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GENERALIZED WEIGHTED COMPOSITION OPERATORS FROM WEIGHTED BERGMAN SPACES INTO ZYGMUND–TYPE SPACES

XIANGLING ZHU

Abstract. The boundedness and the compactness of generalized weighted composition operators from weighted Bergman spaces into Zygmund-type spaces are investigated in this paper. Moreover, we give some estimates for the essential norm of these operators.

1. Introduction

Let $H(\mathbb{D})$ denote the space of analytic functions on the open unit disk $\mathbb{D}$. Let $S(\mathbb{D})$ denote the set of all analytic self-maps of $\mathbb{D}$. The composition operator C_φ with the symbol $\varphi \in S(\mathbb{D})$ is defined by $(C_\varphi f)(z) = f(\varphi(z))$, $f \in H(\mathbb{D})$. See [3, 28] for more information about the theory of composition operators. We denote the set of nonnegative integers by $\mathbb{N}_0$. Let $\varphi \in S(\mathbb{D})$, $u \in H(\mathbb{D})$ and $n \in \mathbb{N}_0$. The generalized weighted composition operator $D^n_{\varphi,u}$ is defined as follows (see [30]).

$$(D^n_{\varphi,u} f)(z) = u(z) \cdot f^{(n)}(\varphi(z)), \quad f \in H(\mathbb{D}), \quad z \in \mathbb{D}.$$

If $n = 0$, then $D^n_{\varphi,u}$ is the weighted composition operator, which is frequently denoted by uC_φ in the literature. When $n = 0$ and $u(z) = 1$, then $D^n_{\varphi,u}$ is just the composition operator C_φ. If $n = 1$, $u(z) = \varphi'(z)$, then $D^n_{\varphi,u} = DC_\varphi$. When $u(z) = 1$, $D^n_{\varphi,u} = C_\varphi D^n$. See [5, 11, 12, 15, 16, 18, 20, 21, 22, 25] for the study of DC_φ and $C_\varphi D^n$. See [6, 9, 17, 23, 24, 26, 27, 29, 30, 31, 32] and the references therein for the study of the operator $D^n_{\varphi,u}$.

Let $v : \mathbb{D} \to \mathbb{R}_+$ be a continuous, strictly positive and bounded function, the, so called, weight. The weight v is called radial, if $v(z) = v(|z|)$ for all $z \in \mathbb{D}$. The weighted space, denoted by H^∞_v, is the set of all $f \in H(\mathbb{D})$ such that

$$\|f\|_v = \sup_{z\in\mathbb{D}} v(z)|f(z)| < \infty.$$

Under the norm $\|\cdot\|_v$, H^∞_v is a Banach space. The associated weight $\widetilde{v}$ of v is defined by

$$\widetilde{v}(z) = \frac{1}{\sup\{|f(z)| : f \in H^\infty_v, \|f\|_v \leqslant 1\}}, \quad z \in \mathbb{D}.$$

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When $v(z) = v_\alpha(z) = (1-|z|^2)^\alpha (0 < \alpha < \infty)$, it is easy to check that $\widetilde{v}_\alpha(z) = v_\alpha(z)$. In this case, we denote H_v^∞ by $H_{v_\alpha}^\infty$.

Let μ be a weight. The Zygmund-type space, denoted by $\mathscr{Z}_\mu$, is the set of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{\mathscr{Z}_\mu} = |f(0)| + |f'(0)| + \sup_{z\in\mathbb{D}} \mu(z)|f''(z)| < \infty.$$

It is also easy to check that $\mathscr{Z}_\mu$ is a Banach space under the norm $\|\cdot\|_{\mathscr{Z}_\mu}$. When $\mu(z) = 1-|z|^2$, $\mathscr{Z}_\mu = \mathscr{Z}$ is the Zygmund space. See [1, 10] for more information on the Zygmund space on the unit disk. See [1, 4, 13, 14] for the study of composition operators on the Zygmund space.

For $0 < p < \infty$ and $\gamma > -1$, the weighted Bergman space, denoted by A_γ^p, is the set of all functions $f \in H(\mathbb{D})$ satisfying

$$\|f\|_{A_\gamma^p}^p = (\gamma+1)\int_{\mathbb{D}} |f(z)|^p (1-|z|^2)^\gamma dA(z) < \infty,$$

where dA is the normalized Lebesgue area measure in $\mathbb{D}$ such that $A(\mathbb{D}) = 1$. When $p = 2$, A_γ^2 is a Hilbert space.

In [24] (see also [27]), Stević has studied the boundedness and compactness of the operator $D_{\varphi,u}^n : \mathscr{B} \to \mathscr{Z}$. Here $\mathscr{B}$ is the classical Bloch space. In [9], Li and Fu gave a new characterization for the boundedness and compactness of the operator $D_{\varphi,u}^n : \mathscr{B} \to \mathscr{Z}$. In [6], motivated by [9, 24], the authors characterized the essential norm of the operator $D_{\varphi,u}^n : \mathscr{B} \to \mathscr{Z}$. In [2], the authors studied the boundedness, compactness and essential norms of weighted composition operators from some Hilbert function spaces (including A_γ^2) into $\mathscr{Z}_\mu$. Recall that the essential norm of a bounded linear operator $T: X \to Y$ is defined as follows.

$$\|T\|_{e,X\to Y} = \inf\{\|T-K\|_{X\to Y} : K \text{ is compact }\},$$

where X and Y are Banach spaces and $\|\cdot\|_{X\to Y}$ is the operator norm.

In [23], Stević studied the operator $D_{\varphi,u}^n$ from mixed-norm spaces to the nth weighted-type space. Among others, he proved the following result (in fact, he proved a more general result, including the one).

THEOREM A. Let μ be radial, non-increasing weight tending to zero at the boundary of $\mathbb{D}$. Let $1 < p < \infty$, $-1 < \gamma < \infty$, $u \in H(\mathbb{D})$, $\varphi \in S(\mathbb{D})$, and $n \in \mathbb{N}_0$. Then the operator $D_{\varphi,u}^n : A_\gamma^p \to \mathscr{Z}_\mu$ is bounded if and only if

$$M_1 := \sup_{z\in\mathbb{D}} \frac{\mu(z)|u''(z)|}{(1-|\varphi(z)|^2)^{\frac{2+\gamma}{p}+n}} < \infty, \tag{1}$$

$$M_2 := \sup_{z\in\mathbb{D}} \frac{\mu(z)|2u'(z)\varphi'(z)+u(z)\varphi''(z)|}{(1-|\varphi(z)|^2)^{\frac{2+\gamma+p}{p}+n}} < \infty \tag{2}$$

and

$$M_3 := \sup_{z\in\mathbb{D}} \frac{\mu(z)|u(z)||\varphi'(z)|^2}{(1-|\varphi(z)|^2)^{\frac{2+\gamma+2p}{p}+n}} < \infty. \tag{3}$$

Motivated by the above observations, in this work we give a new characterization for the boundedness of the operator $D^n_{\varphi,u} : A^p_\gamma \to \mathscr{Z}_\mu$. Moreover, we give some characterizations for the essential norm of the operator $D^n_{\varphi,u} : A^p_\gamma \to \mathscr{Z}_\mu$.

Throughout the paper, we denote by C a positive constant which may differ from one occurrence to the next. In addition, we say that $A \lesssim B$ if there exists a constant C such that $A \leqslant CB$. The symbol $A \approx B$ means that $A \lesssim B \lesssim A$.

2. Main results and proofs

In this section we formulate our main results in this paper. For this purpose, we need some lemmas as follows.

LEMMA 1. [28] *Assume that* $0<p<\infty$, $-1<\gamma<\infty$, $n\in\mathbb{N}_0$ *and* $f\in A^p_\gamma$. *Then there is a positive constant* C *independent of* f *such that*

$$|f^{(n)}(z)| \leqslant C\frac{\|f\|_{A^p_\gamma}}{(1-|z|^2)^{\frac{2+\gamma}{p}+n}}.$$

LEMMA 2. [8] *For* $\alpha>0$, *we have* $\lim_{k\to\infty} k^\alpha \|z^{k-1}\|_{v_\alpha} = (\frac{2\alpha}{e})^\alpha$.

LEMMA 3. [19] *Let* v *and* w *be radial, non-increasing weights tending to zero at the boundary of* $\mathbb{D}$. *Then the following statements hold.*

(a) The weighted composition operator $uC_\varphi : H^\infty_v \to H^\infty_w$ *is bounded if and only if*

$$\sup_{z\in\mathbb{D}} \frac{w(z)}{\widetilde{v}(\varphi(z))}|u(z)| < \infty.$$

(b) Suppose $uC_\varphi : H^\infty_v \to H^\infty_w$ *is bounded. Then*

$$\|uC_\varphi\|_{e,H^\infty_v\to H^\infty_w} = \lim_{s\to 1}\sup_{|\varphi(z)|>s} \frac{w(z)}{\widetilde{v}(\varphi(z))}|u(z)|.$$

LEMMA 4. [7] *Let* v *and* w *be radial, non-increasing weights tending to zero at the boundary of* $\mathbb{D}$. *Then the following statements hold.*

(a) $uC_\varphi : H^\infty_v \to H^\infty_w$ *is bounded if and only if* $\sup_{k\geqslant 0} \frac{\|u\varphi^k\|_w}{\|z^k\|_v} < \infty$, *with the operator norm comparable to the supermum.*

(b) Suppose $uC_\varphi : H^\infty_v \to H^\infty_w$ *is bounded. Then*

$$\|uC_\varphi\|_{e,H^\infty_v\to H^\infty_w} = \limsup_{k\to\infty} \frac{\|u\varphi^k\|_w}{\|z^k\|_v}.$$

The following lemma is proved similar to Lemma 2.3 in [2].

LEMMA 5. *Let μ be radial, non-increasing weight tending to zero at the boundary of $\mathbb{D}$. Let $1 < p < \infty$, $-1 < \gamma < \infty$. Suppose that linear operator $T : A^p_\gamma \to \mathscr{Z}_\mu$ is bounded. Then T is compact if and only if whenever $\{f_n\}$ is bounded in A^p_γ and $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$, $\lim_{n\to\infty} \|Tf_n\|_{\mathscr{Z}_\mu} = 0$.*

THEOREM 1. *Let μ be radial, non-increasing weight tending to zero at the boundary of $\mathbb{D}$. Let $1 < p < \infty$, $-1 < \gamma < \infty$, $u \in H(\mathbb{D})$, $\varphi \in S(\mathbb{D})$, and $n \in \mathbb{N}_0$. Then the operator $D^n_{\varphi,u} : A^p_\gamma \to \mathscr{Z}_\mu$ is bounded if and only if*

$$\sup_{j\geqslant 1} j^{\frac{2+\gamma+p}{p}+n} \|(2u'\varphi' + u\varphi'')\varphi^{j-1}\|_\mu < \infty,$$

$$\sup_{j\geqslant 1} j^{\frac{2+\gamma}{p}+n} \|u''\varphi^{j-1}\|_\mu < \infty \quad \textit{and} \quad \sup_{j\geqslant 1} j^{\frac{2+\gamma+2p}{p}+n} \|u\varphi'^2\varphi^{j-1}\|_\mu < \infty.$$

Proof. By Lemma 3, the inequality in (2) is equivalent to the weighted composition operator $(2u'\varphi' + u\varphi'')C_\varphi : H^\infty_{v_{\frac{2+\gamma+p}{p}+n}} \to H^\infty_\mu$ is bounded, which is equivalent to

$$\sup_{j\geqslant 1} \frac{\|(2u'\varphi' + u\varphi'')\varphi^{j-1}\|_\mu}{\|z^{j-1}\|_{v_{\frac{2+\gamma+p}{p}+n}}} < \infty$$

by Lemma 4. Hence, by Lemma 2

$$\sup_{j\geqslant 1} j^{\frac{2+\gamma+p}{p}+n} \|(2u'\varphi' + u\varphi'')\varphi^{j-1}\|_\mu \approx \sup_{j\geqslant 1} \frac{j^{\frac{2+\gamma+p}{p}+n}\|(2u'\varphi' + u\varphi'')\varphi^{j-1}\|_\mu}{j^{\frac{2+\gamma+p}{p}+n}\|z^{j-1}\|_{v_{\frac{2+\gamma+p}{p}+n}}} < \infty.$$

Similarly, the inequality in (1) is equivalent to

$$\sup_{j\geqslant 1} j^{\frac{2+\gamma}{p}+n} \|u''\varphi^{j-1}\|_\mu \approx \sup_{j\geqslant 1} \frac{j^{\frac{2+\gamma}{p}+n}\|u''\varphi^{j-1}\|_\mu}{j^{\frac{2+\gamma}{p}+n}\|z^{j-1}\|_{v_{\frac{2+\gamma}{p}+n}}} = \sup_{j\geqslant 1} \frac{\|u''\varphi^{j-1}\|_\mu}{\|z^{j-1}\|_{v_{\frac{2+\gamma}{p}+n}}} < \infty$$

and the inequality in (3) is equivalent to

$$\sup_{j\geqslant 1} j^{\frac{2+\gamma+2p}{p}+n} \|u\varphi'^2\varphi^{j-1}\|_\mu \approx \sup_{j\geqslant 1} \frac{j^{\frac{2+\gamma+2p}{p}+n}\|u\varphi'^2\varphi^{j-1}\|_\mu}{j^{\frac{2+\gamma+2p}{p}+n}\|z^{j-1}\|_{v_{\frac{2+\gamma+2p}{p}+n}}}$$

$$= \sup_{j\geqslant 1} \frac{\|u\varphi'^2\varphi^{j-1}\|_\mu}{\|z^{j-1}\|_{v_{\frac{2+\gamma+2p}{p}+n}}} < \infty.$$

This completes the proof of Theorem 1.

THEOREM 2. *Let μ be radial, non-increasing weight tending to zero at the boundary of $\mathbb{D}$. Let $1 < p < \infty$, $-1 < \gamma < \infty$, $u \in H(\mathbb{D})$, $\varphi \in S(\mathbb{D})$, and $n \in \mathbb{N}_0$ such that $D^n_{\varphi,u} : A^p_\gamma \to \mathscr{Z}_\mu$ is bounded. Then*

$$\|D^n_{\varphi,u}\|_{e,A^p_\gamma \to \mathscr{Z}_\mu} \approx \max\Big\{E, F, G\Big\},$$

where

$$E := \limsup_{|\varphi(z)|\to 1} \frac{\mu(z)|2u'(z)\varphi'(z) + u(z)\varphi''(z)|}{(1-|\varphi(z)|^2)^{\frac{2+\gamma+p}{p}+n}},$$

$$F := \limsup_{|\varphi(z)|\to 1} \frac{\mu(z)|u''(z)|}{(1-|\varphi(z)|^2)^{\frac{2+\gamma}{p}+n}}, \qquad G := \limsup_{|\varphi(z)|\to 1} \frac{\mu(z)|u(z)||\varphi'(z)|^2}{(1-|\varphi(z)|^2)^{\frac{2+\gamma+2p}{p}+n}}.$$

Proof. First we prove that $\|D^n_{\varphi,u}\|_{e,A^p_\gamma \to \mathscr{Z}_\mu} \lesssim \max\{E, F, G\}$. For $r \in [0,1)$, set $K_r : H(\mathbb{D}) \to H(\mathbb{D})$ by $(K_r f)(z) = f_r(z) = f(rz)$, $f \in H(\mathbb{D})$. It is clear that $f_r - f \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $r \to 1$. Moreover, the operator K_r is compact on A^p_γ and $\|K_r\|_{A^p_\gamma \to A^p_\gamma} \leqslant 1$. Let $\{r_j\} \subset (0,1)$ be a sequence such that $r_j \to 1$ as $j \to \infty$. Then for each positive integer j, the operator $D^n_{\varphi,u} K_{r_j} : A^p_\gamma \to \mathscr{Z}_\mu$ is compact. Hence

$$\|D^n_{\varphi,u}\|_{e,A^p_\gamma \to \mathscr{Z}_\mu} \leqslant \limsup_{j\to\infty} \|D^n_{\varphi,u} - D^n_{\varphi,u} K_{r_j}\|_{A^p_\gamma \to \mathscr{Z}_\mu}. \tag{4}$$

Therefore, we only need to show that

$$\limsup_{j\to\infty} \|D^n_{\varphi,u} - D^n_{\varphi,u} K_{r_j}\|_{A^p_\gamma \to \mathscr{Z}_\mu} \lesssim \max\{E, F, G\}.$$

For any $f \in A^p_\gamma$ with $\|f\|_{A^p_\gamma} \leqslant 1$, from the facts that

$$\lim_{j\to\infty} |u(0) f^{(n)}(\varphi(0)) - r_j^n u(0) f^{(n)}(r_j \varphi(0))| = 0$$

and

$$\lim_{j\to\infty} |u'(0)(f - f_{r_j})^{(n)}(\varphi(0)) + u(0)(f - f_{r_j})^{(n+1)}(\varphi(0))\varphi'(0)| = 0,$$

we have

$$\begin{aligned}
&\limsup_{j\to\infty} \|(D^n_{\varphi,u} - D^n_{\varphi,u} K_{r_j}) f\|_{\mathscr{Z}_\mu} \\
&= \limsup_{j\to\infty} \mu(z) \big| \big(u \cdot (f - f_{r_j})^{(n)} \circ \varphi\big)''(z) \big| \\
&\leqslant \limsup_{j\to\infty} \sup_{|\varphi(z)| \leqslant r_N} \mu(z) |(f - f_{r_j})^{(n+1)}(\varphi(z))| |2u'(z)\varphi'(z) + u(z)\varphi''(z)| \\
&\quad + \limsup_{j\to\infty} \sup_{|\varphi(z)| > r_N} \mu(z) |(f - f_{r_j})^{(n+1)}(\varphi(z))| |2u'(z)\varphi'(z) + u(z)\varphi''(z)| \\
&\quad + \limsup_{j\to\infty} \sup_{|\varphi(z)| \leqslant r_N} \mu(z) |(f - f_{r_j})^{(n)}(\varphi(z))| |u''(z)| \\
&\quad + \limsup_{j\to\infty} \sup_{|\varphi(z)| > r_N} \mu(z) |(f - f_{r_j})^{(n)}(\varphi(z))| |u''(z)|
\end{aligned}$$

$$
\begin{aligned}
&+\limsup_{j\to\infty}\sup_{|\varphi(z)|\leqslant r_N}\mu(z)|(f-f_{r_j})^{(n+2)}(\varphi(z))||\varphi'(z)|^2|u(z)|\\
&+\limsup_{j\to\infty}\sup_{|\varphi(z)|>r_N}\mu(z)|(f-f_{r_j})^{(n+2)}(\varphi(z))||\varphi'(z)|^2|u(z)|\\
:=&\, P_1+P_2+P_3+P_4+P_5+P_6, \qquad (5)
\end{aligned}
$$

where $N\in\mathbb{N}$ is large enough such that $r_j\geqslant\frac{1}{2}$ for all $j\geqslant N$. Since $r_j^n f_{r_j}^{(n)}-f^{(n)}\to 0$, $r_j^{n+1}f_{r_j}^{(n+1)}-f^{(n+1)}\to 0$ and $r_j^{n+2}f_{r_j}^{(n+2)}-f^{(n+2)}\to 0$ uniformly on compact subsets of $\mathbb{D}$ as $j\to\infty$, we have

$$
\begin{aligned}
P_3&=\limsup_{j\to\infty}\sup_{|\varphi(z)|\leqslant r_N}\mu(z)|(f-f_{r_j})^{(n)}(\varphi(z))||u''(z)|\\
&\leqslant\|u\|_{\mathscr{Z}_\mu}\limsup_{j\to\infty}\sup_{|w|\leqslant r_N}|f^{(n)}(w)-r_j^n f^{(n)}(r_j w)|=0, \qquad (6)
\end{aligned}
$$

$$
\begin{aligned}
P_1&=\limsup_{j\to\infty}\sup_{|\varphi(z)|\leqslant r_N}\mu(z)|(f-f_{r_j})^{(n+1)}(\varphi(z))||2u'(z)\varphi'(z)+u(z)\varphi''(z)|\\
&\leqslant(\|u\varphi\|_{\mathscr{Z}_\mu}+\|u\|_{\mathscr{Z}_\mu})\limsup_{j\to\infty}\sup_{|w|\leqslant r_N}|f^{(n+1)}(w)-r_j^{n+1}f^{(n+1)}(r_j w)|=0 \qquad (7)
\end{aligned}
$$

and

$$
\begin{aligned}
P_5&=\limsup_{j\to\infty}\sup_{|\varphi(z)|\leqslant r_N}\mu(z)|(f-f_{r_j})^{(n+2)}(\varphi(z))||\varphi'(z)|^2|u(z)|\\
&\leqslant(\|u\varphi^2\|_{\mathscr{Z}_\mu}+\|u\varphi\|_{\mathscr{Z}_\mu}+\|u\|_{\mathscr{Z}_\mu})\limsup_{j\to\infty}\sup_{|w|\leqslant r_N}|f^{(n+2)}(w)-r_j^{n+2}f^{(n+2)}(r_j w)|\\
&=0. \qquad (8)
\end{aligned}
$$

Now we estimate P_2. Using the fact that $\|f\|_{A^p_\gamma}\leqslant 1$ and Lemma 1, we have

$$
\begin{aligned}
P_2&=\limsup_{j\to\infty}\sup_{|\varphi(z)|>r_N}\mu(z)|(f-f_{r_j})^{(n+1)}(\varphi(z))||2u'(z)\varphi'(z)+u(z)\varphi''(z)|\\
&\lesssim\limsup_{j\to\infty}\|f-f_{r_j}\|_{A^p_\gamma}\sup_{|\varphi(z)|>r_N}\frac{\mu(z)|2u'(z)\varphi'(z)+u(z)\varphi''(z)|}{(1-|\varphi(z)|^2)^{\frac{2+\gamma+p}{p}+n}}.
\end{aligned}
$$

Taking the limit as $N\to\infty$ we obtain

$$
P_2\lesssim E. \qquad (9)
$$

Similarly, by Lemma 1 we have

$$
\begin{aligned}
P_4&=\limsup_{j\to\infty}\sup_{|\varphi(z)|>r_N}\mu(z)|(f-f_{r_j})^{(n)}(\varphi(z))||u''(z)|\\
&\lesssim\limsup_{j\to\infty}\|f-f_{r_j}\|_{A^p_\gamma}\sup_{|\varphi(z)|>r_N}\frac{\mu(z)|u''(z)|}{(1-|\varphi(z)|^2)^{\frac{2+\gamma}{p}+n}}
\end{aligned}
$$

and

$$P_6 = \limsup_{j\to\infty} \sup_{|\varphi(z)|>r_N} \mu(z)|(f-f_{r_j})^{(n+2)}(\varphi(z))||\varphi'(z)|^2|u(z)|$$
$$\lesssim \limsup_{j\to\infty} \|f-f_{r_j}\|_{A^p_\gamma} \sup_{|\varphi(z)|>r_N} \frac{\mu(z)|\varphi'(z)|^2|u(z)|}{(1-|\varphi(z)|^2)^{\frac{2+\gamma+2p}{p}+n}}.$$

Taking the limits as $N\to\infty$ we obtain

$$P_4 \lesssim F \quad \text{and} \quad P_6 \lesssim G, \tag{10}$$

respectively. Hence, by (5), (6), (7), (8), (9) and (10) we get

$$\limsup_{j\to\infty} \|D^n_{\varphi,u} - D^n_{\varphi,u}K_{r_j}\|_{A^p_\gamma\to\mathscr{Z}_\mu} \lesssim \max\{E,F,G\}, \tag{11}$$

which with (4) implies the implication as desired.

Next, we prove that $\|D^n_{\varphi,u}\|_{e,A^p_\gamma\to\mathscr{Z}_\mu} \gtrsim \max\{E,F,G\}$. Let $\{z_j\}_{j\in\mathbb{N}}$ be a sequence in $\mathbb{D}$ such that $|\varphi(z_j)|\to 1$ as $j\to\infty$. Define

$$k_j(z) = \frac{1-|\varphi(z_j)|^2}{(1-\overline{\varphi(z_j)}z)^{\frac{2+\gamma+p}{p}}} - \frac{(2+\gamma+p)(4+2\gamma+5p+2np)}{(2+\gamma+np)(2+\gamma+3p+np)} \frac{(1-|\varphi(z_j)|^2)^2}{(1-\overline{\varphi(z_j)}z)^{\frac{2+\gamma+2p}{p}}}$$
$$+\frac{(2+\gamma+p)(2+\gamma+2p)}{(2+\gamma+p+np)(2+\gamma+3p+np)} \frac{(1-|\varphi(z_j)|^2)^3}{(1-\overline{\varphi(z_j)}z)^{\frac{2+\gamma+3p}{p}}},$$

$$l_j(z) = \frac{1-|\varphi(z_j)|^2}{(1-\overline{\varphi(z_j)}z)^{\frac{2+\gamma+p}{p}}} - \frac{(4+2\gamma+2p)(2+\gamma+3p+np)}{2p^2+(2+\gamma+p+np)(2+\gamma+4p+np)} \frac{(1-|\varphi(z_j)|^2)^2}{(1-\overline{\varphi(z_j)}z)^{\frac{2+\gamma+2p}{p}}}$$
$$+\frac{(2+\gamma+p)(2+\gamma+2p)}{2p^2+(2+\gamma+p+np)(2+\gamma+4p+np)} \frac{(1-|\varphi(z_j)|^2)^3}{(1-\overline{\varphi(z_j)}z)^{\frac{2+\gamma+3p}{p}}}$$

and

$$m_j(z) = \frac{1-|\varphi(z_j)|^2}{(1-\overline{\varphi(z_j)}z)^{\frac{2+\gamma+p}{p}}} - \frac{4+2\gamma+2p}{2+\gamma+p+np} \frac{(1-|\varphi(z_j)|^2)^2}{(1-\overline{\varphi(z_j)}z)^{\frac{2+\gamma+2p}{p}}}$$
$$+\frac{(2+\gamma+p)(2+\gamma+2p)}{(2+\gamma+p+np)(2+\gamma+2p+np)} \frac{(1-|\varphi(z_j)|^2)^3}{(1-\overline{\varphi(z_j)}z)^{\frac{2+\gamma+3p}{p}}}.$$

It is easy to check that all k_j, l_j and m_j belong to A^p_γ,

$$k_j^{(n)}(\varphi(z_j)) = 0, \;\; k_j^{(n+2)}(\varphi(z_j)) = 0,$$

$$|k_j^{(n+1)}(\varphi(z_j))| = \frac{pq_0}{2+\gamma+3p+np} \frac{|\varphi(z_j)|^{n+1}}{(1-|\varphi(z_j)|^2)^{\frac{2+\gamma+p}{p}+n}},$$

$$l_j^{(n+1)}(\varphi(z_j))=0, \quad l_j^{(n+2)}(\varphi(z_j))=0,$$

$$|l_j^{(n)}(\varphi(z_j))|=\frac{2p^2q_0}{2p^2+(2+\gamma+p+np)(2+\gamma+4p+np)}\frac{|\varphi(z_j)|^n}{(1-|\varphi(z_j)|^2)^{\frac{2+\gamma}{p}+n}},$$

$$m_j^{(n)}(\varphi(z_j))=0, \quad m_j^{(n+1)}(\varphi(z_j))=0, |m_j^{(n+2)}(\varphi(z_j))|=\frac{2q_0|\varphi(z_j)|^{n+2}}{(1-|\varphi(z_j)|^2)^{\frac{2+\gamma+2p}{p}+n}},$$

where $q_0=\prod_{j=0}^{n-1}(\frac{2+\gamma+p}{p}+j)$. Moreover, it is easy to see that k_j, l_j and m_j converge to 0 uniformly on compact subsets of $\mathbb{D}$ as $j\to\infty$. Hence for any compact operator $K: A_\gamma^p \to \mathscr{Z}_\mu$, by Lemma 5 we get

$$\|D_{\varphi,u}^n-K\|_{A_\gamma^p\to\mathscr{Z}_\mu} \gtrsim \limsup_{j\to\infty}\|D_{\varphi,u}^n(k_j)\|_{\mathscr{Z}_\mu}-\limsup_{j\to\infty}\|K(k_j)\|_{\mathscr{Z}_\mu}$$
$$\gtrsim \limsup_{j\to\infty}\frac{\mu(z_j)|2u'(z_j)\varphi'(z_j)+u(z_j)\varphi''(z_j)||\varphi(z_j)|^{n+1}}{(1-|\varphi(z_j)|^2)^{\frac{2+\gamma+p}{p}+n}},$$

$$\|D_{\varphi,u}^n-K\|_{A_\gamma^p\to\mathscr{Z}_\mu} \gtrsim \limsup_{j\to\infty}\frac{\mu(z_j)|u''(z_j)||\varphi(z_j)|^n}{(1-|\varphi(z_j)|^2)^{\frac{2+\gamma}{p}+n}}$$

and

$$\|D_{\varphi,u}^n-K\|_{A_\gamma^p\to\mathscr{Z}_\mu} \gtrsim \limsup_{j\to\infty}\frac{\mu(z_j)|u(z_j)||\varphi'(z_j)|^2|\varphi(z_j)|^{n+2}}{(1-|\varphi(z_j)|^2)^{\frac{2+\gamma+2p}{p}+n}}.$$

Therefore,

$$\begin{aligned}\|D_{\varphi,u}^n\|_{e,A_\gamma^p\to\mathscr{Z}_\mu} &= \inf_K\|D_{\varphi,u}^n-K\|_{A_\gamma^p\to\mathscr{Z}_\mu}\\ &\gtrsim \limsup_{j\to\infty}\frac{\mu(z_j)|2u'(z_j)\varphi'(z_j)+u(z_j)\varphi''(z_j)||\varphi(z_j)|^{n+1}}{(1-|\varphi(z_j)|^2)^{\frac{2+\gamma+p}{p}+n}}\\ &= \limsup_{|\varphi(z)|\to1}\frac{\mu(z)|2u'(z)\varphi'(z)+u(z)\varphi''(z)|}{(1-|\varphi(z)|^2)^{\frac{2+\gamma+p}{p}+n}}=E,\end{aligned}$$

$$\|D_{\varphi,u}^n\|_{e,A_\gamma^p\to\mathscr{Z}_\mu} \gtrsim \limsup_{|\varphi(z)|\to1}\frac{\mu(z)|u''(z)|}{(1-|\varphi(z)|^2)^{\frac{2+\gamma}{p}+n}}=F$$

and

$$\|D_{\varphi,u}^n\|_{e,A_\gamma^p\to\mathscr{Z}_\mu} \gtrsim \limsup_{|\varphi(z)|\to1}\frac{\mu(z)|u(z)||\varphi'(z)|^2}{(1-|\varphi(z)|^2)^{\frac{2+\gamma+2p}{p}+n}}=G.$$

Hence $\|D_{\varphi,u}^n\|_{e,A_\gamma^p\to\mathscr{Z}_\mu} \gtrsim \max\{E,F,G\}$. This completes the proof of Theorem 2.

THEOREM 3. *Let μ be radial, non-increasing weight tending to zero at the boundary of $\mathbb{D}$. Let $1<p<\infty$, $-1<\gamma<\infty$, $u\in H(\mathbb{D})$, $\varphi\in S(\mathbb{D})$, and $n\in\mathbb{N}_0$ such that $D^n_{\varphi,u}:A^p_\gamma\to\mathscr{Z}_\mu$ is bounded. Then*

$$\|D^n_{\varphi,u}\|_{e,A^p_\gamma\to\mathscr{Z}_\mu}\approx\max\{T_1,T_2,T_3\},$$

where

$$T_1:=\limsup_{j\to\infty} j^{\frac{2+\gamma+p}{p}+n}\|(2u'\varphi'+u\varphi'')\varphi^{j-1}\|_\mu,$$

$$T_2:=\limsup_{j\to\infty} j^{\frac{2+\gamma}{p}+n}\|u''\varphi^{j-1}\|_\mu,\ \ T_3:=\limsup_{j\to\infty} j^{\frac{2+\gamma+2p}{p}+n}\|u(\varphi')^2\varphi^{j-1}\|_\mu.$$

Proof. From Theorem 2, Lemmas 2 and 3, we have

$$\|D^n_{\varphi,u}\|_{e,A^p_\gamma\to\mathscr{Z}_\mu}\gtrsim E=\|(2u'\varphi'+u\varphi'')C_\varphi\|_{e,H^\infty_{v_{\frac{2+\gamma+p}{p}+n}}\to H^\infty_\mu}$$

$$=\limsup_{j\to\infty}\frac{\|(2u'\varphi'+u\varphi'')\varphi^{j-1}\|_\mu}{\|z^{j-1}\|_{v_{\frac{2+\gamma+p}{p}+n}}}\approx\limsup_{j\to\infty} j^{\frac{2+\gamma+p}{p}+n}\|(2u'\varphi'+u\varphi'')\varphi^{j-1}\|_\mu=T_1,$$

$$\|D^n_{\varphi,u}\|_{e,A^p_\gamma\to\mathscr{Z}_\mu}\gtrsim F=\|u''C_\varphi\|_{e,H^\infty_{v_{\frac{2+\gamma}{p}+n}}\to H^\infty_\mu}=\limsup_{j\to\infty}\frac{\|u''\varphi^{j-1}\|_\mu}{\|z^{j-1}\|_{v_{\frac{2+\gamma}{p}+n}}}$$

$$\approx\limsup_{j\to\infty} j^{\frac{2+\gamma}{p}+n}\|u''\varphi^{j-1}\|_\mu=T_2$$

and

$$\|D^n_{\varphi,u}\|_{e,A^p_\gamma\to\mathscr{Z}_\mu}\gtrsim G=\|u\varphi'^2C_\varphi\|_{e,H^\infty_{v_{\frac{2+\gamma+2p}{p}+n}}\to H^\infty_\mu}=\limsup_{j\to\infty}\frac{\|u\varphi'^2\varphi^{j-1}\|_\mu}{\|z^{j-1}\|_{v_{\frac{2+\gamma+2p}{p}+n}}}$$

$$\approx\limsup_{j\to\infty} j^{\frac{2+\gamma+2p}{p}+n}\|u\varphi'^2\varphi^{j-1}\|_\mu=T_3.$$

Therefore $\|D^n_{\varphi,u}\|_{e,A^p_\gamma\to\mathscr{Z}_\mu}\gtrsim\max\{T_1,T_2,T_3\}$.

On the other hand, from Theorem A we known that the boundedness of $D^n_{\varphi,u}:A^p_\gamma\to\mathscr{Z}_\mu$ is equivalent to the boundedness of the operators $(2u'\varphi'+u\varphi'')C_\varphi:H^\infty_{v_{\frac{2+\gamma+p}{p}+n}}\to H^\infty_\mu$, $u''C_\varphi:H^\infty_{v_{\frac{2+\gamma}{p}+n}}\to H^\infty_\mu$ and $u\varphi'^2C_\varphi:H^\infty_{v_{\frac{2+\gamma+2p}{p}+n}}\to H^\infty_\mu$. From the above proof, we get

$$\|u''C_\varphi\|_{e,H^\infty_{v_{\frac{2+\gamma}{p}+n}}\to H^\infty_\mu}\approx\limsup_{j\to\infty} j^{\frac{2+\gamma}{p}+n}\|u''\varphi^{j-1}\|_\mu,$$

$$\|(2u'\varphi'+u\varphi'')C_\varphi\|_{e,H^\infty_{v_{\frac{2+\gamma+p}{p}+n}}\to H^\infty_\mu}\approx\limsup_{j\to\infty} j^{\frac{2+\gamma+p}{p}+n}\|(2u'\varphi'+u\varphi'')\varphi^{j-1}\|_\mu$$

and

$$\|u\varphi'^2 C_\varphi\|_{e,H^\infty_{v_{\frac{2+\gamma+2p}{p}+n}}\to H^\infty_\mu} \approx \limsup_{j\to\infty} j^{\frac{2+\gamma+2p}{p}+n}\|u\varphi'^2\varphi^{j-1}\|_\mu.$$

Hence

$$\begin{aligned}\|D^n_{\varphi,u}\|_{e,A^p_\gamma\to\mathscr{Z}_\mu} &\lesssim \|u''C_\varphi\|_{e,H^\infty_{v_{\frac{2+\gamma}{p}+n}}\to H^\infty_\mu} + \|(2u'\varphi'+u\varphi'')C_\varphi\|_{e,H^\infty_{v_{\frac{2+\gamma+p}{p}+n}}\to H^\infty_\mu} \\ &\quad + \|u\varphi'^2 C_\varphi\|_{e,H^\infty_{v_{\frac{2+\gamma+2p}{p}+n}}\to H^\infty_\mu} \\ &\lesssim T_1+T_2+T_3 \lesssim \max\{T_1,T_2,T_3\}.\end{aligned}$$

This completes the proof.

From Theorem 3, we get the following characterization of compactness of the operator $D^n_{\varphi,u}: A^p_\gamma \to \mathscr{Z}_\mu$.

COROLLARY 1. *Let μ be radial, non-increasing weight tending to zero at the boundary of $\mathbb{D}$. Let $0<p<\infty$, $-1<\gamma<\infty$, $u\in H(\mathbb{D})$, $\varphi\in S(\mathbb{D})$, and $n\in\mathbb{N}_0$ such that $D^n_{\varphi,u}: A^p_\gamma \to \mathscr{Z}_\mu$ is bounded. Then the operator $D^n_{\varphi,u}: A^p_\gamma \to \mathscr{Z}_\mu$ is compact if and only if*

$$\begin{aligned}&\limsup_{j\to\infty} j^{\frac{2+\gamma}{p}+n}\|u''\varphi^{j-1}\|_\mu = \limsup_{j\to\infty} j^{\frac{2+\gamma+2p}{p}+n}\|u(\varphi')^2\varphi^{j-1}\|_\mu \\ &= \limsup_{j\to\infty} j^{\frac{2+\gamma+p}{p}+n}\|(2u'\varphi'+u\varphi'')\varphi^{j-1}\|_\mu = 0.\end{aligned}$$

REFERENCES

[1] B. CHOE, H. KOO AND W. SMITH, *Composition operators on small spaces*, Integral Equations Oper. Theory, **56** (2006), 357–380.

[2] F. COLONNA AND M. TJANI, *Essential norms of weighted composition operators from Hilbert function spaces into the Zygmund-type spaces*, Mediterr J. Math., **12** (2015), 1357–1375.

[3] C. COWEN AND B. MACCLUER, *Composition Operators on Spaces of Analytic Functions*, CRC Press, Boca Raton, FL, 1995.

[4] K. ESMAEILI AND M. LINDSTRÖM, *Weighted composition operators between Zygmund type spaces and their essential norms*, Integral Equations Oper. Theory, **75** (2013), 473–490.

[5] R. HIBSCHWEILER AND N. PORTNOY, *Composition followed by differentiation between Bergman and Hardy spaces*, Rocky Mountain J. Math., **35** (2005), 843–855.

[6] Q. HU, Y. SHI, Y. SHI AND X. ZHU, *Essential norm of generalized weighted composition operators from the Bloch space into the Zygmund space*, J. Ineq. Appl., Vol. 2016, Article No. 123, (2016), 16 pages.

[7] O. HYVÄRINEN, M. KEMPPAINEN, M. LINDSTRÖM, A. RAUTIO AND E. SAUKKO, *The essential norm of weighted composition operators on weighted Banach spaces of analytic functions*, Integral Equations Oper. Theory, **72** (2012), 151–157.

[8] O. HYVÄRINEN AND M. LINDSTRÖM, *Estimates of essential norm of weighted composition operators between Bloch-type spaces*, J. Math. Anal. Appl., **393** (2012), 38–44.

[9] H. LI AND X. FU, *A new characterization of generalized weighted composition operators from the Bloch space into the Zygmund space*, J. Funct. Spaces Appl., Volume 2013, Article ID 925901, 12 pages.

[10] S. LI AND S. STEVIĆ, *Volterra type operators on Zygmund spaces*, J. Ineq. Appl., Volume 2007, Article ID 32124, (2007), 10 pages.

[11] S. LI AND S. STEVIĆ, *Composition followed by differentiation between Bloch type spaces*, J. Comput. Anal. Appl., **9** (2007), 195–205.

[12] S. LI AND S. STEVIĆ, *Composition followed by differentiation from mixed-norm spaces to α-Bloch spaces*, Sb. Math., **199** (12) (2008), 1847-1857.

[13] S. LI AND S. STEVIĆ, *Generalized composition operators on Zygmund spaces and Bloch type spaces*, J. Math. Anal. Appl., **338** (2008), 1282–1295.

[14] S. LI AND S. STEVIĆ, *Weighted composition operators from Zygmund spaces into Bloch spaces*, Appl. Math. Comput., **206** (2008), 825–831.

[15] S. LI AND S. STEVIĆ, *Composition followed by differentiation between H^∞ and α-Bloch spaces*, Houston J. Math., **35** (2009), 327–340.

[16] S. LI AND S. STEVIĆ, *Products of composition and differentiation operators from Zygmund spaces to Bloch spaces and Bers spaces*, Appl. Math. Comput., **217** (2010), 3144–3154.

[17] S. LI AND S. STEVIĆ, *Generalized weighted composition operators from α-Bloch spaces into weighted-type spaces*, J. Ineq. Appl., Vol. 2015, Article No. 265, (2015), 12 pages.

[18] Y. LIANG AND Z. ZHOU, *Essential norm of the product of differentiation and composition operators between Bloch-type space*, Arch. Math., **100** (2013), 347–360.

[19] A. MONTES-RODRIGUEZ, *Weighed composition operators on weighted Banach spaces of analytic functions*, J. London Math. Soc., **61** (2000), 872–884.

[20] S. STEVIĆ, *Norm and essential norm of composition followed by differentiation from α-Bloch spaces to H^∞_μ*, Appl. Math. Comput., **207** (2009), 225–229.

[21] S. STEVIĆ, *Products of composition and differentiation operators on the weighted Bergman space*, Bull. Belg. Math. Soc. Simon Stevin, **16** (2009), 623–635.

[22] S. STEVIĆ, *Weighted differentiation composition operators from mixed-norm spaces to weighted-type spaces*, Appl. Math. Comput., **211** (2009), 222–233.

[23] S. STEVIĆ, *Weighted differentiation composition operators from mixed-norm spaces to the nth weighted-type space on the unit disk*, Abstr. Appl. Anal., Vol. 2010, Article ID 246287, (2010), 15 pages.

[24] S. STEVIĆ, *Weighted differentiation composition operators from H^∞ and Bloch spaces to nth weigthed-type spaces on the unit disk*, Appl. Math. Comput., **216** (2010), 3634–3641.

[25] Y. WU AND H. WULAN, *Products of differentiation and composition operators on the Bloch space*, Collet. Math., **63** (2012), 93–107.

[26] W. YANG AND X. ZHU, *Generalized weighted composition operators from area Nevanlinna spaces to Bloch-type spaces*, Taiwanese J. Math., **16** (2012), 869–883.

[27] Y. YU AND Y. LIU, *Weighted differentiation composition operators from H^∞ to Zygmund spaces*, Integral Trans. Spec. Funct., **22** (2011), 507–520.

[28] K. ZHU, *Operator Theory in Function Spaces*, Amer. Math. Soc., second edition, 2007.

[29] X. ZHU, *Products of differentiation, composition and multiplication from Bergman type spaces to Bers type space*, Integ. Tran. Spec. Funct., **18** (2007), 223–231.

[30] X. ZHU, *Generalized weighted composition operators on weighted Bergman spaces*, Numer. Funct. Anal. Opt., **30** (2009), 881–893.

[31] X. ZHU, *Generalized weighted composition operators on Bloch-type spaces*, J. Ineq. Appl., **2015** (2015), 59–68.

[32] X. ZHU, *Essential norm of generalized weighted composition operators on Bloch-type spaces*, Appl. Math. Comput., **274** (2016), 133–142.

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Xiangling Zhu
Department of Mathematics
Jiaying University
514015, Meizhou, Guangdong, China
e-mail: jyuzxl@163.com