Journal of Classical Analysis
Volume 10, Number 1 (2017), 1–14 doi:10.7153/jca-10-01

WEIGHTED SHARING OF NON–LINEAR DIFFERENTIAL POLYNOMIALS SHARING SMALL FUNCTION WITH REGARD TO MULTIPLICITY

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Abstract. With the notion of weighted sharing values we study the uniqueness of meromorphic functions when certain non-zero differential polynomial share a small function with regard to multiplicity. The result of the paper improve and extend some recent result due to Abhijith Banerjee and Pulak Sahoo [3].

1. Introduction

In this paper by meromorphic functions we will always mean meromorphic functions in the complex plane.

Let f and g be two non-constant meromorphic functions and let a be a finite complex number. We say that f and g share a CM, provided that $f-a$ and $g-a$ have the same zeros with the same multiplicities. Similarly, we say that f and g share a IM, provided that $f-a$ and $g-a$ have the same zeros ignoring multiplicities. In addition we say that f and g share ∞ CM, if $\frac{1}{f}$ and $\frac{1}{g}$ share 0 CM, and we say that f and g share ∞ IM, if $\frac{1}{f}$ and $\frac{1}{g}$ share 0 IM.

We adopt the standard notations of value distribution theory (see [8]). We denote by $T(r)$ the maximum of $T(r,f)$ and $T(r,g)$. The notation $S(r)$ denotes any quantity satisfying $S(r)=o(T(r))$ as $r\to\infty$, outside of a possible exceptional set of finite linear measure.

Throughout this paper, we need the following definition.

$$\Theta(a;f)=1-\limsup_{r\to\infty}\frac{\bar{N}(r,a;f)}{T(r,f)},$$

where a is a value in the extended complex plane.

In 1959, Hayman [7] proved the following result.

THEOREM A. *Let f be a transcendental entire function, and let $n(\geqslant 1)$ be an integer. Then $f^nf'=1$ has infinitely many zeros.*

In 2002, Fang and Fang [6] proved the following result.

Mathematics subject classification (2010): Primary: 30D35.
Keywords and phrases: Meromorphic function, weighted sharing, nonlinear differential polynomials etc.

THEOREM B. *Let f and g be two non-constant entire functions, and let $n(\geqslant 8)$ be an integer. If $f^n(f-1)f'$ and $g^n(g-1)g'$ share 1 CM, then $f \equiv g$.*

In the same year Fang [5] investigated the value sharing of more general non-linear differential polynomial than that was considered in Theorem B and obtained the following result.

THEOREM C. *Let f and g be two non-constant entire functions, and let n,k be two positive integers with $n \geqslant 2k+8$. If $[f^n(f-1)]^{(k)}$ and $[g^n(g-1)]^{(k)}$ share 1 CM, then $f \equiv g$.*

In 2004, Lin and Yi [14] considered the case of meromorphic function in Theorem B and obtained the following.

THEOREM D. *Let f and g be two non-constant meromorphic functions with $\Theta(\infty,f) > \frac{2}{n+1}$, and let $n(\geqslant 12)$ be an integer. If $f^n(f-1)f'$ and $g^n(g-1)g'$ share 1 CM, then $f \equiv g$.*

Natural inquisition would be to investigate the situation for meromorphic function in Theorem C. In this direction in 2008, Zhang [20] proved the following result.

THEOREM E. *Suppose that f is a transcendental meromorphic function with finite number of poles, g is a transcendental entire function, and let n,k be two positive integers with $n \geqslant 2k+6$. If $[f^n(f-1)]^{(k)}$ and $[g^n(g-1)]^{(k)}$ share 1 CM, then $f \equiv g$.*

To proceed further we require the following definition known as weighted sharing of values introduced by I. Lahiri [9] which measure how close a shared value is to being shared CM or to being shared IM.

DEFINITION 1. Let k be a non negative integer or infinity. For $a \in \mathbb{C}\bigcup\{\infty\}$ we denote by $E_k(a;f)$ the set of all a-points of f where an a-point of multiplicity m is counted m times if $m \leqslant k$ and $k+1$ times if $m > k$. If $E_k(a;f) = E_k(a;g)$, we say that f,g share the value a with weight k.

The definition implies that if f,g share a value a with weight k, then z_0 is an a-point of f with multiplicity $m(\leqslant k)$ if and only if it is an a-point of g with multiplicity $m(\leqslant k)$ and z_0 is an a-point of f with multiplicity $m(> k)$ if and only if it is an a-point of g with multiplicity $n(> k)$, where m is not necessarily equal to n.

We write f,g share (a,k) to mean that f,g share the value a with weight k. Clearly if f,g share (a,k) then f,g share (a,p) for any integer p, $0 \leqslant p < k$. Also we note that f,g share a IM or CM if and only if f,g share $(a,0)$ or (a,∞) respectively.

In 2009, using the notion of weighted sharing of values, Xu, Yi and Cao [15] proved the following result.

THEOREM F. *Let f and g be two non-constant meromorphic functions, and $n(\geqslant 1)$, $k(\geqslant 1)$ and $l(\geqslant 0)$ be three integers such that $\Theta(\infty,f)+\Theta(\infty,g) > \frac{4}{n}$. Suppose $[f^n(f-1)]^{(k)}$ and $[g^n(g-1)]^{(k)}$ share $(1,l)$. If $l \geqslant 2$ and $n > 5k+11$ or if $l = 1$ and $n > 7k+\frac{23}{2}$, then $f = g$.*

Recently, Li [13] proved the following result which rectify and at the same time improve Theorem F.

THEOREM G. *Let f and g be two non-constant meromorphic functions, and $n(\geqslant 1)$, $k(\geqslant 1)$ and $l(\geqslant 0)$ be three integers such that $\Theta(\infty,f)+\Theta(\infty,g)>\frac{4}{n}$. Suppose $[f^n(f-1)]^{(k)}$ and $[g^n(g-1)]^{(k)}$ share $(1,l)$. If $l\geqslant 2$ and $n>3k+11$ or if $l=1$ and $n>5k+14$, then $f=g$ or $[f^n(f-1)]^{(k)}[g^n(g-1)]^{(k)}=1$.*

In this direction recently Abhijith Banerjee [1] proved the following results first one of which improves Theorem G.

THEOREM H. *Let f and g be two transcendental meromorphic functions and $n(\geqslant 1)$, $k(\geqslant 1)$, $l(\geqslant 0)$ be three integers such that $\Theta(\infty,f)+\Theta(\infty,g)>\frac{4}{n}$. Suppose for two nonzero constants a and b, $[f^n(af+b)]^{(k)}$ and $[g^n(ag+b)]^{(k)}$ share $(1,l)$. If $l\geqslant 2$ and $n\geqslant 3k+9$ or if $l=1$ and $n\geqslant 4k+10$, or if $l=0$ and $n\geqslant 9k+18$, then $f=g$ or $[f^n(af+b)]^{(k)}[g^n(ag+b)]^{(k)}=1$. The possibility $[f^n(af+b)]^{(k)}[g^n(ag+b)]^{(k)}=1$ does not occur for $k=1$.*

THEOREM I. *Let f and g be two transcendental entire functions, and let $n(\geqslant 1)$, $k(\geqslant 1)$, $l(\geqslant 0)$ be three integers. Suppose for two nonzero constants a and b, $[f^n(af+b)]^{(k)}$ and $[g^n(ag+b)]^{(k)}$ share $(1,l)$. If $l\geqslant 2$ and $n\geqslant 2k+6$ or if $l=1$ and $n\geqslant\frac{5k}{2}+7$, or if $l=0$ and $n\geqslant 5k+12$, then $f=g$.*

In 2015, Abhijith Banerjee and Pulak Sahoo [3] obtained the following result.

THEOREM J. *Let f and g be two non-entire transcendental meromorphic functions, and let $n(\geqslant 1)$, $k(\geqslant 1)$, $l(\geqslant 0)$ be three integers such that $\Theta(\infty,f)+\Theta(\infty,g)>\frac{4}{n}$. Suppose for two nonzero constants a and b, $[f^n(af+b)]^{(k)}-P$ and $[g^n(ag+b)]^{(k)}-P$ share $(0,l)$ where $P(\not\equiv 0)$ is a polynomial. If $l\geqslant 2$ and $n\geqslant 3k+9$ or if $l=1$ and $n\geqslant 4k+10$ or if $l=0$ and $n\geqslant 9k+18$, then $f=g$.*

THEOREM K. *Let f and g be two transcendental entire functions, and let $n(\geqslant 1)$, $k(\geqslant 1)$, $l(\geqslant 0)$ be three integers. Suppose for two nonzero constants a and b, $[f^n(af+b)]^{(k)}-P$ and $[g^n(ag+b)]^{(k)}-P$ share $(0,l)$ where $P(\not\equiv 0)$ is a polynomial. If $l\geqslant 2$ and $n\geqslant 2k+6$ or if $l=1$ and $n\geqslant\frac{5k}{2}+7$ or if $l=0$ and $n\geqslant 5k+12$, then $f=g$.*

With the notion of weighted sharing values we study the uniqueness of meromorphic function with certain non-zero differential polynomial share a small function with regard to multiplicity.

We now state our main result.

THEOREM 1. *Let f and g be two non-entire transcendental meromorphic functions, whose zeros and poles are of multiplicities atleast s, where s is a positive integer. Let $n(\geqslant 1)$, $k(\geqslant 1)$, $l(\geqslant 0)$ be three integers such that $\Theta(\infty,f)+\Theta(\infty,g)>\frac{4}{n}$. Suppose for two nonzero constants a and b, $[f^n(af^m+b)]^{(k)}-P$ and $[g^n(ag^m+b)]^{(k)}-P$ share $(0,l)$, where $P(\neq 0)$ is a polynomial. If $l\geqslant 2$ and $n\geqslant\frac{3k+8}{s}+m$, or if $l=1$ and $n\geqslant\frac{4k+9}{s}+\frac{3m}{2}$ or if $l=0$ and $n\geqslant\frac{9k+14}{s}+4m$ then $f=g$.*

THEOREM 2. *Let f and g be two transcendental entire functions, whose zeros and poles are of multiplicities atleast s, where s is a positive integer. Let $n(\geqslant 1)$, $k(\geqslant$*

1), $l(\geqslant 0)$ be three integers such that $\Theta(\infty, f)+\Theta(\infty, g)>\frac{4}{n}$. Suppose for two nonzero constants a and b, $[f^n(af^m+b)]^{(k)}-P$ and $[g^n(ag^m+b)]^{(k)}-P$ share $(0,l)$, where $P(\neq 0)$ is a polynomial. If $l\geqslant 2$ and $n\geqslant \frac{2k+5}{s}+m$, or if $l=1$ and $n\geqslant \frac{5k+10}{2s}+4m$ or if $l=0$ and $n\geqslant \frac{5k+8}{s}+4m$ then $f=g$.

REMARK 1. 1. If we put $m=1$ and $s=1$ in Theorem 1, then Theorem 1 reduces to Theorem J.

2. If we put $m=1$ and $s=1$ in Theorem 2, then Theorem 2 reduces to Theorem K.

Though the standard definitions and notations of the value distribution theory are available in [8], we explain some definitions and notations which are used in the paper.

DEFINITION 2. [10] For $a\in\mathbb{C}\bigcup\{\infty\}$ we denote by $N(r,a;f|=1)$ the counting function of simple a-points of f. For a positive integer p we denote by $N(r,a;f|\leqslant p)$ the counting function of those a-points of f (counted with multiplicities) whose multiplicities are not greater then p. By $\overline{N}(r,a;f|\leqslant p)$ we denote the corresponding reduced counting function.

In an analogous manner we define $N(r,a;f|\geqslant p)$ and $\overline{N}(r,a;f|\geqslant p)$.

DEFINITION 3. [9] Let k be a positive integer or infinity. We denote by $N_k(r,a;f)$ the counting function of a-points of f, where an a-point of multiplicity m is counted m times if $m\leqslant k$ and k times if $m>k$. Then

$$N_k(r,a;f)=\overline{N}(r,a;f)+\overline{N}(r,a;f|\geqslant 2)+\ldots+\overline{N}(r,a;f|\geqslant k).$$

2. Some lemmas

In this section we present some lemmas which will be needed in the sequel. Let F and G be two non-constant meromorphic functions defined in $\mathbb{C}$. We shall denote by H the following function:

$$H=\left(\frac{F''}{F'}-\frac{2F'}{F-1}\right)-\left(\frac{G''}{G'}-\frac{2G'}{G-1}\right).$$

LEMMA 1. [16] *Let f be a transcendental meromorphic function, and let $P_n(f)$ be a differential polynomial in f of the form*

$$P_n(f)=a_nf^n(z)+a_{n-1}f^{n-1}(z)+\ldots+a_1f(z)+a_0.$$

where $a_n(\neq 0)$, $a_{n-1}\ldots a_1, a_0$ are complex numbers. Then

$$T(r,P_n(f))=nT(r,f)+O(1).$$

LEMMA 2. [21] *Let f be a nonconstant meromorphic function, and p,k be positive integers. Then*

$$N_p(r,0;f^{(k)}) \leqslant T(r,f^{(k)}) - T(r,f) + N_{p+k}(r,0;f) + S(r,f), \tag{1}$$

$$N_p(r,0;f^{(k)}) \leqslant k\overline{N}(r,\infty;f) + N_{p+k}(r,0;f) + S(r,f). \tag{2}$$

LEMMA 3. [9] *Let F and G be two non-constant meromorphic functions sharing $(1,2)$. Then one of the following cases holds:*

(i) $T(r) \leqslant N_2(r,0;F) + N_2(r,0;G) + N_2(r,\infty;F) + N_2(r,\infty;G) + S(r)$,

(ii) $F = G$,

(iii) $FG = 1$,

where $T(r)$ denotes the maximum of $T(r,F)$ and $T(r,G)$ and $S(r) = o\{T(r)\}$ as $r \to \infty$, possibly outside a set of finite linear measure.

LEMMA 4. [2] *Let F and G be two non-constant meromorphic functions sharing $(1,1)$ and $H \not\equiv 0$. Then*

$$\begin{aligned} T(r,F) \leqslant\ & N_2(r,0;F) + N_2(r,0;G) + N_2(r,\infty;F) + N_2(r,\infty;G) \\ & + \frac{1}{2}\overline{N}(r,0;F) + \frac{1}{2}\overline{N}(r,\infty;F) + S(r,F) + S(r,G) \end{aligned}$$

LEMMA 5. [2] *Let F and G be two non-constant meromorphic functions sharing $(1,0)$ and $H \not\equiv 0$. Then*

$$\begin{aligned} T(r,F) \leqslant\ & N_2(r,0;F) + N_2(r,0;G) + N_2(r,\infty;F) + N_2(r,\infty;G) \\ & + 2\overline{N}(r,0;F) + \overline{N}(r,0;G) + 2\overline{N}(r,\infty;F) \\ & + \overline{N}(r,\infty;G) + S(r,F) + S(r,G). \end{aligned}$$

LEMMA 6. [12] *If $N(r,0;f^{(k)} \mid f \neq 0)$ denote the counting function of those zeros of $f^{(k)}$ which are not the zeros of f, where a zero of $f^{(k)}$ is counted according to its multiplicity, then*

$$N(r,0;f^{(k)} \mid f \neq 0) \leqslant k\overline{N}(r,\infty;f) + N(r,0;f \mid < k) + k\overline{N}(r,0;f \mid \geqslant k) + S(r,f).$$

LEMMA 7. [8], [17] *Let f be a transcendental meromorphic function, and let $a_1(z), a_2(z)$ be two distinct meromorphic functions such that $T(r,a_i(z)) = S(r,f)$, $i = 1,2$. Then*

$$T(r,f) \leqslant \overline{N}(r,\infty;f) + \overline{N}(r,a_1;f) + \overline{N}(r,a_2;f) + S(r,f).$$

LEMMA 8. [21] *Let f and g be two non-entire transcendental meromorphic functions such that either the zeros and poles of f and g are of multiplicities atleast s, where s is a positive integer or they have no zeros and poles. Let n,k be two positive integers and let P be a nonconstant polynomial. If $n \geqslant \frac{2k+3}{s} + 2m$, then*

$$[f^n(af^m+b)]^{(k)}[g^n(ag^m+b)]^{(k)} \neq P^2,$$

where a,b are any two nonzero constants.

Proof. If possible, let

$$[f^n(af^m+b)]^{(k)}[g^n(ag^m+b)]^{(k)} = P^2. \tag{3}$$

Let $z_1 \notin (z : P(z) = 0)$ be a zero of f with multiplicity $p_1(\geqslant 1)$. Then it follows from (3) that z_1 is a pole of g. Suppose that z_1 is a pole of g of order $q_1(\geqslant 1)$. Then we have

$$np_1 - k = (n+m)q_1 + k \tag{4}$$

from (4) we obtain $np_1 = (n+m)s+2k$, and so

$$p_1 \geqslant (n+m)s-2k.$$

Let $z_2 \notin (z : P(z) = 0)$ be a zero of af^m+b with multiplicity $p_2(\geqslant k+1)$. Then from (3) it follows that z_2 is a pole of g. Suppose that z_2 is a pole of g of order $q_2(\geqslant 1)$. Then we have $p_2 - k = (n+m)q_2 + k$, i.e.,

$$p_2 \geqslant (n+m)s+2k.$$

Let $z_3 \notin (z : P(z) = 0)$ is a zero of af^m+b with multiplicity $p_3(\leqslant k)$, then from (3) it follows that z_3 may be a zero of $[f^n(af^m+b)]^{(k)}$ and if it happens then it will be a pole of g with multiplicity $(n+m)s+k$. Suppose that $z_4 \notin (z : P(z) = 0)$ be a pole of f. Then from (3) it is clear that z_4 is either a zero of $g^n(ag^m+b)$ or a zero of $[g^n(ag^m+b)]^{(k)}$. Therefore

$$\begin{aligned} \overline{N}(r,\infty;f) \leqslant{}& \overline{N}(r,0;g)+\overline{N}(r,0;ag^m+b \mid\leqslant k)+\overline{N}(r,0;ag^m+b \mid\geqslant k+1) \\ &+\overline{N}(r,0;h^{(k)} \mid h\neq 0)+S(r,g), \end{aligned} \tag{5}$$

where $\overline{N}(r,0;h^{(k)} \mid h \neq 0$ denotes the reduced counting function of those zeros of $h^{(k)}$ that are not the zeros of h and $h = g^n(ag^m+b)$.
By Lemma 6 we have

$$\begin{aligned} \overline{N}(r,0;h^{(k)} \mid h\neq 0) &\leqslant \frac{1}{(n+m)s+k}[N(r,0;h^{(k)} \mid h\neq 0] \\ &\leqslant \frac{1}{(n+m)s+k}[k\overline{N}(r,\infty;h)+N(r,0;h \mid< k)+k\overline{N}(r,0;h \mid\geqslant k)] \\ &\leqslant \frac{1}{(n+m)s+k}[k\overline{N}(r,\infty;h)+N_k(r,0;h)] \\ &\leqslant \frac{k}{(n+m)s+k}[\overline{N}(r,\infty;g)+\overline{N}(r,0;g)+\overline{N}(r,0;ag^m+b)] \\ &\leqslant \frac{k}{(n+m)s+k}[\overline{N}(r,\infty;g)+\overline{N}(r,0;g)+\overline{N}(r,0;ag^m+b \mid\leqslant k) \\ &\quad +\overline{N}(r,0;ag^m+b \mid\geqslant k+1)]. \end{aligned}$$

So from (5) we obtain

$$
\begin{aligned}
\overline{N}(r,\infty;f) &\leqslant \left(1+\frac{k}{(n+m)s+k}\right)[\overline{N}(r,0;g)+\overline{N}(r,0;ag^m+b\mid\leqslant k)\\
&\quad +\overline{N}(r,0;ag^m+b\mid\geqslant k+1)]+\frac{k}{(n+m)s+k}\overline{N}(r,\infty;g)+S(r,g)\\
&\leqslant \frac{(n+m)s+2k}{(n+m)s+k}\left[\frac{1}{(n+m)s-2k}+\frac{1}{(n+m)s+k}+\frac{1}{(n+m)s+2k}\right]T(r,g)\\
&\quad +\frac{k}{(n+m)s+k}T(r,g)+S(r,g)\\
&\leqslant \left[\frac{(n+m)s+2k}{[(n+m)s+k][(n+m)s-2k]}+\frac{(n+m)s+2k}{[(n+m)s+k]^2}\right.\\
&\quad \left.+\frac{k+1}{[(n+m)s+k][(n+m)s+2k]}\right]T(r,g)+S(r,g).
\end{aligned}
$$

Using the second fundamental theorem of Nevanlinna we get

$$
\begin{aligned}
T(r,f) &\leqslant \overline{N}(r,\infty;f)+\overline{N}(r,0;f)+\overline{N}(r,0;af^m+b)+S(r,f)\\
&\leqslant \overline{N}(r,\infty;f)+\overline{N}(r,0;f)+\overline{N}(r,0;af^m+b\mid\leqslant k)\\
&\quad +\overline{N}(r,0;af^m+b\mid\geqslant k+1)+S(r,g)\\
&\leqslant \left[\frac{(n+m)s+2k}{[(n+m)s+k][(n+m)s-2k]}+\frac{(n+m)s+2k}{[(n+m)s+k]^2}\right.\\
&\quad \left.+\frac{k+1}{[(n+m)s+k][(n+m)s+2k]}\right]T(r,g)\\
&\quad +\left[\frac{1}{(n+m)s+2k}+\frac{1}{(n+m)s+k}+\frac{1}{(n+m)s+2k}\right]T(r,f)+S(r,f)+S(r,g).
\end{aligned}\tag{6}
$$

Similarly

$$
\begin{aligned}
T(r,g) &\leqslant \left[\frac{(n+m)s+2k}{[(n+m)s+k][(n+m)s-2k]}+\frac{(n+m)s+2k}{[(n+m)s+k]^2}\right.\\
&\quad \left.+\frac{k+1}{[(n+m)s+k][(n+m)s+2k]}\right]T(r,f)\\
&\quad +\left[\frac{1}{(n+m)s+2k}+\frac{1}{(n+m)s+k}+\frac{1}{(n+m)s+2k}\right]T(r,g)+S(r,f)+S(r,g).
\end{aligned}\tag{7}
$$

Adding (6) and (7) we obtain

$$[T(r,f)+T(r,g)] \leqslant \Big[\frac{2[(n+m)s+2k]}{[(n+m)s+k][(n+m)s-2k]}+\frac{2[(n+m)s+2k]}{[(n+m)s+k]^2} + \frac{2[k+1]}{[(n+m)s+k][(n+m)s+2k]}+\frac{2}{(n+m)s+2k} + \frac{2}{(n+m)s+k}+\frac{2}{(n+m)s+2k}][T(r,f)+T(r,g)\Big] + S(r,f)+S(r,g).$$

which is a contradiction. Thus Lemma 8 is proved. □

LEMMA 9. *Let f and g be two transcendental entire function, let n,k are any two positive integers and let P be a non-constant polynomial. Then*

$$[f^n(af^m+b)]^{(k)}[g^n(ag^m+b)]^{(k)} \neq P^2,$$

where a,b are any two nonzero constants.

Proof. Suppose that

$$[f^n(af^m+b)]^{(k)}[g^n(ag^m+b)]^{(k)} = P^2.$$

Let z_0 be a zero of f with multiplicity p. Then clearly z_0 is a zero of P. Since P is a polynomial, f has a finite number of zeros. So we put $f(z)=P_1e^{\alpha}$ where α is a non-constant entire function and P_1 is a polynomial. Now

$$(af^{n+m})^{(k)} = t_1(\alpha',\alpha'',\ldots,\alpha^{(k)},P_1)e^{(n+m)\alpha}, \tag{8}$$

$$(bf^n)^{(k)} = t_0(\alpha',\alpha'',\ldots,\alpha^{(k)},P_1)e^{n\alpha}, \tag{9}$$

where $t_i(\alpha',\alpha'',\ldots,\alpha^{(k)},P_1)$ $(i=0,1,2,\ldots m)$ are differential polynomials in α',α'', $\ldots,\alpha^{(k)}$ with coefficients which are rational functions in P_1 or its derivatives. Obviously

$$t_i(\alpha',\alpha'',\ldots,\alpha^{(k)},P_1)\neq 0$$

for $i=0,1,2,\ldots m$ and

$$[f^n(af^m+b)]^k \neq 0.$$

From (8) and (9) we have

$$t_1(\alpha',\alpha'',\ldots,\alpha^{(k)},P_1)e^{\alpha(z)}+t_0(\alpha',\alpha'',\ldots,\alpha^{(k)},P_1)\neq 0. \tag{10}$$

Since $\alpha(z)$ is an entire function, we obtain $T(r,\alpha^{(j)})=S(r,f)$ for $j=1,2,\ldots,k$. Thus $T(r,t_i)=S(r,f)$ for $i=0,1,2\ldots,m$. So from (10), Lemma 1 and Lemma 7 we obtain

$$\begin{aligned} mT(r,f) &= T(r,t_me^{m\alpha}+t_{m-1}e^{(m-1)\alpha}+\ldots+t_1e^{\alpha})+S(r,f)\\ &\leqslant \overline{N}(r.0;t_me^{m\alpha}+t_{m-1}e^{m-1}\alpha+\ldots+t_1e^{\alpha})+\overline{N}(r,0;t_me^{m\alpha}+\ldots+t_0)+S(r,f)\\ &\leqslant \frac{(m-1)}{s}T(r,f)+S(r,f), \end{aligned}$$

which is a contradiction. This completes the proof of the lemma. □

LEMMA 10. *Let f and g be two transcendental meromorphic (entire) functions such that either the zeros and poles of f and g are of multiplicities atleast s, where s is a positive integer or they have no zeros and poles and let $n(\geqslant 1)$, $k(\geqslant 1)$, be two integers. Suppose that $F = \frac{[f^n(af^m+b)]^{(k)}}{P(z)}$ and $G = \frac{[g^n(ag^m+b)]^{(k)}}{P(z)}$. If there exists two nonzero constants c_1 and c_2 such that $\overline{N}(r,c_1;F) = \overline{N}(r,0;G)$ and $\overline{N}(r,c_2;G) = \overline{N}(r,0;F)$, then $n \leqslant \frac{3k+3}{s}+m$ $(n \leqslant \frac{2k+2}{s}+m)$.*

Proof. We prove the theorem for two transcendental meromorphic functions. By the second fundamental theorem of Nevanlinna we have

$$\begin{aligned} T(r,F) &\leqslant \overline{N}(r,0;F)+\overline{N}(r,\infty;F)+\overline{N}(r,c_1;F)+S(r,F) \\ &\leqslant \overline{N}(r,0;F)+\overline{N}(r,0;G)+\overline{N}(r,\infty;F)+S(r,F). \end{aligned} \tag{11}$$

By (1), (2), (11) and Lemma 1 we obtain

$$\begin{aligned} (n+m)T(r,f) &\leqslant T(r,F)-\overline{N}(r,0;F)+N_{k+1}(r,0;f^n(af^m+b))+O\{\log r\}+S(r,f) \\ &\leqslant \overline{N}(r,0;G)+N_{k+1}(r,0;f^n(af^m+b))+\overline{N}(r,\infty;f)+O\{\log r\}+S(r,f) \\ &\leqslant N_{k+1}(r,0;f^n(af^m+b))+N_{k+1}(r,0;g^n(ag^m+b))+\overline{N}(r,\infty;f) \\ &\quad +k\overline{N}(r,\infty;g)+O\{\log r\}+S(r,f)+S(r,g) \\ &\leqslant \left(\frac{k+2}{s}+m\right)T(r,f)+\left(\frac{2k+1}{s}+m\right)T(r,g)+O\{\log r\}+S(r,f)+S(r,g). \end{aligned} \tag{12}$$

Similarly we obtain

$$(n+m)T(r,g) \leqslant \left(\frac{k+2}{s}+m\right)T(r,g)+\left(\frac{2k+1}{s}+m\right)T(r,f)+O\{\log r\}+S(r,f)+S(r,g). \tag{13}$$

Combining (12), (13) and noting that $O\{\log r\} = O\{T(r,f)\}$ and $O\{\log r\} = O\{T(r,g)\}$ we get

$$\left(n-\frac{3k+3}{s}-m\right)\{T(r,f)+T(r,g)\} \leqslant S(r,f)+S(r,g)$$

which gives $n \leqslant \frac{3k+3}{s}+m$. This completes the proof of the Lemma 10. □

LEMMA 11. *Let f and g be two nonconstant meromorphic functions such that*

$$\Theta(\infty,f)+\Theta(\infty,g) > \frac{4}{n},$$

where $n(\geqslant 3)$ is an integer. Then

$$f^n(af+b) = g^n(ag+b)$$

implies $f = g$, where a,b are two nonzero constants.

Proof. We omit the proof since it can be carried out in the line of Lemma 6 [11]. □

3. Proof of Theorem 1

Proof. Let $F(z)$ and $G(z)$ be given as in Lemma 10. Then $F(z), G(z)$ are non-entire transcendental meromorphic functions that share $(1,l)$ except the zeros of the polynomial $P(z)$. So from (1) we obtain

$$\begin{aligned} N_2(r,0;F) &\leqslant N_2(r,0;[f^n(af^m+b)]^{(k)} + S(r,f) \\ &\leqslant T(r,[f^n(af^m+b)]^{(k)} - (n+m)T(r,f) + N_{k+2}(r,0;f^n(af^m+b)) + S(r,f) \\ &\leqslant T(r,F) - (n+m)T(r,f) + N_{k+2}(r,0;f^n(af^m+b)) + O\{\log r\} + S(r,f). \end{aligned} \tag{14}$$

Again by (2) we have

$$N_2(r,0;G) \leqslant k\overline{N}(r,\infty;f) + N_{k+2}(r,0;g^n(ag^m+b)) + S(r,g). \tag{15}$$

From (14) we get

$$(n+m)T(r,f) \leqslant T(r,F) + N_{k+2}(r,0;f^n(af^m+b)) - N_2(r,0;F) + O\{\log r\} + S(r,f). \tag{16}$$

Now, we consider the following three cases.

Case 1. Let $l \geqslant 2$. Let (i) of Lemma 3 holds. Then using (15) we obtain from (16)

$$\begin{aligned} (n+m)T(r,f) &\leqslant N_2(r,0;G) + N_2(r,\infty;F) + N_2(r,\infty;G) + N_{k+2}(r,0;f^n(af^m+b)) \\ &\quad + O\{\log r\} + S(r,f) + S(r,g) \\ &\leqslant N_{k+2}(r,0;f^n(af^m+b)) + N_{k+2}(r,0;g^n(ag^m+b)) + 2\overline{N}(r,\infty;f) \\ &\quad + (k+2)\overline{N}(r,\infty;g) + O\{\log r\} + S(r,f) + S(r,g) \\ &\leqslant (k+m+2)\{T(r,f)+T(r,g)\} + 2\overline{N}(r,\infty;f) + (k+2)\overline{N}(r,\infty;g) \\ &\quad + O\{\log r\} + S(r,f) + S(r,g) \\ &\leqslant \left[\left(\frac{k+4}{s}+m\right) - 2\Theta(\infty;f) + \varepsilon\right]T(r,f) \\ &\quad + \left[\left(\frac{2k+4}{s}+m\right) - \left(\frac{k+2}{s}\right)\Theta(\infty,g) + \varepsilon\right]T(r,g) + S(r,f) + S(r,g) \\ &\leqslant \left[\left(\frac{3k+8}{s}+2m\right) - 2\Theta(\infty,f) - 2\Theta(\infty,g)\right. \\ &\quad \left. - k\min\{\Theta(\infty,f),\Theta(\infty,g)\} + 2\varepsilon\right]T(r) + S(r). \end{aligned} \tag{17}$$

In a similar way we can obtain

$$\begin{aligned} (n+m)T(r,g) \leqslant \Big[\Big(\frac{3k+8}{s}+2m\Big) - 2\Theta(\infty,f) - 2\Theta(\infty,g) \\ - k\min\{\Theta(\infty,f),\Theta(\infty,g)\} + 2\varepsilon\Big]T(r) + S(r). \end{aligned} \tag{18}$$

From (17) and (18) we obtain

$$\left[n - \left(\frac{3k+8}{s}\right) - m + 2\Theta(\infty,f) + 2\Theta(\infty,g) + k\min\{\Theta(\infty,f)\Theta(\infty,g)\} - 2\varepsilon\right]T(r) \leqslant S(r)$$

contradicting with the fact that $n \geqslant \frac{3k+8}{s}+m$, for $m=1$ we have $\Theta(\infty,f)+\Theta(\infty,g)>\frac{4}{n}$ and $\varepsilon>0$ be arbitrary. So by Lemma 3 either $FG\equiv 1$ or $F=G$. Let $FG=1$. Then

$$[f^n(af^m+b)]^{(k)}[g^n(ag^m+b)]^{(k)}=P^2,$$

a contradiction by Lemma 8. So we have $F=G$. That is

$$[f^n(af^m+b)]^{(k)}=[g^n(ag^m+b)]^{(k)}.$$

Integrating we get

$$[f^n(af^m+b)]^{(k-1)}=[g^n(ag^m+b)]^{(k-1)}+C_{k-1},$$

where C_{k-1} is a constant. If $C_{k-1}\neq 0$, from Lemma 10 we obtain $n\leqslant \frac{3k+3}{s}+m$, a contradiction. Hence $C_{k-1}=0$. Repeating k times and substituting $m=1$, we obtain

$$f^n(af^m+b)=g^n(ag^m+b). \tag{19}$$

Now the result follows from Lemma 11.

Case 2. Let $l=1$ and $H\not\equiv 0$. Using Lemma 4 and (15) we obtain from (16),

$$\begin{aligned}(n+m)T(r,f) &\leqslant N_2(r,0;G)+N_2(r,\infty;F)+N_2(r,\infty;G)+\frac{1}{2}\overline{N}(r,0;F)\\ &\quad+\frac{1}{2}\overline{N}(r,\infty;F)+N_{k+2}(r,0;f^n(af^m+b))+O\{\log r\}+S(r,f)+S(r,g)\\ &\leqslant N_{k+2}(r,0;f^n(af^m+b))+N_{k+2}(r,0;g^n(ag^m+b))\\ &\quad+\frac{1}{2}N_{k+1}(r,0;f^n(af^m+b))+\frac{k+5}{2}\overline{N}(r,\infty;f)\\ &\quad+(k+2)\overline{N}(r,\infty;g)+O\{\log r\}+S(r,f)+S(r,g)\\ &\leqslant (k+m+2)\{T(r,f)+T(r,g)\}+\frac{k+m+1}{2}T(r,f)+\frac{k+5}{2}\overline{N}(r,\infty;f)\\ &\quad+(k+2)\overline{N}(r,\infty;g)+O\{\log r\}+S(r,f)+S(r,g)\\ &\leqslant \left[\frac{2k+5}{s}+\frac{3m}{2}-\left(\frac{k}{2}+3\right)\Theta(\infty,f)-\frac{1}{2}\Theta(\infty,f)+\varepsilon\right]T(r,f)\\ &\quad+\left[\left(\frac{2k+5}{s}+m\right)-\left(\frac{k}{2}+2\right)\Theta(\infty,g)-\frac{k}{2}\Theta(\infty,f)+\varepsilon\right]T(r,g)\\ &\quad+O\{\log r\}+S(r,f)+S(r,g)\\ &\leqslant \left[\frac{4k+9}{s}+\frac{5m}{2}-\left(\frac{k+5}{2}\right)(\Theta(\infty,f)+\Theta(\infty,g))+2\varepsilon\right]T(r)+S(r).\end{aligned} \tag{20}$$

Similarly

$$(n+m)T(r,g)\leqslant \left[\frac{4k+9}{s}+\frac{5m}{2}-\frac{k+5}{2}(\Theta(\infty,f)+\Theta(\infty,g))+2\varepsilon\right]T(r)+S(r). \tag{21}$$

combining (20) and (21) we obtain

$$\left[n-\frac{4k+9}{s}-\frac{5m}{2}+m+\frac{k+5}{2}(\Theta(\infty,f)+\Theta(\infty,g))+2\varepsilon\right]T(r)\leqslant S(r),$$

a contradiction. Since $n \geqslant \frac{4k+9}{s} + \frac{3m}{2}$, for $m = 1$ we have $\Theta(\infty, f) + \Theta(\infty, g) > \frac{4}{n}$ and $\varepsilon > 0$ be arbitrary. We now assume that $H \equiv 0$. That is

$$\left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right) = 0.$$

Integrating both sides of the above equality twice we get

$$\frac{1}{F-1} = \frac{A}{G-1} + B, \tag{22}$$

where $A(\neq 0)$ and B are constants. From (22) it is clear that F, G share the value 1 CM and so they share 1 with weight two. Hence we have $n \geqslant \frac{3k+8}{s} + m$. Now we discuss the following three subcases.

Subcase 1. Let $B \neq 0$ and $A = B$. Then from (22) we get

$$\frac{1}{F-1} = \frac{BG}{G-1}. \tag{23}$$

If $B = -1$, then from (23) we obtain

$$FG = 1,$$

a contradiction by Lemma 8.

If $B \neq -1$, from (23), we have $\frac{1}{F} = \frac{BG}{(1+B)G-1}$ and so $\overline{N}(r, \frac{1}{1+B}; G) = \overline{N}(r, 0; F)$. Now from the second fundamental theorem of Nevanlinna, we get

$$\begin{aligned} T(r,G) &\leqslant \overline{N}(r,0;G) + \overline{N}\left(r, \frac{1}{1+B}; G\right) + \overline{N}(r,\infty;G) + S(r,G) \\ &\leqslant \overline{N}(r,0;F) + \overline{N}(r,0;G) + \overline{N}(r,\infty;G) + S(r,G). \end{aligned}$$

Using (1) and (2) we obtain from above inequality

$$\begin{aligned} T(r,G) \leqslant\ & N_{k+1}(r,0;f^n(af^m+b)) + k\overline{N}(r,\infty;f) + T(r,G) \\ & + N_{k+1}(r,0;g^n(ag^m+b)) - (n+m)T(r,g) + \overline{N}(r,\infty;g) \\ & + O\{\log r\} + S(r,g). \end{aligned}$$

Hence

$$(n+m)T(r,g) \leqslant \left(\frac{2k+1}{s} + m\right)T(r,f) + \left(\frac{k+2}{a} + m\right)T(r,g) + S(r,g).$$

Thus we obtain

$$\left(n - \frac{3k+3}{s} - 2m + m\right)\{T(r,f) + T(r,g)\} \leqslant S(r,f) + S(r,g),$$

$$\left(n - \frac{3k+3}{s} - m\right)\{T(r,f) + T(r,g)\} \leqslant S(r,f) + S(r,g),$$

which contradicts as $n \geqslant \frac{3k+3}{s} + m$.

Subcase 2. Let $B \neq 0$ and $A \neq B$. Then from (22) we get $F = \frac{(B+1)G-(B-A+1)}{BG+(A-B)}$ and so $\overline{N}(r, \frac{B-A+1}{B+1});\ G = \overline{N}(r,0;F)$. Proceeding as in Subcase 1 we obtain a contradiction.

Subcase 3. Let $B=0$ and $A\neq B$. Then from (22) $\overline{N}(r,\frac{A-1}{A};F)=\overline{N}(r,0;G)$ and $\overline{N}(r,1-A;G)=\overline{N}(r,0;F)$. So by Lemma 10 we have $n\leqslant\frac{3k+3}{s}+m$, a contradiction. Thus $A=1$ and hence $F\equiv G$. Now using the same technique as used in case 1 we can obtain (19) which by Lemma 11 gives $f=g$.

Case 3. Let $l=0$ and $H\not\equiv 0$, using Lemma 5 and (15) we obtain from (16)

$$\begin{aligned}(n+m)T(r,f)\leqslant\; & N_2(r,0;G)+N_2(r,\infty;F)+N_2(r,\infty;G)+2\overline{N}(r,0;F)\\ & +\overline{N}(r,0;G)+N_{k+2}(r,0;f^nP(f))+2\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)\\ & +O\{\log r\}+S(r,f)+S(r,g)\\ \leqslant\; & N_{k+2}(r,0;f^nP(f))+N_{k+2}(r,0;g^nP(g))+2N_{k+2}(r,0;f^nP(f))\\ & +N_{k+1}(r,0;g^nP(g))+(2k+4)\overline{N}(r,\infty;f)+(2k+3)\overline{N}(r,\infty;g)\\ & +O\{\log r\}+S(r,f)+S(r,g)\\ \leqslant\; & \left[\left(\frac{5k+8}{s}+3m\right)-(2k+4)\Theta(\infty;f)-\varepsilon\right]T(r,f)\\ & +\left[\left(\frac{4k+6}{s}+2m\right)-(2k+3)\Theta(\infty;g)-\varepsilon\right]T(r,g)+O\{\log r\}\\ & +S(r,f)+S(r,g)\\ & +\left[\left(\frac{9k+14}{s}+5m\right)-(2k+3)[\Theta(\infty;f)+\Theta(\infty,g)]\right.\\ & \left.-\min\{\Theta(\infty,f)\Theta(\infty;g)\}+2\varepsilon\right]T(r)+S(r).\end{aligned}\tag{24}$$

Similarly

$$\begin{aligned}(n+m)T(r,g)\leqslant & \left[\left(\frac{9k+14}{s}+5m\right)-(2k+3)[\Theta(\infty;f)+\Theta(\infty,g)]\right.\\ & \left.-\min\{\Theta(\infty,f)\Theta(\infty;g)\}+2\varepsilon\right]T(r)+S(r).\end{aligned}\tag{25}$$

From (24) and (25) we get

$$\begin{aligned}&\left[\left(n-\frac{9k+14}{s}-5m+m\right)+(2k+3)(\Theta(\infty,f)+\Theta(\infty;g))\right.\\ &\qquad\left.+\min\{\Theta(\infty;f)\Theta(\infty;g)\}-2\varepsilon\right]T(r)\leqslant S(r),\end{aligned}$$

contradicts with the facts that $n\geqslant\frac{9k+14}{s}+4m$, for $m=1$ we have $\Theta(\infty,f)+\Theta(\infty,g)>\frac{4}{n}$ and $\varepsilon>0$ be arbitrary. we now assume that $H\equiv 0$. Then proceeding in a similar manner as in case 2 we obtain $f=g$. This completes the proof of the Theorem 1. □

4. Proof of Theorem 2

Proof. Noting that $\overline{N}(r,\infty;f)=0$, $\overline{N}(r,\infty;g)=0$ and using Lemma 9 instead of Lemma 8 and proceeding in the like manner as the proof of Theorem 1 we obtain the result of the Theorem 2. □

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(Received August 18, 2016)

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