Journal of Classical Analysis
Volume 10, Number 1 (2017), 39–47 doi:10.7153/jca-10-04

ANALYTIC FUNCTIONS DEFINED BY A PRODUCT OF EXPRESSIONS HAVING GEOMETRIC MEANING

F. M. JIMOH AND K. O. BABALOLA

Abstract. We define a new class, $\mathscr{J}_n^{\alpha}(\beta)$, of analytic functions by a product of certain expressions having geometric meaning. We establish univalence of the new class, obtain its integral representations, sufficient inclusion conditions and coefficient inequalities. Examples are given.

1. Introduction

Let A be the class of functions of the form

$$f(z) = z + a_2 z^2 + \cdots$$

which are analytic in the unit disk $E = |z| < 1$. Denote by P, the class of functions

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \ldots$$

analytic in E and satisfying Re $p(z) > 0$, and by $P(\beta)$ if Re $p(z) > \beta$ for some real number $0 \leqslant \beta < 1$.

In [7], Singh investigated a subclass $B_1(\alpha)$ of univalent Bazilevic maps (of type α) which consists of functions f satisfying

$$\operatorname{Re} \frac{f(z)^{\alpha-1} f'(z)}{z^{\alpha-1}} > 0, \quad z \in E.$$

The parameter $\alpha \geqslant 0$ is real. Thus the case $\alpha = 0$ coincides with the class S^* of starlike maps of the disk E, defined by Re $zf'(z)/f(z) > 0$.

Abdulhalim [1] generalized the class $B_1(\alpha)$ to $B_n(\alpha)$ consisting of functions satisfying

$$\operatorname{Re} \frac{D^n f(z)^{\alpha}}{z^{\alpha}} > 0, \quad z \in E$$

where D^n is the Salagean derivative [6] defined by

$$D^n f(z) = D(D^{n-1} f(z)) = z[D^{n-1} f(z)]', \quad n \in N_0 = \{0, 1, 2, \ldots\}$$

with $D^0 f(z) = f(z)$. Powers mean principal determinations only. He proved that $B_{n+1}(\alpha) \subset B_n(\alpha)$ which implies that for $n \geqslant 1$, the class contains only univalent maps of the unit disk.

Mathematics subject classification (2010): 30C45.
Keywords and phrases: Bazilevic functions, product of expressions having geometric meaning, bounded turning, starlike, analytic and univalent functions.

A further generalization, $T_n^\alpha(\beta)$, was done by Opoola [5] (and slightly modified in [3]) which consists of functions satisfying

$$\operatorname{Re}\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha} > \beta$$

where β is a real number such that $0 \leqslant \beta < 1$, $\alpha > 0$ is real and D^n is the Salagean derivative, and powers also mean principal determinations only. He established the univalence of functions in this class for $n \geqslant 1$ by proving the inclusion $T_{n+1}^\alpha(\beta) \subset T_n^\alpha(\beta)$.

In this article, we say:

DEFINITION 1. A function $f \in A$ is said to be in the class $\mathscr{J}_n^\alpha(\beta)$ if and only if

$$\operatorname{Re}\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha}\frac{D^{n+1} f(z)^\alpha}{\alpha^{n+1} z^\alpha} > \beta. \tag{1}$$

All parameters have their usual definitions and specifications.

REMARK 1. The new class of functions interestingly includes many well known ones as special cases. For instance, of note are the following particular cases defining families of univalent maps of the disk.

(a) If $n = 0$ and $\alpha = 0$ we have the class of starlike functions of order β. The admissibility of the case $\alpha = 0$ (though for $n = 0$ only) is a striking feature of the new class, a departure from its restriction in $T_n^\alpha(\beta)$! Univalence in this case is well known.

(b) If $n = 0$ and $\alpha = \frac{1}{2}$, the class $\mathscr{J}_n^\alpha(\beta)$ reduces to the class of functions with bounded turning of order β.

(c) If $n = 0$ and $\alpha = 1$ we have the class of functions satisfying

$$\operatorname{Re}\frac{f(z)f'(z)}{z} > \beta$$

which consists of univalent Bazilevic maps of type 2.

(d) If $n = 1$ and $\alpha = 1$ then we have the class of analytic functions satisfying

$$\operatorname{Re} f'(f'(z) + zf''(z)) > \beta.$$

Furthermore, the geometric condition (1) can also be written as

$$\operatorname{Re}\frac{D((D^n f(z)^\alpha)^2)}{2\alpha^{2n+1}z^{2\alpha}} > \beta \tag{2}$$

which shall be employed interchangeably with (1) depending on convenience.

2. Preliminary lemmas

In our investigation, the following lemmas will be useful.

LEMMA 1. [3] *Let* $u = u_1 + u_2 i$, $v = v_1 + v_2 i$ *and* $\psi(u,v)$ *a complex valued function satisfying:*

(a) $\psi(u,v)$ *is continuous in a domain* Ω *of* $\mathbb{C}^2$,

(b) $(1,0) \in \Omega$ *and* $\operatorname{Re} \psi(1,0) > 0$,

(c) $\operatorname{Re} \psi(\zeta + (1-\zeta)u_2 i, v_1) \leqslant \zeta$ *when* $(\zeta + (1-\zeta)u_2 i, v_1) \in \Omega$ *and* $2v_1 \leqslant -(1-\zeta)(1+u_2^2)$ *for real* $0 \leqslant \zeta < 1$.

If $p \in P$ *such that* $(p(z), zp'(z)) \in \Omega$ *and* $\operatorname{Re} \psi(p(z), zp'(z)) > \zeta$ *for* $z \in E$. *Then* $\operatorname{Re} p(z) > \zeta$ *in* E.

LEMMA 2. [2] *Let* $p(z)$ *be analytic in* E *with* $p(0) = 1$. *Suppose that*

$$\operatorname{Re}\left(1 + \frac{zp'(z)}{p(z)}\right) > \frac{3\beta - 1}{2\beta}, \quad z \in E.$$

Then Re $p(z) > 2^{1-\frac{1}{\beta}}$, $\frac{1}{2} \leqslant \beta < 1$, $z \in E$ *and the constant* $2^{1-\frac{1}{\beta}}$ *is the best possible.*

We shall also require the well known Caratheodory inequality $|p_k| \leqslant 2$, $k = 1,2,3\ldots$, together with one of its consequences, which is the following lemma.

LEMMA 3. [4] *Let* $p \in P$. *Then for any real or complex number* σ, *we have sharp inequalities*

$$\left| p_2 - \sigma \frac{p_1^2}{2} \right| \leqslant 2 \max\{1, |1-\sigma|\}.$$

3. Main results

The main results of this paper are presented as follows.

THEOREM 1. $\mathscr{J}_n^{\alpha}(\beta) \subset T_n^{\alpha}(\beta)$, $\alpha > 0$.

Proof. Let

$$p(z) = \frac{D^n f(z)^{\alpha}}{\alpha^n z^{\alpha}}.$$

Then from (2) we have

$$\operatorname{Re}\left(p(z)^2 + \frac{zp(z)p'(z)}{\alpha}\right) > \beta.$$

Now define

$$\varphi(u,v) = u^2 + \frac{uv}{\alpha}, \quad \alpha > 0.$$

Then $\varphi(u,v)$ clearly satisfies the conditions (a) and (b) of Lemma 1. Furthermore whenever $2v_1 < -(1-\beta)(1+u_2^2)$, we have

$$\text{Re } \varphi(\beta + (1-\beta)u_2 i, v_1) = \beta^2 - (1-\beta)^2 u_2^2 - \frac{\beta(1-\beta)(1+u_2^2)}{2\alpha} < \beta^2 < \beta.$$

Hence by Lemma 1 we have Re $p(z) > \beta$. That is Re $D^n f(z)^\alpha / (\alpha^n z^\alpha) > \beta$ which proves the result. $\square$

COROLLARY 1. *For $n \geqslant 1$, the class $\mathscr{J}_n^\alpha(\beta)$ consists of univalent functions only. In particular, the class of functions defined by Re $f'(f' + zf''(z)) > \beta$ in Remark* 1 (d) *also consists of univalent functions only.*

THEOREM 2. *Let $f \in \mathscr{J}_n^\alpha(\beta)$. Then $f(z)$ has the integral representation*

$$f(z) = \left(I_n \left(2\alpha^{2n+1} \int_0^z t^{2\alpha-1} p(t) dt \right)^{\frac{1}{2}} \right)^{\frac{1}{\alpha}}.$$

The integral, I_n, is the anti-derivative of D^n also defined by Salagean [6] as

$$I_n = I(I_{n-1} f(z)) = \int_0^z \frac{I_{n-1} f(t)}{t} dt$$

with $I_0 f(z) = f(z)$ and such that $I_n(D^n f(z)) = D^n(I_n f(z)) = f(z)$. Now the proof is as follows.

Proof. Since $f \in \mathscr{J}_n^\alpha(\beta)$ there exists $p \in P(\beta)$ such that from (2)

$$\frac{D((D^n f(z)^\alpha)^2)}{2\alpha^{2n+1} z^{2\alpha}} = p(z)$$

and

$$D^n f(z)^\alpha = \left(2\alpha^{2n+1} \int_0^z t^{2\alpha-1} p(t) dt \right)^{\frac{1}{2}}.$$

Applying the integral operator I_n on the last equation, we have

$$f(z)^\alpha = I_n \left(2\alpha^{2n+1} \int_0^z t^{2\alpha-1} p(t) dt \right)^{\frac{1}{2}}$$

and so

$$f(z) = \left(I_n \left(2\alpha^{2n+1} \int_0^z t^{2\alpha-1} p(t) dt \right)^{\frac{1}{2}} \right)^{\frac{1}{\alpha}}$$

as required. $\square$

THEOREM 3. *If* $f \in A$ *satisfies*

$$\operatorname{Re}\left(\frac{D^{n+1}f(z)^{\alpha}}{D^{n}f(z)^{\alpha}}+\frac{D^{n+2}f(z)^{\alpha}}{D^{n+1}f(z)^{\alpha}}\right)>\frac{4\alpha\beta+\beta-1}{2\beta}. \tag{3}$$

Then

$$\operatorname{Re}\frac{D^{n}f(z)^{\alpha}}{\alpha^{n}z^{\alpha}}\frac{D^{n+1}f(z)^{\alpha}}{\alpha^{n+1}z^{\alpha}}>2^{1-\frac{1}{\beta}},\ \ \frac{1}{2}\leqslant\beta<1,\ \ z\in E.$$

Proof. Let

$$p(z)=\frac{D^{n}f(z)^{\alpha}D^{n+1}f(z)^{\alpha}}{\alpha^{2n+1}z^{2\alpha}}.$$

Then we obtain

$$\frac{zp'(z)}{p(z)}=\frac{D^{n+1}f(z)^{\alpha}}{D^{n}f(z)^{\alpha}}+\frac{D^{n+2}f(z)^{\alpha}}{D^{n+1}f(z)^{\alpha}}-2\alpha.$$

By the hypothesis of the theorem,

$$\operatorname{Re}\left(1+\frac{zp'(z)}{p(z)}\right)=\operatorname{Re}\left(1+\frac{D^{n+1}f(z)^{\alpha}}{D^{n}f(z)^{\alpha}}+\frac{D^{n+2}f(z)^{\alpha}}{D^{n+1}f(z)^{\alpha}}-2\alpha\right)>\frac{3\beta-1}{2\beta}$$

which is equivalent to

$$\operatorname{Re}\left(\frac{D^{n+1}f(z)^{\alpha}}{D^{n}f(z)^{\alpha}}+\frac{D^{n+2}f(z)^{\alpha}}{D^{n+1}f(z)^{\alpha}}\right)>\frac{4\alpha\beta+\beta-1}{2\beta}.$$

Thus by Lemma 2, Re $p(z)>2^{1-\frac{1}{\beta}}$, $\frac{1}{2}\leqslant\beta<1$, which proves the result. □

COROLLARY 2. *If* $f \in A$ *satisfies the condition* (3), *then* $f \in \mathscr{I}_n^{\alpha}(2^{1-\frac{1}{\beta}})$.

By putting $n=0$, $\alpha=1$ and $\beta=1/2$, we have

COROLLARY 3. *Suppose*

$$\operatorname{Re}\left(\frac{zf''(z)}{f(z)}+\frac{zf'(z)}{f(z)}\right)>\frac{1}{2}.$$

Then

$$\frac{f'(z)f(z)}{z}>\frac{1}{2}.$$

If $n=\alpha=0$ and $\beta=1/2$, we have the following.

COROLLARY 4. *Suppose*

$$\operatorname{Re}\left(\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right)>-\frac{3}{2}.$$

Then

$$\operatorname{Re}\frac{zf'(z)}{f(z)}>\frac{1}{2}.$$

That is f *is starlike of order* $\frac{1}{2}$.

If $n=0$ and $\alpha=1/2$, we have

COROLLARY 5. *Suppose*

$$\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right)>\frac{3\beta-1}{2\beta}.$$

Then

$$\operatorname{Re} f'(z)>2^{1-\frac{1}{\beta}}.$$

If $\beta=1/2$ in the last corollary, we have

COROLLARY 6. *Suppose*

$$\operatorname{Re}\left(\frac{zf''(z)}{f'(z)}+1\right)>\frac{1}{2}.$$

Then

$$\operatorname{Re} f'(z)>\frac{1}{2}.$$

THEOREM 4. *Let* $f\in\mathscr{J}_n^{\alpha}(\beta)$. *Then*

$$|a_2|\leqslant\frac{2\alpha^n(1-\beta)}{(\alpha+1)^n(2\alpha+1)}$$

$$|a_3|\leqslant\frac{\alpha^n(1-\beta)}{(\alpha+1)(\alpha+2)^n}\max\{1,|1-\sigma_1|\}$$

where

$$\sigma_1=\frac{2\alpha(1-\beta)[(\alpha+1)^{2n}-\alpha^{n-1}(1-\alpha)(\alpha+2)^n]}{(\alpha+1)^{2n-1}(2\alpha+1)^2}.$$

Proof. For $f\in\mathscr{J}_n^{\alpha}(\beta)$, there exists $p\in P$ such that

$$((D^nf(z)^{\alpha})^2)'=2\alpha^{2n+1}z^{2\alpha-1}(\beta+(1-\beta)p(z)) \tag{4}$$

Expanding the right hand side of (4) in series form we have

$$\begin{aligned}[(D^nf(z)^{\alpha})^2]'&=2\alpha^{2n+1}z^{2\alpha-1}(\beta+(1-\beta)p(z))\\&=2\alpha^{2n+1}z^{2\alpha-1}+2\alpha^{2n+1}(1-\beta)p_1(z)z^{2\alpha}+2\alpha^{2n+1}(1-\beta)p_2(z)z^{2\alpha+1}+\dots\end{aligned} \tag{5}$$

Then integrating we have

$$(D^nf(z)^{\alpha})^2=\alpha^{2n}z^{2\alpha}+\frac{2\alpha^{2n+1}(1-\beta)p_1}{2\alpha+1}z^{2\alpha+1}+\frac{\alpha^{2n+1}(1-\beta)p_2}{\alpha+1}z^{2(\alpha+1)}+\dots$$

so that

$$D^n f(z)^\alpha = \alpha^n z^\alpha + \frac{\alpha^{n+1}(1-\beta)p_1}{2\alpha+1} z^{\alpha+1} + \frac{\alpha^{n+1}(1-\beta)(2\alpha+1)^2 p_2 - \alpha^{n+2}(1-\beta)^2(\alpha+1)p_1^2}{2(\alpha+1)(2\alpha+1)^2} z^{\alpha+2} + \dots$$

Applying the integral operator I_n on the above, we obtain

$$f(z)^\alpha = z^\alpha + \frac{\alpha^{n+1}(1-\beta)p_1}{(\alpha+1)^n(2\alpha+1)} z^{\alpha+1} + \frac{\alpha^{n+1}(1-\beta)(2\alpha+1)^2 p_2 - \alpha^{n+2}(1-\beta)^2(\alpha+1)p_1^2}{2(\alpha+1)(\alpha+2)^n(2\alpha+1)^2} z^{\alpha+2} + \dots$$

Hence we have

$$f(z) = z + \frac{\alpha^n(1-\beta)p_1}{(\alpha+1)^n(2\alpha+1)} z^2 + \frac{\alpha^n(1-\beta)}{2(\alpha+1)(\alpha+2)^n}\left(p_2 - \sigma_1 \frac{p_1^2}{2}\right) z^3 + \dots$$

where

$$\sigma_1 = \frac{2\alpha(1-\beta)[(\alpha+1)^{2n} - \alpha^{n-1}(1-\alpha)(\alpha+2)^n]}{(\alpha+1)^{2n-1}(2\alpha+1)^2}.$$

Comparing the coefficients of both sides of the last equation we have

$$a_2 = \frac{\alpha^n(1-\beta)p_1}{(\alpha+1)^n(2\alpha+1)}. \tag{6}$$

Thus applying the Caratheodory inequality, $|p_k| \leqslant 2$, to a_2 we obtain the desired bound.

Next we have

$$a_3 = \frac{\alpha^n(1-\beta)}{2(\alpha+1)(\alpha+2)^n}\left(p_2 - \sigma_1 \frac{p_1^2}{2}\right). \tag{7}$$

Taking $\sigma = \sigma_1$ in Lemma 3 we have $\left|p_2 - \sigma_1 \frac{p_1^2}{2}\right| \leqslant 2\max\{1, |1-\sigma_1|\}$ so that

$$|a_3| \leqslant \frac{\alpha^n(1-\beta)}{(\alpha+1)(\alpha+2)^n} \max\{1, |1-\sigma_1|\}$$

as desired. This completes the proof of the theorem. □

THEOREM 5. *Let $f \in \mathscr{J}_n^\alpha(\beta)$. Then for any real or complex number λ*

$$|a_3 - \lambda a_2^2| = \frac{(1-\beta)\alpha^n}{(\alpha+1)(\alpha+2)^n} \max\{1, |1-\sigma_2|\}$$

where

$$\sigma_2 = \frac{2\alpha(1-\beta)[\lambda\alpha^{2n-1} + (\alpha+1)^{2n+1} + (\alpha-1)(\alpha+1)(\alpha+2)^n\alpha^{n-1}]}{(\alpha+1)^{2n}(2\alpha+1)^2}.$$

Proof. From Theorem 4 using equations (6) and (7) we obtain

$$|a_3-\lambda a_2^2| = \frac{(1-\beta)\alpha^n}{(\alpha+2)^n(2\alpha+2)}\left|p_2-\sigma_2\frac{p_1^2}{2}\right|$$

where

$$\sigma_2 = \frac{2\alpha(1-\beta)[\lambda\alpha^{2n-1}+(\alpha+1)^{2n+1}+(\alpha-1)(\alpha+1)(\alpha+2)^n\alpha^{n-1}}{(\alpha+1)^{2n}(2\alpha+1)^2}.$$

Thus taking $\sigma=\sigma_2$ in Lemma 3, we have

$$|a_3-\lambda a_2^2| = \frac{(1-\beta)\alpha^n}{(\alpha+1)(\alpha+2)^n}\max\{1,|1-\sigma_2|\}$$

as required. □

Finally if we define

$$\frac{D^nf_j(z)^\alpha}{\alpha^nz^\alpha}\frac{D^{n+1}f_j(z)^\alpha}{\alpha^{n+1}z^\alpha} = \begin{cases}\beta+(1-\beta)(1+z), \text{ if } j=1,\\ \beta+(1-\beta)(1-z), \text{ if } j=2,\\ \beta+(1-\beta)\frac{1+z}{1-z}, \quad \text{ if } j=3.\end{cases}$$

Then right hand sides of the above equations are all functions in $P(\beta)$ so that the following functions $f_j(z)$ $j=1,2,3$ given by:

$$\begin{aligned}f_1(z) &= \left\{I_n\left(\alpha^nz^\alpha\frac{1+2\alpha(1+z)}{2\alpha+1}\right)^{\frac{1}{2}}\right\}^{\frac{1}{\alpha}}\\ &= z+\frac{\alpha^n(1-\beta)}{(\alpha+1)^n(2\alpha+1)}z^2+\frac{(1-\beta)^2\alpha^{n+1}[\alpha^{n-1}(1-\alpha)(\alpha+2)^n-(\alpha+1)^{2n}]}{2(\alpha+1)^{2n}(\alpha+2)^n(2\alpha+1)^2}z^3+\dots\end{aligned}$$

$$\begin{aligned}f_2(z) &= \left\{I_n\left(\alpha^nz^\alpha\frac{1+2\alpha(1-z)}{2\alpha+1}\right)^{\frac{1}{2}}\right\}^{\frac{1}{\alpha}}\\ &= z-\frac{\alpha^n(1-\beta)}{(\alpha+1)^n(2\alpha+1)}z^2+\frac{(1-\beta)^2\alpha^{n+1}[\alpha^{n-1}(1-\alpha)(\alpha+2)^n-(\alpha+1)^{2n}]}{2(\alpha+1)^{2n}(\alpha+2)^n(2\alpha+1)^2}z^3-\dots\end{aligned}$$

$$\begin{aligned}f_3(z) &= \left\{I_n\left(2\alpha^{2n+1}\int_0^z\frac{t^{2\alpha-1}+t^{2\alpha}}{1-t}dt\right)^{\frac{1}{2}}\right\}^{\frac{1}{\alpha}}\\ &= z+\frac{2\alpha^n(1-\beta)}{(2\alpha+1)(\alpha+1)^n}z^2\\ &\quad+(1-\beta)\frac{(\alpha+1)^{2n-1}[2\alpha(\alpha+1)(1+\beta)+1]+2\alpha^n(1-\beta)(1-\alpha)(\alpha+2)^n}{\alpha^{-n}(\alpha+1)^{2n}(2\alpha+1)^2(\alpha+2)^n}z^3+\dots\end{aligned}$$

are examples of functions in the class $\mathscr{J}_n^\alpha(\beta)$.

REFERENCES

[1] S. ABDULHALIM, *On a class of analytic functions involving Salagean differential operator*, Tamkang J. Math., **23**, (1) (1992), 51–58.
[2] K. O. BABALOLA, *On λ-Pseudo-Starlike functions*, Journal of Classical Analysis, **3**, (2) (2013), 137–147.
[3] K. O. BABALOLA AND T. O. OPOOLA, *Iterated integral transforms of Caratheodory functions and their applications to analytic and Univalent functions*, Tamkang Journal of Mathematics, **37**, (4) (2006), 355–366.
[4] K. O. BABALOLA AND T. O. OPOOLA, *On the Coefficients of a Certain Class of Analytic Functions*, Advances in Inequalities for Series, (2008), 1–13.
[5] T. O. OPOOLA, *On a new subclass of univalent functions*, Mathematica (Cluj) Tome, **36**, (59), no. 2, (1994), 195–200.
[6] G. S. SALAGEAN, *Subclasses of univalent functions*, Lecture Notes in Mathematics. Springer-Verlag, Berlin, Heidelberg and New York, **1013**, (1983), 362–372.
[7] R. SINGH, *On Bazilevic functions*, Proc. Amer. Math. Soc., **38**, (1973), 261–271.

(Received February 8, 2016)

F. M. Jimoh
Department of Physical Sciences
Al-Hikmah University
Ilorin, Nigeria
e-mail: folashittu@yahoo.com

K. O. Babalola
Department of Mathematics
University of Ilorin
Ilorin, Nigeria
e-mail: kobabalola@gmail.com