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PARTIAL SUMS OF NORMALIZED DINI FUNCTIONS

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Abstract. Some lower bounds are proposed for the quotients of normalized Dini functions and their partial sum, as well as for the quotients of the derivative of normalized Dini functions and their partial sums. Our study is motivated by some earlier results one similar lower bounds for quotients of convex univalent functions and their partial sum.

1. Introduction

Special functions, like Bessel functions, play an important role in applied mathematics and physics. Because of their remarkable properties the Bessel functions have been studied by many scientists. The Bessel function of the first kind J_ν is defined by (see [2] and [12])

$$J_\nu(z) = \sum_{n\geqslant 0} \frac{(-1)^n}{n!\Gamma(\nu+n+1)}\left(\frac{z}{2}\right)^{2n+\nu}, \tag{1}$$

where Γ stands for Euler gamma function, and it is a particular solution of the second-order linear homogeneous differential equation

$$z^2y''(z)+zy'(z)+(z^2-\nu^2)y(z)=0, \tag{2}$$

where $\nu \in \mathbb{C}$.

In this paper, we consider the normalized Dini function $w_{\alpha,\nu} : \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\} \to \mathbb{C}$, defined by (see [4])

$$\begin{aligned} w_{\alpha,\nu}(z) &= \frac{2^\nu}{\alpha}\Gamma(\nu+1)z^{1-\frac{\nu}{2}}((\alpha-\nu)J_\nu(\sqrt{z})+\sqrt{z}J'_\nu(\sqrt{z})) \\ &= \sum_{n\geqslant 0}\frac{(-1)^n(2n+\alpha)\Gamma(\nu+1)z^{n+1}}{\alpha 4^n n!\Gamma(\nu+n+1)} \\ &= \sum_{n\geqslant 0}\frac{(-1)^n(2n+\alpha)}{\alpha 4^n n!(\nu+1)_n}z^{n+1}, \end{aligned} \tag{3}$$

where $(\mu)_n = \frac{\Gamma(\mu+n)}{\Gamma(\mu)} = \mu(\mu+1)\dots(\mu+n-1)$ stands for the Pochhammer symbol. Motivated by the papers of Silverman [10] and Silvia [11] in this paper we investigate the ratio of a function of the form (3) to its sequence of partial sums $(w_{\alpha,\nu}(z))_m =$

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Paper JCA-09-13

$z+\sum_{n=1}^{m} a_n z^{n+1}$ when the coefficients of $w_{\alpha,\nu}$ satisfy some conditions. We will obtain some lower bounds for

$$\Re\left[\frac{w_{\alpha,\nu}(z)}{(w_{\alpha,\nu})_m(z)}\right],\quad \Re\left[\frac{(w_{\alpha,\nu})_m(z)}{w_{\alpha,\nu}(z)}\right],\quad \Re\left[\frac{w'_{\alpha,\nu}(z)}{(w_{\alpha,\nu})'_m(z)}\right] \quad \text{and} \quad \Re\left[\frac{(w_{\alpha,\nu})'_m(z)}{w'_{\alpha,\nu}(z)}\right].$$

For interesting developments on the partial sums of some special functions and some classes of analytic functions, the readers can refer to the works of Sheil-Small [9], Brickman et al.[5], Silvia [11], Silverman [10], Owa et al. [8], Orhan and Yağmur [7], Çağlar and Deniz [6], Aktaş and Orhan [1].

The next result is a preliminary one, which will be used in the proof of the main results.

LEMMA 1. *If $\alpha>0$ and $\nu>-\frac{7}{8}$, then the function*

$$w_{\alpha,\nu}:\mathbb{D}\to\mathbb{C},$$

given by (3), *satisfies the following inequalities:*

$$|w_{\alpha,\nu}(z)|\leqslant 1+\frac{2(16\nu+8\alpha\nu+7\alpha+16)}{(8\nu+7)^2\alpha}, \tag{4}$$

$$\left|w'_{\alpha,\nu}(z)\right|\leqslant 1+\frac{256(2+\alpha)\nu^2+16(64+29\alpha)\nu+210\alpha+512}{(8\nu+7)^3\alpha}. \tag{5}$$

Proof. By using the well-known triangle inequality:

$$|z_1+z_2|\leqslant|z_1|+|z_2| \tag{6}$$

and the inequalities

$$(\nu+1)^n\leqslant(\nu+1)_n \quad \text{and} \quad 2^{n-1}\leqslant n! \tag{7}$$

for $n\in\mathbb{N}=\{1,2,\ldots\}$, we get

$$\begin{aligned}
|w_{\alpha,\nu}(z)| &= |z+\sum_{n\geqslant 1}\frac{(-1)^n(2n+\alpha)}{\alpha 4^n n!(\nu+1)_n}z^{n+1}|\leqslant 1+\sum_{n\geqslant 1}\frac{2n+\alpha}{\alpha 4^n 2^{n-1}(\nu+1)^n}\\
&= 1+\frac{1}{2\alpha(\nu+1)}\sum_{n\geqslant 1}\frac{n}{(8(\nu+1))^{n-1}}+\frac{1}{4(\nu+1)}\sum_{n\geqslant 1}\left(\frac{1}{8(\nu+1)}\right)^{n-1}\\
&= 1+\frac{2(16\nu+8\alpha\nu+7\alpha+16)}{(8\nu+7)^2\alpha}.
\end{aligned}$$

If we consider the inequality (5), then we have

$$\begin{aligned}\left|w'_{\alpha,\nu}(z)\right| &= \left|1+\sum_{n\geqslant 1}\frac{(-1)^n(2n+\alpha)(n+1)}{\alpha 4^n n!(\nu+1)_n}z^n\right| \leqslant 1+\sum_{n\geqslant 1}\frac{2n^2+(2+\alpha)n+\alpha}{\alpha 4^n 2^{n-1}(\nu+1)^n}\\ &= 1+\frac{1}{(\nu+1)}\left[\frac{1}{2\alpha}\sum_{n\geqslant 1}\frac{n^2}{(8(\nu+1))^{n-1}}+\frac{(2+\alpha)}{4\alpha}\sum_{n\geqslant 1}\frac{n}{(8(\nu+1))^{n-1}}\right]\\ &\quad+\frac{1}{4(\nu+1)}\sum_{n\geqslant 1}\left(\frac{1}{8(\nu+1)}\right)^{n-1}\\ &= 1+\frac{256(2+\alpha)\nu^2+16(64+29\alpha)\nu+210\alpha+512}{(8\nu+7)^3\alpha}. \qquad \square\end{aligned}$$

2. Lower bounds for quotients of Dini function and its partial sum

Our first main result is the following theorem.

THEOREM 1. *Let* $\alpha > 0$, $\nu > -\frac{7}{8}$, *the function* $w_{\alpha,\nu}:\mathbb{D}\to\mathbb{C}$ *be defined by* (3) *and its sequence of partial sums by* $(w_{\alpha,\nu}(z))_m = z+\sum_{n=1}^{m} a_n z^{n+1}$, *where* $a_n = \frac{(-1)^n(2n+\alpha)}{\alpha 4^n n!(\nu+1)_n}$. *If the inequality*

$$64\alpha\nu^2+96\alpha\nu+35\alpha-32\nu-32\geqslant 0$$

is valid, then we have the followings:

$$\Re\left[\frac{w_{\alpha,\nu}(z)}{(w_{\alpha,\nu})_m(z)}\right]\geqslant\frac{64\alpha\nu^2+32(3\alpha-1)\nu+35\alpha-32}{(8\nu+7)^2\alpha}, \tag{8}$$

$$\Re\left[\frac{(w_{\alpha,\nu})_m(z)}{w_{\alpha,\nu}(z)}\right]\geqslant\frac{(8\nu+7)^2\alpha}{64\alpha\nu^2+32(4\alpha+1)\nu+63\alpha+32}. \tag{9}$$

Proof. From the inequality (4) in case $z=-1$ we get that

$$1+\sum_{n\geqslant 1}|a_n|\leqslant 1+\frac{2(16\nu+8\alpha\nu+7\alpha+16)}{(8\nu+7)^2\alpha},$$

which is equivalent to

$$\frac{(8\nu+7)^2\alpha}{2(16\nu+8\alpha\nu+7\alpha+16)}\sum_{n\geqslant 1}|a_n|\leqslant 1.$$

Now, we write

$$\frac{(8\nu+7)^2\alpha}{2(16\nu+8\alpha\nu+7\alpha+16)}\left[\frac{w_{\alpha,\nu}(z)}{(w_{\alpha,\nu})_m(z)}-\left(1-\frac{2(16\nu+8\alpha\nu+7\alpha+16)}{(8\nu+7)^2\alpha}\right)\right]$$

$$=\frac{1+\sum_{n=1}^{m}a_nz^n+\frac{(8\nu+7)^2\alpha}{2(16\nu+8\alpha\nu+7\alpha+16)}\sum_{n\geqslant m+1}a_nz^n}{1+\sum_{n=1}^{m}a_nz^n}=\frac{1+A(z)}{1+B(z)}. \tag{10}$$

Set

$$\frac{1+A(z)}{1+B(z)}=\frac{1+p(z)}{1-p(z)}$$

so that

$$p(z)=\frac{A(z)-B(z)}{2+A(z)+B(z)}.$$

The inequality $|p(z)|\leqslant 1$ implies (8). Then

$$p(z)=\frac{\frac{(8\nu+7)^2\alpha}{2(16\nu+8\alpha\nu+7\alpha+16)}\sum_{n\geqslant m+1}a_nz^n}{2+2\sum_{n=1}^{m}a_nz^n+\frac{(8\nu+7)^2\alpha}{2(16\nu+8\alpha\nu+7\alpha+16)}\sum_{n\geqslant m+1}a_nz^n}$$

and

$$|p(z)|\leqslant\frac{\frac{(8\nu+7)^2\alpha}{2(16\nu+8\alpha\nu+7\alpha+16)}\sum_{n\geqslant m+1}|a_n|}{2-2\sum_{n=1}^{m}|a_n|-\frac{(8\nu+7)^2\alpha}{2(16\nu+8\alpha\nu+7\alpha+16)}\sum_{n\geqslant m+1}|a_n|}.$$

The inequality

$$\sum_{n=1}^{m}|a_n|+\frac{(8\nu+7)^2\alpha}{2(16\nu+8\alpha\nu+7\alpha+16)}\sum_{n\geqslant m+1}|a_n|\leqslant 1 \tag{11}$$

implies that $|p(z)|\leqslant 1$. It suffices to show that the left hand side of (11) is bounded above by

$$\frac{(8\nu+7)^2\alpha}{2(16\nu+8\alpha\nu+7\alpha+16)}\sum_{n\geqslant 1}|a_n|,$$

which is equivalent to

$$\frac{64\alpha\nu^2+32(3\alpha-1)\nu+35\alpha-32}{2(16\nu+8\alpha\nu+7\alpha+16)}\sum_{n=1}^{m}|a_n|\geqslant 0.$$

To prove the result (9), we write

$$\left(\frac{(8\nu+7)^2\alpha}{2(16\nu+8\alpha\nu+7\alpha+16)}+1\right)\left[\frac{(w_{\alpha,\nu})_m(z)}{w_{\alpha,\nu}(z)}-\frac{(8\nu+7)^2\alpha}{64\alpha\nu^2+32(4\alpha+1)\nu+63\alpha+32}\right]$$
$$=\frac{1+\sum\limits_{n=1}^{m}a_nz^n-\frac{(8\nu+7)^2\alpha}{2(16\nu+8\alpha\nu+7\alpha+16)}\sum\limits_{n\geqslant m+1}a_nz^n}{1+\sum\limits_{n\geqslant 1}a_nz^n}=\frac{1+p(z)}{1-p(z)}. \tag{12}$$

Then from (12) we get

$$p(z)=\frac{-\frac{64\alpha\nu^2+32(4\alpha+1)\nu+63\alpha+32}{2(16\nu+8\alpha\nu+7\alpha+16)}\sum\limits_{n\geqslant m+1}a_nz^n}{2+2\sum\limits_{n=1}^{m}a_nz^n-\frac{64\alpha\nu^2+32(3\alpha-1)\nu+35\alpha-32}{2(16\nu+8\alpha\nu+7\alpha+16)}\sum\limits_{n\geqslant m+1}a_nz^n}$$

and

$$|p(z)|\leqslant\frac{\frac{64\alpha\nu^2+32(4\alpha+1)\nu+63\alpha+32}{2(16\nu+8\alpha\nu+7\alpha+16)}\sum\limits_{n\geqslant m+1}|a_n|}{2-2\sum\limits_{n=1}^{m}|a_n|-\frac{64\alpha\nu^2+32(3\alpha-1)\nu+35\alpha-32}{2(16\nu+8\alpha\nu+7\alpha+16)}\sum\limits_{n\geqslant m+1}|a_n|}.$$

The inequality

$$\sum_{n=1}^{m}|a_n|+\frac{(8\nu+7)^2\alpha}{2(16\nu+8\alpha\nu+7\alpha+16)}\sum_{n\geqslant m+1}|a_n|\leqslant 1 \tag{13}$$

implies that $|p(z)|\leqslant 1$. Since the left hand side of (13) is bounded above by

$$\frac{(8\nu+7)^2\alpha}{2(16\nu+8\alpha\nu+7\alpha+16)}\sum_{n\geqslant 1}|a_n|$$

the proof is completed. □

The next result is analogous to our first main result.

THEOREM 2. *Let* $\alpha>0,\ \nu>-\frac{7}{8}$, *the function* $w_{\alpha,\nu}:\mathbb{D}\to\mathbb{C}$ *be defined by* (3) *and its sequence of partial sums by* $(w_{\alpha,\nu}(z))_m=z+\sum\limits_{n=1}^{m}a_nz^{n+1}$. *If the inequality*

$$512\alpha\nu^3+64(17\alpha-8)\nu^2+8(89\alpha-128)\nu+133\alpha-512\geqslant 0$$

is valid, then we have the followings:

$$\Re\left[\frac{w'_{\alpha,\nu}(z)}{(w_{\alpha,\nu})'_m(z)}\right]\geqslant\frac{512\alpha\nu^3+64(17\alpha-8)\nu^2+8(89\alpha-128)\nu+133\alpha-512}{256(2+\alpha)\nu^2+16(64+29\alpha)\nu+210\alpha+512}, \tag{14}$$

$$\Re\left[\frac{(w_{\alpha,\nu})'_m(z)}{w'_{\alpha,\nu}(z)}\right]\geqslant\frac{(8\nu+7)^3\alpha}{512\alpha\nu^3+64(25\alpha+8)\nu^2+8(205\alpha+128)\nu+553\alpha+512}. \tag{15}$$

Proof. From the inequality (5) in Lemma 1 we write that

$$1+\sum_{n\geqslant 1}(n+1)\left|a_n\right| \leqslant 1+\frac{1}{\delta},$$

which is equivalent to

$$\delta\cdot\sum_{n\geqslant 1}(n+1)\left|a_n\right| \leqslant 1,$$

where

$$\delta=\frac{(8\nu+7)^3\,\alpha}{256\,(2+\alpha)\,\nu^2+16\,(64+29\alpha)\,\nu+210\alpha+512}.$$

Now, we write

$$\delta\cdot\left[\frac{w'_{\alpha,\nu}(z)}{\left(w_{\alpha,\nu}\right)'_m(z)}-\left(1-\frac{1}{\delta}\right)\right]$$
$$=\frac{1+\sum\limits_{n=1}^{m}(n+1)\,a_nz^n+\delta\sum\limits_{n\geqslant m+1}(n+1)\,a_nz^n}{1+\sum\limits_{n=1}^{m}(n+1)\,a_nz^n}=\frac{1+p(z)}{1-p(z)},$$

where

$$|p(z)|\leqslant\frac{\delta\cdot\sum\limits_{n\geqslant m+1}(n+1)\left|a_n\right|}{2-2\sum\limits_{n=1}^{m}(n+1)\left|a_n\right|-\delta\cdot\sum\limits_{n\geqslant m+1}(n+1)\left|a_n\right|}\leqslant 1.$$

The last inequality is equivalent to

$$\sum_{n=1}^{m}(n+1)\left|a_n\right|+\delta\cdot\sum_{n\geqslant m+1}(n+1)\left|a_n\right|\leqslant 1. \tag{16}$$

It suffices to show that the left hand side of (16) is bounded above by

$$\delta\cdot\sum_{n\geqslant 1}(n+1)\left|a_n\right|,$$

which is equivalent to

$$\frac{512\alpha\nu^3+64\,(17\alpha-8)\,\nu^2+8\,(89\alpha-128)\,\nu+133\alpha-512}{256\,(2+\alpha)\,\nu^2+16\,(64+29\alpha)\,\nu+210\alpha+512}\sum_{n=1}^{m}(n+1)\left|a_n\right|\geqslant 0.$$

To prove the result (15), we write

$$\begin{aligned}
&\frac{1+p(z)}{1-p(z)}\\
&=(\delta+1)\cdot\left[\frac{\left(w_{\alpha,\nu}\right)_m'(z)}{w_{\alpha,\nu}'(z)}-\frac{(8\nu+7)^3\,\alpha}{512\alpha\nu^3+64\,(25\alpha+8)\,\nu^2+8\,(205\alpha+128)\,\nu+553\alpha+512}\right]\\
&=\frac{1+\sum\limits_{n=1}^{m}(n+1)\,a_nz^n-\delta\cdot\sum\limits_{n\geqslant m+1}(n+1)\,a_nz^n}{1+\sum\limits_{n\geqslant 1}(n+1)\,a_nz^n}
\end{aligned}$$

where

$$\begin{aligned}
|p(z)|&\leqslant\frac{\frac{512\alpha\nu^3+64(25\alpha+8)\nu^2+8(205\alpha+128)\nu+553\alpha+512}{256(2+\alpha)\nu^2+16(64+29\alpha)\nu+210\alpha+512}\sum\limits_{n\geqslant m+1}(n+1)\,|a_n|}{2-2\sum\limits_{n=1}^{m}(n+1)\,|a_n|-\frac{512\alpha\nu^3+64(17\alpha-8)\nu^2+8(89\alpha-128)\nu+133\alpha-512}{256(2+\alpha)\nu^2+16(64+29\alpha)\nu+210\alpha+512}\sum\limits_{n\geqslant m+1}(n+1)\,|a_n|}\\
&\leqslant 1.
\end{aligned}$$

The last inequality is equivalent to

$$\sum_{n=1}^{m}(n+1)\,|a_n|+\delta\cdot\sum_{n\geqslant m+1}(n+1)\,|a_n|\leqslant 1. \tag{17}$$

Since the left hand side of (17) is bounded above by $\delta\cdot\sum\limits_{n\geqslant 1}(n+1)\,|a_n|$, the proof is completed. □

For some special cases of ν it is known that (see [3])

$$J_{\frac{1}{2}}(z)=\sqrt{\frac{2}{\pi z}}\sin z$$

and

$$J_{\frac{3}{2}}(z)=\sqrt{\frac{2}{\pi z}}\left(\frac{\sin z}{z}-\cos z\right).$$

By using some special cases of α and ν in Theorem 1 and Theorem 2, we can obtain the following corollaries:

a. If we take $\alpha=1,\ \nu=\frac{1}{2}$ and $m=0$, we have

$$w_{1,\frac{1}{2}}(z)=\frac{\sqrt{z}}{2}\left\{\sin\sqrt{z}+\cos\sqrt{z}\right\}-\frac{\sin\sqrt{z}}{4},$$

$$w_{1,\frac{1}{2}}'(z)=\frac{1}{4\sqrt{z}}\left\{\left(1-\sqrt{z}\right)\sin\sqrt{z}+\frac{(1+2\sqrt{z})\cos\sqrt{z}}{2}\right\},$$

$$\left(w_{1,\frac{1}{2}}\right)_0(z)=z \quad \text{and} \quad \left(w_{1,\frac{1}{2}}\right)_0'(z)=1.$$

In view of Theorem 1 we get the following inequalities:

$$\Re\left[\frac{1}{2\sqrt{z}}\left(sin\sqrt{z}+cos\sqrt{z}\right)-\frac{sin\sqrt{z}}{4z}\right]\geqslant\frac{51}{121},$$

$$\Re\left[\frac{4z}{(2\sqrt{z}-1)\,sin\sqrt{z}+2\sqrt{z}cos\sqrt{z}}\right]\geqslant\frac{121}{191}.$$

b. If we take $\alpha=\frac{3}{2},\ \nu=\frac{1}{2}$ and $m=0,$ we have

$$w_{\frac{3}{2},\frac{1}{2}}(z)=\frac{\sqrt{z}}{3}(2\sin\sqrt{z}+\cos\sqrt{z})-\frac{\sin\sqrt{z}}{6},$$

$$w_{\frac{3}{2},\frac{1}{2}}'(z)=\frac{4\sin\sqrt{z}+\cos\sqrt{z}}{12\sqrt{z}}+\frac{2\cos\sqrt{z}-\sin\sqrt{z}}{6},$$

$$\left(w_{\frac{3}{2},\frac{1}{2}}\right)_0(z)=z \quad \text{and} \quad \left(w_{\frac{3}{2},\frac{1}{2}}\right)_0'(z)=1.$$

In view of Theorem 2 we get the following inequalities:

$$\Re\left[\frac{4\sin\sqrt{z}+\cos\sqrt{z}}{12\sqrt{z}}+\frac{2\cos\sqrt{z}-\sin\sqrt{z}}{6}\right]\geqslant\frac{57}{1274},$$

$$\Re\left[\frac{12\sqrt{z}}{(4\sqrt{z}+1)\cos\sqrt{z}-2\left(\sqrt{z}-2\right)\sin\sqrt{z}}\right]\geqslant\frac{1331}{2605}.$$

c. If we take $\alpha=5,\ \nu=\frac{3}{2}$ and $m=0,$ we have

$$w_{5,\frac{3}{2}}(z)=\frac{21}{10}\left(\frac{\sin\sqrt{z}}{\sqrt{z}}-\cos\sqrt{z}\right)+\frac{9}{20}\left(\frac{\cos\sqrt{z}}{\sqrt{z}}-\frac{\sin\sqrt{z}}{z}\right)+\frac{3}{10}\sin\sqrt{z},$$

$$w_{5,\frac{3}{2}}'(z)=\frac{21}{20}\left(\frac{\cos\sqrt{z}}{z}-\frac{\sin\sqrt{z}}{z\sqrt{z}}+\frac{\sin\sqrt{z}}{\sqrt{z}}\right)+\frac{9}{20}\left(\frac{\sin\sqrt{z}}{z^2}-\frac{\cos\sqrt{z}}{z\sqrt{z}}\right)$$
$$-\frac{9}{40}\frac{\sin\sqrt{z}}{z}+\frac{3}{20}\frac{\cos\sqrt{z}}{\sqrt{z}},$$

$$\left(w_{5,\frac{3}{2}}\right)_0(z)=z \quad \text{and} \quad \left(w_{5,\frac{3}{2}}\right)_0'(z)=1.$$

In view of Theorem 1 we get the following inequalities:

$$\Re\left[\frac{14z\left(\sin\sqrt{z}-\sqrt{z}\cos\sqrt{z}\right)+(2z-3)\sqrt{z}\sin\sqrt{z}+3z\cos\sqrt{z}}{z^{5/2}}\right]\geqslant\frac{307}{361},$$

$$\Re\left[\frac{z^{5/2}}{14z\left(\sin\sqrt{z}-\sqrt{z}\cos\sqrt{z}\right)+(2z-3)\sqrt{z}\sin\sqrt{z}+3z\cos\sqrt{z}}\right]\geqslant\frac{361}{415}.$$

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