_n < -1 < x_{n-1} < \cdots < x_2 < 1 < x_1$ so the corresponding t-zeros satisfy $t_1 < 0 < t_2 < \ldots < t_n$. With $f(t) = {}_2F_1\left(-n, n+2\lambda; \lambda+1/2; t\right)$, we have

$$S_1 = \frac{n(n+2\lambda)}{\lambda+1/2}, \quad S_2 = -S_1\left[\frac{2(n-1)(n+2\lambda+1)}{2\lambda+3} - S_1\right].$$

Applying the case $m = 1$ of (12), leads to

$$-|S_1|^{-1} < t_1 < -S_2^{-1/2} < S_1/S_2.$$

When converted to inequalities for $x_1 = 1-2t_1$, these inequalities give

$$1 - 2S_1/S_2 < 1 + 2S_2^{-1/2} < x_{1,n}(\lambda) < 1 + 2/|S_1|,$$

or equivalently:

THEOREM 3. *For each fixed* λ, $-3/2 < \lambda < -1/2$, *let* $x_{1,n}(\lambda)$ *denote the largest* x*-zero of* $C_n^{(\lambda)}(x)$. *Then*

$$\left[1 + \frac{(2\lambda+1)(2\lambda+3)}{2(n-1)(n+2\lambda+1)}\right]^{-1} < 1 - \frac{2(2\lambda+1)\sqrt{2\lambda+3}}{\sqrt{n(2\lambda+n)(4\lambda^2+4n\lambda+2n^2+4\lambda+1)}}$$
$$< x_{1,n}(\lambda) < 1 - \frac{2\lambda+1}{n(n+2\lambda)}. \tag{15}$$

Remark. Although the situation is not as simple as that indicated in Lemma 1, it is likely that some of the bounds in Lemma 2 (and its application to Theorem 3) improve, although they become more complicated, with increasing m.

4. Comparison of bounds

It is of interest to compare the bounds in Sections 2 and 3, as well as to compare them with similar bounds valid for $\lambda > -1/2$.

For $n \geqslant 6$, the upper bound in (15) is sharper than that given by Theorem 1. That is

$$1 - \frac{2\lambda+1}{n(n+2\lambda)} < \sqrt{\frac{(n-1)}{(2\lambda+n)}}, \quad -\frac{3}{2} \leqslant \lambda < -\frac{1}{2}, \; n \geqslant 6, \tag{16}$$

with equality for $\lambda = -1/2$. The required inequality can be written $f(\lambda) > 0$ where

$$f(\lambda) = n^2(n+2\lambda)(n-1) - (n^2+2\lambda n - 2\lambda - 1)^2,$$

$$f'(\lambda) = 2(n-1)(-n^2 - 4\lambda n + 4\lambda + 2), \; f''(\lambda) = -8(n-1)^2 < 0.$$

Thus $f'(\lambda)$ is decreasing, and since $f'(-3/2) = -2(n-1)((n-3)^2-5) < 0$ we find that $f(\lambda)$ is decreasing on $(-3/2, -1/2)$. Finally since $f(-1/2) = 0$, we see that $f(\lambda) > 0$ for $-3/2 < \lambda < -1/2$.

For $n \geqslant 2$, the lower bound given by Theorem 2 is sharper than the smaller lower bound in (15). This statement is equivalent to the inequality

$$1+\frac{(2\lambda+1)(2\lambda+3)}{2(n-1)(n+2\lambda+1)} < \left[1+\frac{(2\lambda+1)(2\lambda+3)}{(n-1)(n+2\lambda+1)}\right]^{-1/2} \tag{17}$$

The number $a = [(2\lambda+1)(2\lambda+3)]/[(n-1)(n+2\lambda+1)]$ satisfies $-1 < a < 0$ for the values of n and λ concerned and it is a simple matter to show that, in this case, $1+a/2 < 1/\sqrt{1+a}$.

We can use the case $m = 1$ of Lemma 1 to show that the first and last expressions in (15) continue to provide lower and upper bounds for $x_{1,n}(\lambda)$ for $\lambda > -1/2$. Thus we have

$$\left[1+\frac{(2\lambda+1)(2\lambda+3)}{2(n-1)(n+2\lambda+1)}\right]^{-1} < \; x_{1,n}(\lambda) < \; 1-\frac{2\lambda+1}{n(n+2\lambda)} \tag{18}$$

for $\lambda > -3/2$, $\lambda \neq -1/2$. Both inequalities become equalities for $\lambda = -1/2$.

5. A continuity-based proof of an interlacing property

In [9, Theorem 3.1(i)], we showed that for $-3/2 < \lambda < -1/2$, the zeros of $C_n^{(\lambda)}(x)$ interlace with the zeros of $(1-x^2)C_{n-1}^{(\lambda)}(x)$. The same result holds for $\lambda > -1/2$ as can be seen by adding the two points ± 1 to the well-known interlacing of the zeros (all in $(-1,1)$) of $C_{n-1}^{(\lambda)}(x)$ and $C_n^{(\lambda)}(x)$. Here we show how the interlacing for $\lambda > -1/2$ may be used to get the interlacing for $-3/2 < \lambda < -1/2$.

The proof is based on the idea that interlacing between the zeros of $C_n^{(\lambda)}(x)$ and $C_{n+1}^{(\lambda)}(x)$ can break down or change only when these two functions have a common zero. First of all, we prove some lemmas.

LEMMA 3. *The only common zeros of $C_n^{(\lambda)}(x)$ and $C_{n+1}^{(\lambda)}(x)$ occur where $|x| = 1$ and $\lambda = -1/2, -3/2, \ldots, -\lfloor n/2 \rfloor + 1/2$.*

Proof. Because of parity considerations, the polynomials cannot have a common zero at 0. Let $\lambda = \lambda_0$ be fixed. The recurrence relation [19, (4.7.17)]

$$nC_n^{(\lambda)}(x) = 2(n+\lambda-1)xC_{n-1}^{(\lambda)}(x) - (n+2\lambda-2)C_{n-2}^{(\lambda)}(x),\ n = 2,3,4,\ldots \tag{19}$$

shows that, if $C_n^{(\lambda_0)}(x)$ and $C_{n-1}^{(\lambda_0)}(x)$ had a common zero x_0 for a value of x satisfying $0 < |x| \neq 1$, then x_0 would also be a zero of $C_{n-2}^{(\lambda_0)}(x)$. Repeating this argument, we would find that $C_{n-3}^{(\lambda_0)}(x_0) = 0$, and, eventually, $C_1^{(\lambda_0)}(x_0) = 0$, which is impossible.

Thus the only possible common zeros of $C_n^{(\lambda)}(x)$ and $C_{n+1}^{(\lambda)}(x)$ occur for $x = \pm 1$. From [12, Ch. 5] the zeros of $C_n^{(\lambda)}$ are continuous functions of λ and, as λ varies, the only values of λ for which interlacing breaks down are where $C_n^{(\lambda)}(x)$ and $C_{n+1}^{(\lambda)}(x)$

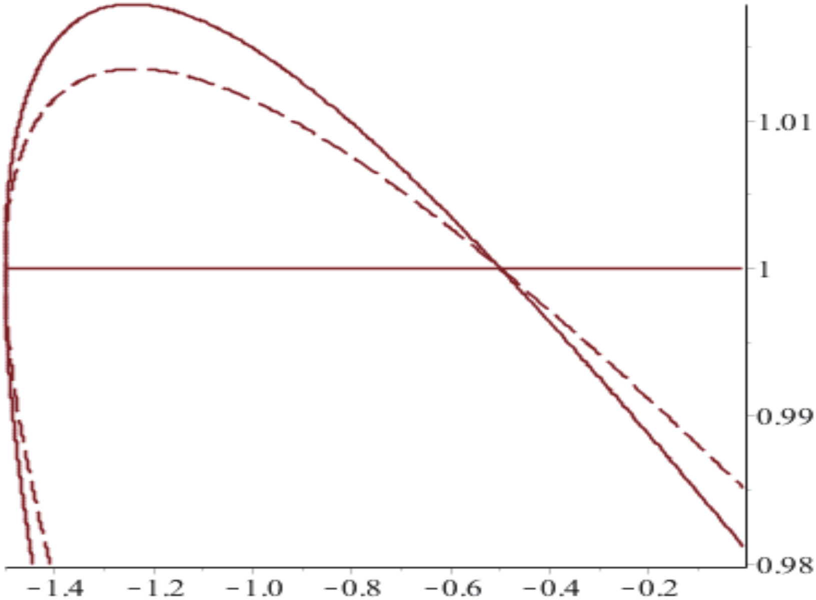

Figure 1: Largest zeros of $C_8^{(\lambda)}(x)$ (solid curves) and $C_9^{(\lambda)}(x)$ (dashed curves), as functions of λ, $-3/2 < \lambda < 0$.

have common zeros. Now [19, (4.7.6)], $C_n^{(\lambda)}(x)$ is a multiple of ${}_2F_1(-n, n+2\lambda; \lambda + 1/2; (1-x)/2)$. It vanishes at $x = 1$ only for $\lambda = -1/2, -3/2, \ldots, -\lfloor n/2 \rfloor + 1/2$ and $C_{n+1}^{(\lambda)}(x)$ vanishes at $x = 1$ only for $\lambda = -1/2, -3/2, \ldots, -\lfloor (n+1)/2 \rfloor + 1/2$. The statement of the Lemma follows. □

LEMMA 4. *If $x_{1,n}(\lambda)$ is the largest zero of $C_n^{(\lambda)}(x)$, then*

$$x'_{1,n}(-1/2) = -2/(n^2 - n).$$

In particular, $x'_{1,n}(-1/2)$ increases with n, $n = 2, 3, \ldots$.

Proof. The functions giving upper and lower bounds in (18) both have derivatives equal to $-2/(n^2 - n)$ at the point $\lambda = -1/2$. Then (18) implies that $x'_{1,n}(-1/2)$ must have the same value. □

We are now ready to show how to extend the property of the interlacing of real zeros of $(1-x^2)C_n^{(\lambda)}(x)$ and $C_{n+1}^{(\lambda)}(x)$ from the case $\lambda > -1/2$ to the case $-3/2 < \lambda < -1/2$. To keep things simple we consider positive zeros only.

We write the zeros in decreasing order. For $\lambda > -1/2$, we have

$$1 > x_{1,n}(\lambda) > x_{2,n}(\lambda) > x_{3,n}(\lambda) > \ldots, \tag{20}$$

and the interlacing property

$$1 > x_{1,n+1}(\lambda) > x_{1,n}(\lambda) > x_{2,n+1}(\lambda) > x_{2,n}(\lambda) > \dots. \tag{21}$$

From Lemma 2, the slope of $x_{1,n}(\lambda)$, for $\lambda = -1/2$, is a negative increasing function of n and so the functions $1, x_{1,n+1}(\lambda)$ and $x_{1,n}(\lambda)$ have their order reversed in the inequality (21) as λ passes through the value $-1/2$. Thus, for $-3/2 < \lambda < -1/2$, we find that the order of the first three terms in (21) are reversed, that is

$$\boxed{x_{1,n}(\lambda) > x_{1,n+1}(\lambda) > 1} > x_{2,n+1}(\lambda) > x_{2,n}(\lambda) > \dots, \tag{22}$$

with the other inequalities remaining the same, since the values $\lambda = -1/2$ and $x = 1$ constitute the only double zero within the range considered. Thus the interlacing of the zeros of $(1-x^2)C_n^{(\lambda)}(x)$ and $C_{n+1}^{(\lambda)}(x)$ persists for $-3/2 < \lambda < -1/2$.

Figure 1, produced using Maple, illustrates the change of order of $x_{1,n}(\lambda), x_{1,n+1}(\lambda)$ and 1 as lambda passes through $-1/2$ in the case $n = 8$.

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REFERENCES

[1] C. BREZINSKI, K. DRIVER AND M. REDIVO-ZAGLIA, *Quasi-orthogonality with applications to some families of classical orthogonal polynomials*, Appl. Numer. Math. **48** (2004), 157–168.

[2] T.S. CHIHARA, *On quasi–orthogonal polynomials*, Proc. Amer. Math. Soc. **8** (1957), 765–767.

[3] D. DICKINSON, *On quasi-orthogonal polynomials*, Proc. Amer. Math. Soc. **12** (1961), 185–194.

[4] D. K. DIMITROV AND G. K. NIKOLOV, *Sharp bounds for the extreme zeros of classical orthogonal polynomials*, J. Approx. Theory **162** no. 1 (2010), 1793–1804.

[5] A. DRAUX, *On quasi-orthogonal polynomials*, J. Approx. Theory **62** no.1 (1990), 1–14.

[6] K. DRIVER, *Interlacing of zeros of Gegenbauer polynomials of non-consecutive degree from different sequences*, Numer. Math. **120** (2012), 35–44.

[7] K. DRIVER AND P. DUREN, *Trajectories of zeros of hypergeometric polynomials $F(-n,b;2b;z)$ for $b < -1/2$*, Constr. Approx. **17** (2001), 169–179.

[8] K. DRIVER AND M. E. MULDOON, *Interlacing properties and bounds for zeros of some quasi-orthogonal Laguerre polynomials*, Comput. Methods Funct. Theory **15** (2015), 645–654.

[9] K. DRIVER AND M. E. MULDOON, *Zeros of quasi-orthogonal ultraspherical polynomials*, Indag. Math. **27** (2016), 930–944.

[10] L. FEJÉR , *Mechanische Quadraturen mit positiven Cotesschen Zahlen*, Math. Z. **37** (1933), 287–309.

[11] D. P. GUPTA AND M. E. MULDOON, *Inequalities for the smallest zeros of Laguerre polynomials and their q-analogues*, JIPAM. J. Inequal. Pure Appl. Math. **8** no. 1 (2007), Article 24.

[12] P. HARTMAN, *Ordinary Differential Equations*, Wiley, 1964.

[13] M. E. H. ISMAIL AND M. E. MULDOON, *Bounds for the small real and purely imaginary zeros of Bessel and related functions*, Methods Appl. Anal. **2** (1995), 1-21.

[14] P. MARONI, *Une caractérisation des polynômes orthogonaux semi-classiques*. C. R. Acad. Sci. Paris, Sér. I **301** (1985), 269–272.

[15] P. MARONI, *Prolégomènes à l'étude des polynômes orthogonaux semi-classiques*, Ann. Mat. Pura Appl. IV Ser. **149** (1987), 165–183.

[16] P. MARONI, *Une théorie algébrique des polynômes orthogonaux. Application aux polynômes orthogonaux semi-classiques*, Orthogonal polynomials and their applications: Proceedings of the Third International Symposium (Erice, 1990), 95–130, IMACS Ann. Comput. Appl. Math, **9** (1991), Baltzer, Basel, 1991.

[17] M. RIESZ, *Sur le problème des moments, III,* Ark. f. Mat., Astronom. och Fys. **17** no. 16 (1923), 1–52.
[18] J.A. SHOHAT, *On mechanical quadratures, in particular, with positive coefficients*, Trans. Amer. Math. Soc. **42** (1937), 461–496.
[19] G. SZEGŐ, *Orthogonal Polynomials*, American Mathematical Society Colloquium Publications, vol 23, 4th ed.,1975.
[20] G. N. Watson. *A treatise on the Theory of Bessel Functions*, 2nd ed., Cambridge University Press, 1944.

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