GENERALIZED WEIGHTED COMPOSITION OPERATORS FROM WEIGHTED BERGMAN SPACES INTO ZYGMUND–TYPE SPACES

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Abstract. The boundedness and the compactness of generalized weighted composition operators from weighted Bergman spaces into Zygmund-type spaces are investigated in this paper. Moreover, we give some estimates for the essential norm of these operators.

1. Introduction

Let $H(D)$ denote the space of analytic functions on the open unit disk $D$. Let $S(D)$ denote the set of all analytic self-maps of $D$. The composition operator $C_\varphi$ with the symbol $\varphi \in S(D)$ is defined by $(C_\varphi f)(z) = f(\varphi(z)),$ $f \in H(D)$. See [3, 28] for more information about the theory of composition operators. We denote the set of nonnegative integers by $\mathbb{N}_0$. Let $\varphi \in S(D)$, $u \in H(D)$ and $n \in \mathbb{N}_0$. The generalized weighted composition operator $D^n_{\varphi,u}$ is defined as follows (see [30]).

$$(D^n_{\varphi,u} f)(z) = u(z) \cdot f^{(n)}(\varphi(z)), \quad f \in H(D), \quad z \in D.$$  

If $n = 0$, then $D^n_{\varphi,u}$ is the weighted composition operator, which is frequently denoted by $uC_\varphi$ in the literature. When $n = 0$ and $u(z) = 1$, then $D^n_{\varphi,u}$ is just the composition operator $C_\varphi$. If $n = 1$, $u(z) = \varphi'(z)$, then $D^n_{\varphi,u} = DC_\varphi$. When $u(z) = 1$, $D^n_{\varphi,u} = C_\varphi D^n$. See [5, 11, 12, 15, 16, 18, 20, 21, 22, 25] for the study of $DC_\varphi$ and $C_\varphi D^n$. See [6, 9, 17, 23, 24, 26, 27, 29, 30, 31, 32] and the references therein for the study of the operator $D^n_{\varphi,u}$.

Let $v : D \to \mathbb{R}_+$ be a continuous, strictly positive and bounded function, the, so called, weight. The weight $v$ is called radial, if $v(z) = v(|z|)$ for all $z \in D$. The weighted space, denoted by $H^\infty_v$, is the set of all $f \in H(D)$ such that

$$\|f\|_v = \sup_{z \in D} v(z) |f(z)| < \infty.$$  

Under the norm $\| \cdot \|_v$, $H^\infty_v$ is a Banach space. The associated weight $\tilde{v}$ of $v$ is defined by

$$\tilde{v}(z) = \frac{1}{\sup\{ |f(z)| : f \in H^\infty_v, \|f\|_v \leq 1 \}}, \quad z \in D.$$  


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When \( v(z) = v_\alpha(z) = (1 - |z|^2)^\alpha (0 < \alpha < \infty) \), it is easy to check that \( \tilde{v}_\alpha(z) = v_\alpha(z) \).
In this case, we denote \( H_\alpha^{\infty} \) by \( H_\alpha^{\infty} \).

Let \( \mu \) be a weight. The Zygmund-type space, denoted by \( \mathcal{Z}_\mu \), is the set of all \( f \in H(\mathbb{D}) \) such that

\[
\| f \|_{\mathcal{Z}_\mu} = |f(0)| + |f'(0)| + \sup_{z \in \mathbb{D}} \mu(z)|f''(z)| < \infty.
\]

It is also easy to check that \( \mathcal{Z}_\mu \) is a Banach space under the norm \( \| \cdot \|_{\mathcal{Z}_\mu} \). When \( \mu(z) = 1 - |z|^2 \), \( \mathcal{Z}_\mu = \mathcal{Z} \) is the Zygmund space. See [1, 10] for more information on the Zygmund space on the unit disk. See [1, 4, 13, 14] for the study of composition operators on the Zygmund space.

For \( 0 < p < \infty \) and \( \gamma > -1 \), the weighted Bergman space, denoted by \( A_\gamma^p \), is the set of all functions \( f \in H(\mathbb{D}) \) satisfying

\[
\| f \|_{A_\gamma^p} = (\gamma + 1) \left( \int_{\mathbb{D}} |f(z)|^p (1 - |z|^2)^\gamma dA(z) \right)^{1/p} < \infty,
\]

where \( dA \) is the normalized Lebesgue area measure in \( \mathbb{D} \) such that \( A(\mathbb{D}) = 1 \). When \( p = 2 \), \( A_2^\gamma \) is a Hilbert space.

In [24] (see also [27]), Stević has studied the boundedness and compactness of the operator \( D_{\varphi,u}^n : \mathcal{B} \to \mathcal{Z} \). Here \( \mathcal{B} \) is the classical Bloch space. In [9], Li and Fu gave a new characterization for the boundedness and compactness of the operator \( D_{\varphi,u}^n : \mathcal{B} \to \mathcal{Z} \). In [6], motivated by [9, 24], the authors characterized the essential norm of the operator \( D_{\varphi,u}^n : \mathcal{B} \to \mathcal{Z} \). In [2], the authors studied the boundedness, compactness and essential norms of weighted composition operators from some Hilbert function spaces (including \( A_2^\gamma \)) into \( \mathcal{Z}_\mu \). Recall that the essential norm of a bounded linear operator \( T : X \to Y \) is defined as follows.

\[
\| T \|_{e,X \to Y} = \inf \{ \| T - K \|_{X \to Y} : K \text{ is compact} \},
\]

where \( X \) and \( Y \) are Banach spaces and \( \| \cdot \|_{X \to Y} \) is the operator norm.

In [23], Stević studied the operator \( D_{\varphi,u}^n \) from mixed-norm spaces to the \( n \)th weighted-type space. Among others, he proved the following result (in fact, he proved a more general result, including the one).

**THEOREM A.** Let \( \mu \) be radial, non-increasing weight tending to zero at the boundary of \( \mathbb{D} \). Let \( 1 < p < \infty \), \( -1 < \gamma < \infty \), \( u \in H(\mathbb{D}) \), \( \varphi \in S(\mathbb{D}) \), and \( n \in \mathbb{N}_0 \). Then the operator \( D_{\varphi,u}^n : A_\gamma^p \to \mathcal{Z}_\mu \) is bounded if and only if

\[
M_1 := \sup_{z \in \mathbb{D}} \frac{\mu(z)|u''(z)|}{(1 - |\varphi(z)|^2)^{\frac{2+\gamma}{p}+n}} < \infty, \tag{1}
\]

\[
M_2 := \sup_{z \in \mathbb{D}} \frac{\mu(z)|2u'(z)\varphi'(z) + u(z)\varphi''(z)|}{(1 - |\varphi(z)|^2)^{\frac{2+\gamma+p}{p}+n}} < \infty. \tag{2}
\]
and

$$M_3 := \sup_{z \in \mathbb{D}} \frac{\mu(z)|u(z)||\phi'(z)|^2}{(1 - |\phi(z)|^2)^{\frac{2\gamma + 2p}{p} + n}} < \infty. \quad (3)$$

Motivated by the above observations, in this work we give a new characterization for the boundedness of the operator $D_{\phi,u}^p : A^p_\gamma \to \mathcal{L}_\mu$. Moreover, we give some characterizations for the essential norm of the operator $D_{\phi,u}^p : A^p_\gamma \to \mathcal{L}_\mu$.

Throughout the paper, we denote by $C$ a positive constant which may differ from one occurrence to the next. In addition, we say that $A \lesssim B$ if there exists a constant $C$ such that $A \leq CB$. The symbol $A \approx B$ means that $A \lesssim B \lesssim A$.

2. Main results and proofs

In this section we formulate our main results in this paper. For this purpose, we need some lemmas as follows.

**Lemma 1.** [28] Assume that $0 < p < \infty$, $-1 < \gamma < \infty$, $n \in \mathbb{N}_0$ and $f \in A^p_\gamma$. Then there is a positive constant $C$ independent of $f$ such that

$$|f^{(n)}(z)| \leq C \frac{\|f\|_{A^p_\gamma}}{(1 - |z|^2)^{\frac{2\gamma + n}{p}}}. \quad (4)$$

**Lemma 2.** [8] For $\alpha > 0$, we have $\lim_{k \to \infty} k^{\alpha} \|z^k\|_{v^{\alpha}} = \left(\frac{2\alpha}{c}\right)^\alpha$.

**Lemma 3.** [19] Let $v$ and $w$ be radial, non-increasing weights tending to zero at the boundary of $\mathbb{D}$. Then the following statements hold.

(a) The weighted composition operator $uC_{\phi} : H^\infty_v \to H^\infty_w$ is bounded if and only if

$$\sup_{z \in \mathbb{D}} \frac{w(z)}{v(\phi(z))}|u(z)| < \infty.$$

(b) Suppose $uC_{\phi} : H^\infty_v \to H^\infty_w$ is bounded. Then

$$\|uC_{\phi}\|_{e,H^\infty_v \to H^\infty_w} = \limsup_{k \to \infty} \|u\phi^k\|_w \|z^k\|_v.$$
The following lemma is proved similar to Lemma 2.3 in [2].

**Lemma 5.** Let μ be radial, non-increasing weight tending to zero at the boundary of \( \mathbb{D} \). Let \( 1 < p < \infty \), \( -1 < \gamma < \infty \). Suppose that linear operator \( T : A^p_\gamma \to \mathcal{L}_\mu \) is bounded. Then \( T \) is compact if and only if whenever \( \{ f_n \} \) is bounded in \( A^p_\gamma \) and \( f_n \to 0 \) uniformly on compact subsets of \( \mathbb{D} \), \( \lim_{n \to \infty} \| Tf_n \| \mathcal{L}_\mu = 0 \).

**Theorem 1.** Let μ be radial, non-increasing weight tending to zero at the boundary of \( \mathbb{D} \). Let \( 1 < p < \infty \), \( -1 < \gamma < \infty \), \( u \in H(\mathbb{D}) \), \( \varphi \in S(\mathbb{D}) \), and \( n \in \mathbb{N}_0 \). Then the operator \( D^n_{\varphi,u} : A^p_\gamma \to \mathcal{L}_\mu \) is bounded if and only if

\[
\sup_{j \geq 1} \frac{2 + \gamma}{p} + n \| (2u^\prime \varphi + u\varphi''') \varphi^{j-1} \|_\mu < \infty,
\]

\[
\sup_{j \geq 1} \frac{2 + \gamma}{p} + n \| u'' \varphi^{j-1} \|_\mu < \infty \quad \text{and} \quad \sup_{j \geq 1} \frac{2 + \gamma + 2p}{p} + n \| u \varphi^2 \varphi^{j-1} \|_\mu < \infty.
\]

**Proof.** By Lemma 3, the inequality in (2) is equivalent to the weighted composition operator \( (2u^\prime \varphi + u\varphi''')C_\varphi : H^\infty_{v^{\gamma + p} + n} \to H^\infty_\mu \) is bounded, which is equivalent to

\[
\sup_{j \geq 1} \frac{2 + \gamma}{p} + n \| (2u^\prime \varphi + u\varphi''') \varphi^{j-1} \|_\mu < \infty
\]

by Lemma 4. Hence, by Lemma 2

\[
\sup_{j \geq 1} \frac{2 + \gamma + p}{p} + n \| (2u^\prime \varphi + u\varphi''') \varphi^{j-1} \|_\mu = \sup_{j \geq 1} \frac{2 + \gamma}{p} + n \| (2u^\prime \varphi + u\varphi''') \varphi^{j-1} \|_\mu < \infty.
\]

Similarly, the inequality in (1) is equivalent to

\[
\sup_{j \geq 1} \frac{2 + \gamma}{p} + n \| u'' \varphi^{j-1} \|_\mu \approx \sup_{j \geq 1} \frac{2 + \gamma}{p} + n \| u'' \varphi^{j-1} \|_\mu = \sup_{j \geq 1} \frac{2 + \gamma + 2p}{p} + n \| u'' \varphi^{j-1} \|_\mu < \infty
\]

and the inequality in (3) is equivalent to

\[
\sup_{j \geq 1} \frac{2 + \gamma}{p} + n \| u \varphi^2 \varphi^{j-1} \|_\mu = \sup_{j \geq 1} \frac{2 + \gamma + 2p}{p} + n \| u \varphi^2 \varphi^{j-1} \|_\mu = \sup_{j \geq 1} \frac{2 + \gamma + 2p}{p} + n \| u \varphi^2 \varphi^{j-1} \|_\mu < \infty.
\]

This completes the proof of Theorem 1.
THEOREM 2. Let $\mu$ be radial, non-increasing weight tending to zero at the boundary of $\mathbb{D}$. Let $1 < p < \infty$, $-1 < \gamma < \infty$, $u \in H(\mathbb{D})$, $\varphi \in S(\mathbb{D})$, and $n \in \mathbb{N}_0$ such that $D^n_{\varphi,u} : A^p_\gamma \to \mathcal{X}_\mu$ is bounded. Then

$$\|D^n_{\varphi,u}\|_{e; A^p_\gamma \to \mathcal{X}_\mu} \approx \max \left\{ E, F, G \right\},$$

where

$$E := \limsup_{|z| \to 1} \frac{\mu(z)|2u'(z)\varphi'(z) + u(z)\varphi''(z)|}{(1 - |\varphi(z)|^2)^{\frac{2+\gamma+p}{p} + n}},$$

$$F := \limsup_{|\varphi(z)| \to 1} \frac{\mu(z)|u''(z)|}{(1 - |\varphi(z)|^2)^{\frac{2+\gamma+p}{p} + n}},$$

$$G := \limsup_{|\varphi(z)| \to 1} \frac{\mu(z)|u(z)||\varphi'(z)|^2}{(1 - |\varphi(z)|^2)^{\frac{2+\gamma+p}{p} + n}}.$$

Proof. First we prove that $\|D^n_{\varphi,u}\|_{e; A^p_\gamma \to \mathcal{X}_\mu} \lesssim \max \{ E, F, G \}$. For $r \in [0,1)$, set $K_r : H(\mathbb{D}) \to H(\mathbb{D})$ by $(K_rf)(z) = f(rz)$, $f \in H(\mathbb{D})$. It is clear that $f_r - f \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $r \to 1$. Moreover, the operator $K_r$ is compact on $A^p_\gamma$ and $\|K_r\|_{A^p_\gamma \to A^p_\gamma} \leq 1$. Let $\{r_j\} \subset (0,1)$ be a sequence such that $r_j \to 1$ as $j \to \infty$. Then for each positive integer $j$, the operator $D^n_{\varphi,u}K_{r_j} : A^p_\gamma \to \mathcal{X}_\mu$ is compact. Hence

$$\|D^n_{\varphi,u}\|_{e; A^p_\gamma \to \mathcal{X}_\mu} \leq \limsup_{j \to \infty} \|D^n_{\varphi,u} - D^n_{\varphi,u}K_{r_j}\|_{A^p_\gamma \to \mathcal{X}_\mu}.$$ (4)

Therefore, we only need to show that

$$\limsup_{j \to \infty} \|D^n_{\varphi,u} - D^n_{\varphi,u}K_{r_j}\|_{A^p_\gamma \to \mathcal{X}_\mu} \lesssim \max \{ E, F, G \}.$$

For any $f \in A^p_\gamma$ with $\|f\|_{A^p_\gamma} \leq 1$, from the facts that

$$\lim_{j \to \infty} |u(0)f^{(n)}(\varphi(0)) - r_j^n u(0)f^{(n)}(r_j \varphi(0))| = 0$$

and

$$\lim_{j \to \infty} |u'(0)(f - f_{r_j})^{(n)}(\varphi(0)) + u(0)(f - f_{r_j})^{(n+1)}(\varphi(0))\varphi'(0)| = 0,$$

we have

$$\limsup_{j \to \infty} \| (D^n_{\varphi,u} - D^n_{\varphi,u}K_{r_j})f \|_{\mathcal{X}_\mu} = \limsup_{j \to \infty} \mu(z)|(u \cdot (f - f_{r_j})^{(n)} \circ \varphi)''(z)| \leq \limsup_{j \to \infty} \sup_{|\varphi(z)| \leq r_N} \mu(z)|(f - f_{r_j})^{(n+1)}(\varphi(z))|2u'(z)\varphi'(z) + u(z)\varphi''(z)|$$

$$+ \limsup_{j \to \infty} \sup_{|\varphi(z)| > r_N} \mu(z)|(f - f_{r_j})^{(n+1)}(\varphi(z))|2u'(z)\varphi'(z) + u(z)\varphi''(z)|$$

$$+ \limsup_{j \to \infty} \sup_{|\varphi(z)| \leq r_N} \mu(z)|(f - f_{r_j})^{(n)}(\varphi(z))|u''(z)|$$

$$+ \limsup_{j \to \infty} \sup_{|\varphi(z)| > r_N} \mu(z)|(f - f_{r_j})^{(n)}(\varphi(z))|u''(z)|$$
\[ + \limsup_{j \to \infty} \sup_{\|\varphi(z)\| \leq r_N} \mu(z) |(f - f_{r_j})^{(n+2)}(\varphi(z))| |\varphi'(z)|^2 |u(z)| \\
+ \limsup_{j \to \infty} \sup_{\|\varphi(z)\| > r_N} \mu(z) |(f - f_{r_j})^{(n+2)}(\varphi(z))| |\varphi'(z)|^2 |u(z)| \\
:= P_1 + P_2 + P_3 + P_4 + P_5 + P_6, \tag{5} \]

where \( N \in \mathbb{N} \) is large enough such that \( r_j \geq \frac{1}{2} \) for all \( j \geq N \). Since \( r_j^n f_{r_{j+1}}^{(n)} - f^{(n)} \to 0 \), \( r_j^{n+1} f_{r_{j+1}}^{(n+1)} - f^{(n+1)} \to 0 \) and \( r_j^{n+2} f_{r_{j+1}}^{(n+2)} - f^{(n+2)} \to 0 \) uniformly on compact subsets of \( \mathbb{D} \) as \( j \to \infty \), we have

\[ P_3 = \limsup_{j \to \infty} \sup_{\|\varphi(z)\| \leq r_N} \mu(z) |(f - f_{r_j})^{(n)}(\varphi(z))| |u''(z)| \leq \|u\|_{L_\mu} \sup_{\|\varphi(z)\| \leq r_N} |f^{(n)}(w)| - r_j^n f^{(n)}(r_jw) = 0, \tag{6} \]

and

\[ P_1 = \limsup_{j \to \infty} \sup_{\|\varphi(z)\| \leq r_N} \mu(z) |(f - f_{r_j})^{(n+1)}(\varphi(z))| 2|u'(z)||\varphi'(z) + u(z)\varphi''(z)| \leq (\|u\|_{L_\mu} + \|u\|_{L_{\mu^2}}) \limsup_{j \to \infty} \sup_{\|\varphi(z)\| \leq r_N} |f^{(n+1)}(w)| - r_j^{n+1} f^{(n+1)}(r_jw) = 0 \tag{7} \]

and

\[ P_5 = \limsup_{j \to \infty} \sup_{\|\varphi(z)\| \leq r_N} \mu(z) |(f - f_{r_j})^{(n+2)}(\varphi(z))| |\varphi'(z)|^2 |u(z)| \leq (\|u\varphi^2\|_{L_{\mu^2}} + \|u\varphi\|_{L_{\mu^2}} + \|u\|_{L_{\mu^2}}) \limsup_{j \to \infty} \sup_{\|\varphi(z)\| \leq r_N} |f^{(n+2)}(w)| - r_j^{n+2} f^{(n+2)}(r_jw) = 0. \tag{8} \]

Now we estimate \( P_2 \). Using the fact that \( \|f\|_{A^p_\varphi} \leq 1 \) and Lemma 1, we have

\[ P_2 = \limsup_{j \to \infty} \sup_{\|\varphi(z)\| > r_N} \mu(z) |(f - f_{r_j})^{(n+1)}(\varphi(z))| 2|u'(z)||\varphi'(z) + u(z)\varphi''(z)| \leq \|f - f_{r_j}\|_{A^p_\varphi} \sup_{\|\varphi(z)\| > r_N} \frac{\mu(z)|2u'(z)||\varphi'(z) + u(z)\varphi''(z)|}{(1 - |\varphi(z)|^2)^{\frac{2+\gamma}{p} + n}}. \]

Taking the limit as \( N \to \infty \) we obtain

\[ P_2 \lesssim E. \tag{9} \]

Similarly, by Lemma 1 we have

\[ P_4 = \limsup_{j \to \infty} \sup_{\|\varphi(z)\| > r_N} \mu(z) |(f - f_{r_j})^{(n)}(\varphi(z))| |u''(z)| \lesssim \limsup_{j \to \infty} \|f - f_{r_j}\|_{A^p_\varphi} \sup_{\|\varphi(z)\| > r_N} \frac{\mu(z)|u''(z)|}{(1 - |\varphi(z)|^2)^{\frac{2+\gamma}{p} + n}}. \]
\( P_6 = \limsup_{j \to \infty} \sup_{|\varphi(z)| > r_N} \mu(z) |(f - f_{r_j})(n+2)(\varphi(z))| |\varphi'(z)|^2 |u(z)| \)

\( \lesssim \limsup_{j \to \infty} \|f - f_{r_j}\|_{A_\gamma^p} \sup_{|\varphi(z)| > r_N} \frac{\mu(z) |\varphi'(z)|^2 |u(z)|}{(1 - |\varphi(z)|^2)^{\frac{2+\gamma+2p}{p}+n}}. \)

Taking the limits as \( N \to \infty \) we obtain

\[ P_4 \lesssim F \quad \text{and} \quad P_6 \lesssim G, \quad (10) \]

respectively. Hence, by (5), (6), (7), (8), (9) and (10) we get

\[ \limsup_{j \to \infty} \|D^n_{\varphi,u} - D^n_{\varphi,u} K_{r_j} \|_{A_\gamma^p \to \mathcal{Z}_p} \lesssim \max \{E,F,G\}, \quad (11) \]

which with (4) implies the implication as desired.

Next, we prove that \( \|D^n_{\varphi,u} \|_{A_\gamma^p \to \mathcal{Z}_p} \geq \max \{E,F,G\} \). Let \( \{z_j\}_{j \in \mathbb{N}} \) be a sequence in \( \mathbb{D} \) such that \( |\varphi(z_j)| \to 1 \) as \( j \to \infty \). Define

\[
k_j(z) = \frac{1 - |\varphi(z_j)|^2}{(1 - \varphi(z_j)z)^{\frac{2+\gamma+p}{p}}} - \frac{(2 + \gamma + p)(4 + 2 \gamma + 5p + 2np)}{(2 + \gamma + np)(2 + \gamma + 3p + np)} \frac{(1 - |\varphi(z_j)|^2)^2}{(1 - \varphi(z_j)z)^{\frac{2+\gamma+2p}{p}}} \]

\[ + \frac{(2 + \gamma + p)(2 + \gamma + 2p)}{(2 + \gamma + p + np)(2 + \gamma + 3p + np)} \frac{(1 - |\varphi(z_j)|^2)^3}{(1 - \varphi(z_j)z)^{\frac{2+\gamma+3p}{p}}}, \]

\[
l_j(z) = \frac{(1 - |\varphi(z_j)|^2)^2}{(1 - \varphi(z_j)z)^{\frac{2+\gamma+p}{p}}} - \frac{(4 + 2 \gamma + 2p)(2 + \gamma + 3p + np)}{2p^2 + (2 + \gamma + p + np)(2 + \gamma + 4p + np)} \frac{(1 - |\varphi(z_j)|^2)^2}{(1 - \varphi(z_j)z)^{\frac{2+\gamma+2p}{p}}} \]

\[ + \frac{(2 + \gamma + p)(2 + \gamma + 2p)}{2p^2 + (2 + \gamma + p + np)(2 + \gamma + 4p + np)} \frac{(1 - |\varphi(z_j)|^2)^3}{(1 - \varphi(z_j)z)^{\frac{2+\gamma+3p}{p}}}, \]

and

\[
m_j(z) = \frac{1 - |\varphi(z_j)|^2}{(1 - \varphi(z_j)z)^{\frac{2+\gamma+p}{p}}} - \frac{4 + 2 \gamma + 2p}{2 + \gamma + p + np} \frac{(1 - |\varphi(z_j)|^2)^2}{(1 - \varphi(z_j)z)^{\frac{2+\gamma+2p}{p}}} \]

\[ + \frac{(2 + \gamma + p)(2 + \gamma + 2p)}{(2 + \gamma + p + np)(2 + \gamma + 2p + np)} \frac{(1 - |\varphi(z_j)|^2)^3}{(1 - \varphi(z_j)z)^{\frac{2+\gamma+3p}{p}}}. \]

It is easy to check that all \( k_j, l_j \) and \( m_j \) belong to \( A_\gamma^p \),

\[ k_j^{(n)}(\varphi(z_j)) = 0, \quad k_j^{(n+2)}(\varphi(z_j)) = 0, \]

\[ |k_j^{(n+1)}(\varphi(z_j))| = \frac{pq_0}{2 + \gamma + 3p + np} \frac{|\varphi(z_j)|^{n+1}}{(1 - |\varphi(z_j)|^2)^{\frac{2+\gamma+p}{p}+n}}, \]
\[ l_j^{(n+1)}(\phi(z_j)) = 0, \quad l_j^{(n+2)}(\phi(z_j)) = 0, \]

\[ |l_j^{(n)}(\phi(z_j))| = \frac{2p^2 q_0}{2p^2 + (2 + \gamma + p + np)(2 + \gamma + 4p + np)} \frac{|\phi(z_j)|n}{(1 - |\phi(z_j)|^2)^{\frac{2+\gamma+p}{p}+n}}, \]

\[ m_j^{(n)}(\phi(z_j)) = 0, \quad m_j^{(n+1)}(\phi(z_j)) = 0, \quad |m_j^{(n+2)}(\phi(z_j))| = \frac{2q_0 |\phi(z_j)|n+2}{(1 - |\phi(z_j)|^2)^{\frac{2+\gamma+2p}{p}+n}}, \]

where \( q_0 = \prod_{j=0}^{n-1} \left( \frac{2+\gamma+p}{p} + j \right) \). Moreover, it is easy to see that \( k_j, l_j \) and \( m_j \) converge to 0 uniformly on compact subsets of \( \mathbb{D} \) as \( j \to \infty \). Hence for any compact operator \( K : A^p_{\gamma} \to \mathcal{X}_{\mu} \), by Lemma 5 we get

\[
\| D^n_{\phi,u} - K \|_{A^p_{\gamma} \to \mathcal{X}_{\mu}} \geq \limsup_{j \to \infty} \| D^n_{\phi,u}(k_j) \|_{\mathcal{X}_{\mu}} - \limsup_{j \to \infty} \| K(k_j) \|_{\mathcal{X}_{\mu}}
\]

\[ \geq \limsup_{j \to \infty} \frac{\mu(z_j)|2u'(z_j)\varphi'(z_j) + u(z_j)\varphi''(z_j)||\phi(z_j)|n+1}{(1 - |\phi(z_j)|^2)^{\frac{2+\gamma+p}{p}+n}}, \]

and

\[ \| D^n_{\phi,u} - K \|_{A^p_{\gamma} \to \mathcal{X}_{\mu}} \geq \limsup_{j \to \infty} \frac{\mu(z_j)|u''(z_j)||\phi(z_j)|n}{(1 - |\phi(z_j)|^2)^{\frac{2+\gamma+2p}{p}+n}}. \]

Therefore,

\[ \| D^n_{\phi,u} \|_{e,A^p_{\gamma} \to \mathcal{X}_{\mu}} = \inf_K \| D^n_{\phi,u} - K \|_{A^p_{\gamma} \to \mathcal{X}_{\mu}} \]

\[ \geq \limsup_{j \to \infty} \frac{\mu(z_j)|2u'(z_j)\varphi'(z_j) + u(z_j)\varphi''(z_j)||\phi(z_j)|n+1}{(1 - |\phi(z_j)|^2)^{\frac{2+\gamma+p}{p}+n}} = \limsup_{|\phi(z)| \to 1} \frac{\mu(z)|2u'(z)\varphi'(z) + u(z)\varphi''(z)|}{(1 - |\phi(z)|^2)^{\frac{2+\gamma+p}{p}+n}} = E, \]

\[ \| D^n_{\phi,u} \|_{e,A^p_{\gamma} \to \mathcal{X}_{\mu}} \geq \limsup_{|\phi(z)| \to 1} \frac{\mu(z)|u''(z)|}{(1 - |\phi(z)|^2)^{\frac{2+\gamma+p}{p}+n}} = F \]

and

\[ \| D^n_{\phi,u} \|_{e,A^p_{\gamma} \to \mathcal{X}_{\mu}} \geq \limsup_{|\phi(z)| \to 1} \frac{\mu(z)|u(z)||\phi'(z)|^2}{(1 - |\phi(z)|^2)^{\frac{2+\gamma+2p}{p}+n}} = G. \]

Hence \( \| D^n_{\phi,u} \|_{e,A^p_{\gamma} \to \mathcal{X}_{\mu}} \geq \max \{ E, F, G \} \). This completes the proof of Theorem 2.
THEOREM 3. Let μ be radial, non-increasing weight tending to zero at the boundary of \( \mathbb{D} \). Let \( 1 < p < \infty \), \(-1 < \gamma < \infty\), \( u \in H(\mathbb{D}) \), \( \varphi \in S(\mathbb{D}) \), and \( n \in \mathbb{N}_0 \) such that \( D^n_{\varphi,u} : \mathcal{A}_f \rightarrow \mathcal{F}_\mu \) is bounded. Then

\[
\| D^n_{\varphi,u} \|_{e, \mathcal{A}_f} \rightarrow \mathcal{F}_\mu \cong \max \{ T_1, T_2, T_3 \},
\]

where

\[
T_1 := \limsup_{j \to \infty} j^{2+\gamma+p+n} \| (2u' \varphi' + u\varphi'') \varphi_j^{-1} \|_\mu,
\]

\[
T_2 := \limsup_{j \to \infty} j^{2+\gamma+n} \| u'' \varphi_j^{-1} \|_\mu,
\]

\[
T_3 := \limsup_{j \to \infty} j^{2+\gamma+2p+n} \| (u \varphi')^2 \varphi_j^{-1} \|_\mu.
\]

Proof. From Theorem 2, Lemmas 2 and 3, we have

\[
\| D^n_{\varphi,u} \|_{e, \mathcal{A}_f} \rightarrow \mathcal{F}_\mu \cong E = \| (2u' \varphi' + u\varphi'') C_\varphi |_{e, \mathcal{A}_f} \rightarrow \mathcal{F}_\mu \cong \limsup_{j \to \infty} j^{2+\gamma+p+n} \| (2u' \varphi' + u\varphi'') \varphi_j^{-1} \|_\mu = T_1,
\]

\[
\| D^n_{\varphi,u} \|_{e, \mathcal{A}_f} \rightarrow \mathcal{F}_\mu \cong F = \| u'' C_\varphi |_{e, \mathcal{A}_f} \rightarrow \mathcal{F}_\mu \cong \limsup_{j \to \infty} j^{2+\gamma+n} \| u'' \varphi_j^{-1} \|_\mu = T_2,
\]

and

\[
\| D^n_{\varphi,u} \|_{e, \mathcal{A}_f} \rightarrow \mathcal{F}_\mu \cong G = \| u\varphi'^2 C_\varphi |_{e, \mathcal{A}_f} \rightarrow \mathcal{F}_\mu \cong \limsup_{j \to \infty} j^{2+\gamma+2p+n} \| u\varphi'^2 \varphi_j^{-1} \|_\mu = T_3.
\]

Therefore \( \| D^n_{\varphi,u} \|_{e, \mathcal{A}_f} \rightarrow \mathcal{F}_\mu \cong \max \{ T_1, T_2, T_3 \} \).

On the other hand, from Theorem A we known that the boundedness of \( D^n_{\varphi,u} : \mathcal{A}_f \rightarrow \mathcal{F}_\mu \) is equivalent to the boundedness of the operators \((2u' \varphi' + u\varphi'') C_\varphi |_{e, \mathcal{A}_f} \rightarrow \mathcal{F}_\mu \cong \limsup_{j \to \infty} j^{2+\gamma+p+n} \| (2u' \varphi' + u\varphi'') \varphi_j^{-1} \|_\mu = T_1\). From the above proof, we get

\[
\| u'' C_\varphi |_{e, \mathcal{A}_f} \rightarrow \mathcal{F}_\mu \cong \limsup_{j \to \infty} j^{2+\gamma+n} \| u'' \varphi_j^{-1} \|_\mu,
\]

\[
\| (2u' \varphi' + u\varphi'') C_\varphi |_{e, \mathcal{A}_f} \rightarrow \mathcal{F}_\mu \cong \limsup_{j \to \infty} j^{2+\gamma+2p+n} \| (2u' \varphi' + u\varphi'') \varphi_j^{-1} \|_\mu.
\]
and
\[ \|u \varphi'^2 \varphi\|_{e_n L_\gamma} \approx \limsup_{j \to \infty} j^{2+\gamma+2p/p} \|u \varphi'^2 \varphi^{j-1}\|_\mu. \]

Hence
\[ \|D^n_{\varphi, u}\|_{e_n \gamma \to \mathcal{Z}_\mu} \lesssim \|u'' \varphi\|_{e_n L_\gamma}^{2+\gamma+2p/p} + \|2u' \varphi' + u \varphi''\|_{e_n L_\gamma}^{2+\gamma+2p/p} \]
\[ + \|u \varphi'^2 \varphi\|_{e_n L_\gamma} \approx \|u\|_{e_n L_\gamma}^{2+\gamma+2p/p} \lesssim T_1 + T_2 + T_3 \lesssim \max\{T_1, T_2, T_3\}. \]

This completes the proof.

From Theorem 3, we get the following characterization of compactness of the operator \( D^n_{\varphi, u} : A^n_\gamma \to \mathcal{Z}_\mu \).

**Corollary 1.** Let \( \mu \), be radial, non-increasing weight tending to zero at the boundary of \( \mathbb{D} \). Let \( 0 < p < \infty \), \(-1 < \gamma < \infty\), \( u \in H(\mathbb{D}) \), \( \varphi \in S(\mathbb{D}) \), and \( n \in \mathbb{N}_0 \) such that \( \|D^n_{\varphi, u}\|_{e_n \gamma \to \mathcal{Z}_\mu} \) is bounded. Then the operator \( D^n_{\varphi, u} : A^n_\gamma \to \mathcal{Z}_\mu \) is compact if and only if
\[ \limsup_{j \to \infty} j^{2+\gamma+2p/p} \|u' \varphi^{j-1}\|_\mu = \limsup_{j \to \infty} j^{2+\gamma+2p/p} \|u(\varphi')^2 \varphi^{j-1}\|_\mu = \limsup_{j \to \infty} j^{2+\gamma+2p/p} \|2u' \varphi' + u \varphi''\) \varphi^{j-1}\|_\mu = 0. \]

**References**


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