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ORLICZ–QUASI–CAUCHY DOUBLE SEQUENCE SPACES FOR RH–REGULAR MATRIX

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Abstract. In this research paper we pioneer, a new double sequence space called Orlicz-Quasi-Cauchy double sequence space, $(M,A,\Delta)_{OQC}$. We investigate continuity type properties of Orlicz double function defined on a double subset $A\times A$ of R^2 into R and study some vital results related to uniform continuity. In the last we also make an attempt to prove some topological properties of $(M,A,\Delta)_{OQC}$.

1. Introduction

An Orlicz function M is a function, which is continuous, non-decreasing and convex with $M(0)=0$, $M(x)>0$ for $x>0$ and $M(x)\longrightarrow\infty$ as $x\longrightarrow\infty$.

Lindenstrauss and Tzafriri [13] used the idea of Orlicz function to define the following sequence space. Let w be the space of all real or complex sequences $x=(x_k)$, then

$$\ell_M=\Big\{x\in w:\sum_{k=1}^{\infty}M\Big(\frac{|x_k|}{\rho}\Big)<\infty,\ \text{for some}\ \rho>0\Big\}$$

which is called as an Orlicz sequence space. The space ℓ_M is a Banach space with the norm

$$||x||=\inf\Big\{\rho>0:\sum_{k=1}^{\infty}M\Big(\frac{|x_k|}{\rho}\Big)\leqslant 1\Big\}.$$

It is shown in [13] that every Orlicz sequence space ℓ_M contains a subspace isomorphic to $\ell_p (p\geqslant 1)$. The Δ_2-condition is equivalent to $M(Lx)\leqslant kLM(x)$ for all values of $x\geqslant 0$ and for $L>1$.

The study of single Quasi-Cauchy sequences have been discussed in ([9], [10], [17]). In ([9], [10]) Çakallı and Çakallı and Hazarika have introduced and studied the statistical Quasi-Cauchy sequences and Ideal Quasi-Cauchy sequences respectively. Also in [17], Hazarika studied ϕ-statistical Quasi-Cauchy sequences in details.

The notion of difference sequence spaces was introduced by Kızmaz [12],who studied the difference sequence spaces $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$. For Z a given sequence space, we have

$$Z(\Delta)=\{x=(x_k)\in w:(\Delta x_k)\in Z\}$$

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for $Z = c, c_0$ and l_∞ where $\Delta x = (\Delta x_k) = (x_k - x_{k+1})$. The difference sequence spaces was also discussed in (Altınok et. al. [3], Isik [18], Tripathy et al. [34]). Similarly, we can define difference operators on double sequence spaces as:

$$\begin{aligned}\Delta x_{k,l} &= (x_{k,l} - x_{k,l+1}) - (x_{k+1,l} - x_{k+1,l+1}) \\ &= x_{k,l} - x_{k,l+1} - x_{k+1,l} + x_{k+1,l+1}\end{aligned}$$

In 1900, Pringsheim [24] introduced the concept of convergence of real double sequences. Four year later, Hardy [15] introduced the notion of regular convergence for double sequences.

The initial work on double sequences is found in Bromwich [5]. Later on Móricz [19], Móricz and Rhoades [20], Tripathy ([32], [33]), Başarır and Sonalcan [4] and many others. Quite recently, Zeltser [35] in her Ph.D thesis has essentially studied both the theory of topological double sequence spaces and the theory of summability of double sequences. Mursaleen and Edely [23] have recently introduced the statistical convergence and Cauchy convergence for double sequences and given the relation between statistical convergent and strongly Cesàro summable double sequences. Nextly, Mursaleen [21] and Mursaleen and Edely [22] have defined the almost strong regularity of matrices for double sequences and applied these matrices to establish a core theorem and introduced the M-core for double sequences and determined those four dimensional matrices transforming every bounded double sequences $x = (x_{jk})$ into one whose core is a subset of the M-core of x. Recently Quasi Cauchy double sequence spaces was studied by Patterson and Cakalli [28]. A considerable number of papers which appeared in recent years study double sequences from various point of view (see [1], [2], [6], [7], [8], [11], [16], [26], [27], [29], [30]).

DEFINITION 1. A double sequence $x = \{x_{k,l}\}$ is Cauchy provided that, given an $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that $|x_{k,l} - x_{s,t}| < \varepsilon$ whenever $k, l, s, t > N$.

DEFINITION 2. By the convergence of a double sequence we mean the convergence in the Pringsheim sense i.e. a double sequence $x = (x_{k,l})$ has *Pringsheim limit* L (denoted by $P - \lim x = L$) provided that given $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that $|x_{k,l} - L| < \varepsilon$ whenever $k, l > n$. We shall write more briefly as P-convergent.

If $\lim x = \infty$, (equivalently, for every $\varepsilon > 0$ there are $n_1, n_2 \in N$) such that $|x_{m,n}| > M$ whenever $m > n_1, n > n_2$, then $x = \{x_{m,n}\}$ is said to be definitely divergent. A double sequence $x = \{x_{m,n}\}$ is bounded if there is an $M > 0$ such that $|x_{m,n}| < M$ for all $m, n \in N$. Notice that a P-convergent double sequence need not be bounded.

DEFINITION 3. [25] A double sequence y is a double subsequence of x provided that there exist increasing index sequences $\{n_j\}$ and $\{k_j\}$ such that if $\{x_j\} = \{x_{n_j,k_j}\}$, then y is formed by

$$\begin{array}{cccc} x_1 & x_2 & x_5 & x_{10} \\ x_4 & x_3 & x_6 & - \\ x_9 & x_8 & x_7 & - \\ - & - & - & - \end{array}$$

DEFINITION 4. Let $A = (a_{mnkl})$ denote a four dimensional summability method that maps the complex double sequences x into the double sequence Ax where the mn^{th} term to Ax is as follows:

$$(Ax)_{mn} = \sum_{k,l=1,1}^{\infty,\infty} a_{mnkl} x_{k,l}.$$

Such a transformation is said to be non-negative if a_{mnkl} is nonnegative for all n,m,k and l.

DEFINITION 5. A 4-dimensional matrix A is said to be RH-regular if it maps every bounded P-convergent sequence into a P-convergent sequence with the same P-limit.

An RH-regular matrix is characterized as follows:

LEMMA 1. (Robison [31] and Hamilton [14]) *A four dimensional matrix A is RH-regular if and only if*

RH_1: $P-\lim_{mn} a_{mnkl} = 0$ *for each* $(k,l) \in \mathbb{N}^2$;

RH_2: $P-\lim_{mn} \sum_{k,l} |a_{mnkl}| = 1$;

RH_3: $P-\lim_{mn} \sum_{l} |a_{mnkl}| = 0$ *for each* k;

RH_4: $P-\lim_{mn} \sum_{k} |a_{mnkl}| = 0$ *for each* l;

RH_5: $\sum_{k,l} |a_{mnkl}|$ *is P-convergent;*

RH_6: *there exists finite positive integers A and B such that* $\sum_{j,k>B} a_{mnkl} < A$ *for each* $(m,n) \in \mathbb{N}^2$.

Let M be an Orlicz function and $A = (a_{mnkl})$ is a non-negative four dimensional RH-regular matrix. A double sequence $x = (x_{k,l})$ is called Orlicz-Quasi-Cauchy if for given $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that

$$\frac{1}{nm} \sum_{k,l=0,0}^{\infty,\infty} a_{mnkl} \left[M|\Delta x_{k,l}| \right] < \varepsilon$$

where $\Delta x_{k,l} = \max_{r,s=1 \text{ and/or } 0} \{x_{k,l} - x_{k+r,l+s}\}$, whenever $k,l > N$. This Orlicz-Quasi-Cauchy double sequence space is denoted by $(M,A,\Delta)_{OQC}$ and is defined as

$$(M,A,\Delta)_{OQC} = \left\{ x = (x_{k,l}) : \frac{1}{nm} \sum_{k,l=0,0}^{\infty,\infty} a_{mnkl} \left[M|\Delta x_{k,l}| \right] < \varepsilon \right\}$$

Any P-convergent double Orlicz sequence is Orlicz-Quasi-Cauchy, so any regularly convergent double Orlicz sequence is Orlicz-Quasi-Cauchy. Any Cauchy double Orlicz sequence is Orlicz-Quasi-Cauchy. Any subsequence of a P-convergent double sequence is P-convergent. Any subsequence of a Cauchy double Orlicz sequence is

Cauchy. But for Orlicz-Quasi-Cauchy double sequence, the situation is different. There are subsequence of Orlicz-Quasi-Cauchy double sequence which are not Orlicz-Quasi-Cauchy.

DEFINITION 6. A double Orlicz function M defined on a double subset $A \times A$ of R^2 into R is called double sequentially continuous at a point L of $A \times A$ if $M(\Delta x)$ is P-convergent to $M(L)$, whenever $\Delta x = (\Delta x_{k,l})$ is a P-convergent double Orlicz sequence of points in $A \times A$ with P-limit L. If M is double Orlicz sequentially continuous at every point of $A \times A$, we say M is double Orlicz sequentially continuous on $A \times A$.

We note that any continuous function at a point L of $A \times A$ is also double Orlicz sequentially continuous on $A \times A$. The converse is also true.

2. Continuity and uniformly continuity properties

In this section we made an effort to obtain interesting results related to continuity, sequentially continuity and uniform continuity of double Orlicz-Quasi-Cauchy sequences.

THEOREM 1. *If double Orlicz function M defined on a double subset $A \times A$ of R^2 is double Orlicz sequentially continuous at L, then it is continuous.*

Proof. Suppose M is not continuous at L. Then there is an $\varepsilon_0 > 0$ such that for any $\delta > 0$ there exists an x_δ so that $|\Delta x_1(\delta) - L_1| < \delta$ and $|\Delta x_2(\delta) - L_2| < \delta$ but $|M(\Delta x_1(\delta) - \Delta x_2(\delta)) - M(L)| \geqslant \varepsilon_0$. It is easy to construct a convergent double Orlicz sequence with limit L whose transformed sequence is not convergent to $M(L)$. Thus M is not double Orlicz sequentially continuous at L. This contradiction completes the proof of the theorem. □

From the above theorem we conclude that a double-Orlicz function M defined on a double subset $A \times A$ of R^2 is double Orlicz sequentially continuous at a point L if and only if it is continuous.

THEOREM 2. *If double-Orlicz function M defined on a double subset $A \times A$ of R^2 preserves double Orlicz-Quasi-Cauchy sequences from $A \times A$, then it is continuous.*

Proof. Suppose that M preserves double Orlicz-Quasi-Cauchy sequences from $A \times A$. Let $x = (x_{i,j})$ be a double sequence define by

$$\begin{matrix} x_{1,1} & x_{1,2} & x_{1,3} & \cdots \\ x_{2,1} & x_{2,2} & x_{2,3} & \cdots \\ x_{3,1} & x_{3,2} & x_{3,3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{matrix}$$

be any P-convergent double sequence with P-limit L. Applying the double difference operator, the above double sequence reduced to

$$\begin{array}{ccccccc} \Delta x_{1,1}L & \Delta x_{1,2}L & \Delta x_{1,3}L & \cdots & & & \\ L & L & L & L & L & L & \cdots \\ \Delta x_{2,1}L & \Delta x_{2,2}L & \Delta x_{3,3}L & \cdots & & & \\ L & L & L & L & L & L & \cdots \\ \Delta x_{3,1}L & \Delta x_{3,2}L & \Delta x_{3,3}L & \cdots & & & \\ L & L & L & L & L & L & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

is also P-convergent with P-limit L. Since any convergent double-Orlicz sequence space is Orlicz-Quasi-Cauchy sequence, so the transformed sequence $M(\Delta x) = M|\Delta x_{k,l}|$ of the sequence Δx is Orlicz-Quasi-Cauchy. Thus it follows that

$$\begin{array}{ccccccc} M|\Delta x_{1,1}|M(L) & M|\Delta x_{1,2}|M(L) & M|\Delta x_{1,3}|M(L) & \cdots & & & \\ M(L) & M(L) & M(L) & M(L) & M(L) & M(L) & \cdots \\ M|\Delta x_{2,1}|M(L) & M|\Delta x_{2,2}|M(L) & M|\Delta x_{3,3}|M(L) & \cdots & & & \\ M(L) & M(L) & M(L) & M(L) & M(L) & M(L) & \cdots \\ M|\Delta x_{3,1}|M(L) & M|\Delta x_{3,2}|M(L) & M|\Delta x_{3,3}|M(L) & \cdots & & & \\ M(L) & M(L) & M(L) & M(L) & M(L) & M(L) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \end{array}$$

Using the non-negative four dimensional RH-regular matrix on this double sequence space, we get Orlicz-Quasi-Cauchy double sequence. From this it follows that $\{M|\Delta x_{i,j}|\}$ is a P-convergent Orlicz double sequence with P-limit $M(L)$. By combining the result of Theorem 1 we get that an Orlicz function M is Continuous. This completes the proof. $\square$

COROLLARY 1. *Suppose that $I \times I$ is a two dimentional interval and*

$$\begin{array}{ccccccc} a_{1,1} & b_{1,1} & a_{1,2} & b_{1,2} & a_{1,3} & b_{1,3} & \cdots \\ d_{1,1} & c_{1,1} & d_{1,2} & c_{1,2} & d_{1,3} & c_{1,3} & \cdots \\ a_{2,1} & b_{2,1} & a_{2,2} & b_{2,2} & a_{2,3} & b_{2,3} & \cdots \\ d_{2,1} & c_{2,1} & d_{2,2} & c_{2,2} & d_{2,3} & c_{2,3} & \cdots \\ a_{3,1} & b_{3,1} & a_{3,2} & b_{3,2} & a_{3,3} & b_{3,3} & \cdots \\ d_{3,1} & c_{3,1} & d_{3,2} & c_{3,2} & d_{3,3} & c_{3,3} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

is a double sequence of ordered pairs in $I \times I$ with

$$\lim_i |a_{i,j} - b_{i,j}| = \lim_i |a_{i,j} - c_{i,j}| = \lim_i |a_{i,j} - d_{i,j}| = 0$$

then there exists a double Orlicz quasi Cauchy sequence $\{M|\Delta x_{i,j}|\}$ with the property that for any order pair of integers $(i,j) : i,j > 1$, then there exists an ordered pair $(\bar{i},\bar{j}); \bar{i},\bar{j} > 1$ such that

$$(a_{i,j}, b_{i,j}) = (\Delta x_{\bar{i},\bar{j}}, \Delta x_{\bar{i},\bar{j}+1})$$

$$(a_{i,j}, c_{i,j}) = (\Delta x_{\bar{i},\bar{j}}, \Delta x_{\bar{i}+1,\bar{j}+1})$$
$$(a_{i,j}, d_{i,j}) = (\Delta x_{\bar{i},\bar{j}}, \Delta x_{\bar{i}+1,\bar{j}}).$$

Proof. See [28]. □

THEOREM 3. *Suppose that $I \times I$ is a two dimensional interval. Then a two dimensional Orlicz real-valued function is uniformly continuous on $I \times I$ if and only if it is defined on $I \times I$ and preserves double Orlicz-Quasi-Cauchy sequences from $I \times I$.*

Proof. Clearly, the two dimensional uniformly continuous function preserves double Orlicz-Quasi-Cauchy sequence.

Conversely, Suppose that M defined on $I \times I$ is not uniformaly continuous. Then there exists an $\varepsilon > 0$ such that for any $\delta > 0$ there exist (a,b), $(\bar{a},\bar{b}) \in I \times I$, with $\sqrt{(a-\bar{a})^2 + (b-\bar{b})^2} < \delta$ but $|M(a,b) - M(\bar{a},b)| \geqslant \varepsilon$, $|M(a,b) - M(a,\bar{b})| \geqslant \varepsilon$, and $|M(a,b) - M(\bar{a},\bar{b})| \geqslant \varepsilon$, respectively. Then by the corollary here exists a double Orlicz-Quasi-Cauchy sequence such that for any ordered pair $(i,j) : i \geqslant 1,\ j \geqslant 1$, then there exists an ordered pairs integers $(\bar{i},\bar{j})$ with $a_{i,j} = \Delta x_{\bar{i},\bar{j}}$ and $b_{i,j} = \Delta x_{\bar{i+1},\bar{j+1}}$. This implies that

$$|M(\Delta x_{\bar{i}}, \Delta x_{\bar{j}}) - M(\Delta x_{\bar{i}+1}, \Delta x_{\bar{j}})| \geqslant \varepsilon,$$
$$|M(\Delta x_{\bar{i}}, \Delta x_{\bar{j}}) - M(\Delta x_{\bar{i}}, \Delta x_{\bar{j}+1})| \geqslant \varepsilon,$$

and

$$|M(\Delta x_{\bar{i}}, \Delta x_{\bar{j}}) - M(\Delta x_{\bar{i}+1}, \Delta x_{\bar{j}+1})| \geqslant \varepsilon,$$

Thus, $\{M(\Delta x_i, \Delta x_j)\}$ is not Orlicz-Quasi-Cauchy. Therefore, M does not preserves double Orlicz Quasi Cauchy sequence. □

THEOREM 4. *Suppose that M is Orlicz double function defined on bounded double interval $I \times I$. Then M, an Orlicz function is uniformaly continuous on $I \times I$ if and only if the image under M of any Cauchy double sequence in $I \times I$ is Orlicz-Quasi-Cauchy.*

Proof. By using the concept of Theorem 4, clearly the image of any double Cauchy under factorable function is Orlicz-Quasi-Cauchy. Conversely, suppose the image of every Cauchy double sequence is Orlicz-Quasi-Cauchy but the Orlicz function to be uniformly continuous. Then there exists an $\varepsilon > 0$ such that for any $\delta > 0$ there exist $(x,y), (\bar{x},\bar{y}) \in I \times I$, with $\sqrt{(x-\bar{x})^2 + (y-\bar{y})^2} < \delta$ but $|M(x,y) - M(\bar{x},y)| \geqslant \varepsilon$, $|M(x,y) - M(x,\bar{y})| \geqslant \varepsilon$, and $|M(x,y) - M(\bar{x},\bar{y})| \geqslant \varepsilon$, respectively.

For each $(m,n) : m,n \geqslant 1$, for fix double sequence $(\Delta x_m, \Delta y_n)$ and $(\Delta \bar{x_m}, \Delta \bar{y_n})$ in $I \times I$ with $\sqrt{(\Delta x_m - \Delta \bar{x_m})^2 + (\Delta y_n - \Delta \bar{y_n})^2} < \delta$

$$|M(\Delta x_m, \Delta y_n) - M(\Delta \bar{x_m}, \Delta y_n)| \geqslant \varepsilon,$$

$$|M(\Delta x_m, \Delta y_n) - M(\Delta x_m, \Delta \bar{y_n})| \geqslant \varepsilon,$$

and

$$|M(\Delta x_m, \Delta y_n) - M(\Delta x_{\bar{m}}, \Delta y_{\bar{n}})| \geqslant \varepsilon,$$

respectively. Since $I \times I$ is bounded there exists a P-convergent subsequence by a simple extension of Bolzano-Weierstrass theorem, say $\{\Delta x_{k,l}\}$. The following double sequence

$$\begin{matrix} \Delta x_{1,1} & \Delta y_{1,2} & \Delta x_{1,3} & \Delta y_{1,4} & \Delta x_{1,5} & \Delta y_{1,6} & \cdots \\ \Delta y_{2,1} & \Delta x_{2,2} & \Delta y_{2,3} & \Delta x_{2,4} & \Delta y_{2,5} & \Delta x_{2,6} & \cdots \\ \Delta x_{3,1} & \Delta y_{3,2} & \Delta x_{3,3} & \Delta y_{3,4} & \Delta x_{3,5} & \Delta y_{3,6} & \cdots \\ \Delta y_{4,1} & \Delta x_{4,2} & \Delta y_{4,3} & \Delta x_{4,4} & \Delta y_{4,5} & \Delta x_{4,6} & \cdots \\ \Delta x_{5,1} & \Delta y_{5,2} & \Delta x_{5,3} & \Delta y_{5,4} & \Delta x_{5,5} & \Delta y_{5,6} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{matrix}$$

is P-convergent. However the following image of Cauchy sequence

$$\begin{matrix} M(\Delta x_1, \Delta x_1) & M(\Delta y_1, \Delta y_2) & M(\Delta x_1, \Delta x_3) & M(\Delta y_1, \Delta y_4) & M(\Delta x_1, \Delta x_5) & M(\Delta y_1, \Delta y_6) & \cdots \\ M(\Delta y_2, \Delta y_1) & M(\Delta x_2, \Delta x_2) & M(\Delta y_2, \Delta y_3) & M(\Delta x_2, \Delta x_4) & M(\Delta y_2, \Delta y_5) & M(\Delta x_2, \Delta x_6) & \cdots \\ M(\Delta x_3, \Delta x_1) & M(\Delta y_3, \Delta y_2) & M(\Delta x_3, \Delta x_3) & M(\Delta y_3, \Delta y_4) & M(\Delta x_3, \Delta x_5) & M(\Delta y_3, \Delta y_6) & \cdots \\ M(\Delta y_4, \Delta y_1) & M(\Delta x_4, \Delta x_2) & M(\Delta y_4, \Delta y_3) & M(\Delta x_4, \Delta x_4) & M(\Delta y_4, \Delta y_5) & M(\Delta x_4, \Delta x_6) & \cdots \\ M(\Delta x_5, \Delta x_1) & M(\Delta y_5, \Delta y_2) & M(\Delta x_5, \Delta x_3) & M(\Delta y_5, \Delta y_4) & M(\Delta x_5, \Delta x_5) & M(\Delta y_5, \Delta y_6) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{matrix}$$

is not a Orlicz Quasi Cauchy. Thus we have a contradiction. □

3. Topological properties of Orlicz-Quasi-Cauchy double sequence space

This sections is devoted to the study of some topological properties like linearity and completeness of $(M,A,\Delta)_{OQC}$.

THEOREM 5. *Let $A = (a_{mnkl})$ be RH-regular matrix, M be an Orlicz function. Then an Orlicz-Quasi-Cauchy double sequence space, $(M,A,\Delta)_{OQC}$ is a linear space over the field of complex numbers $\mathbb{C}$.*

Proof. Let $x,y \in (M,A,\Delta)_{OQC}$ and for $\alpha, \beta \in \mathbb{C}$ there exist integers M_α and N_β such that $|\alpha| < M_\alpha$ and $|\beta| < N_\beta$. Since M is an Orlicz functions, so we have

$$\frac{1}{mn}\sum_{k,l=0,0}^{\infty,\infty} a_{mnkl}\left[M|\alpha\Delta x_{k,l} + \beta\Delta y_{k,l}|\right] \leqslant M_\alpha \frac{1}{mn}\sum_{k,l=0,0}^{\infty,\infty} a_{mnkl}\left[M|\Delta x_{k,l}|\right] + N_\beta \frac{1}{nm}\sum_{k,l=0,0}^{\infty,\infty} a_{mnkl}\left[M|\Delta y_{k,l}|\right].$$

Thus, $\alpha x + \beta y \in (M,A,\Delta)_{OQC}$ for all k,l. Hence $(M,A,\Delta)_{OQC}$ is a linear space. This completes the proof of the theorem. □

THEOREM 6. *Let M be an Orlicz function, $A=(a_{mnkl})$ be a RH-regular 4-dimensional matrix summability method such that* $\sup_{mn}\sum_{k,l=0,0}^{\infty,\infty} a_{mnkl}<\infty$*, Then $(M,A,\Delta)_{OQC}$ is a complete topological linear space with the paranorm defined by*

$$g(x)=\sup_{mn}\sum_{k,l=0,0}^{\infty,\infty} a_{mnkl}\left[M|\Delta x_{k,l}|\right].$$

Proof. Let (x^q_{kl}) be a Cauchy sequence in $(M,A,\Delta)_{OQC}$. Then, we write $g(x^q-x^t)\to 0$ as $q,t\to\infty$ for all m,n, we have

$$\sum_{k,l=0,0}^{\infty,\infty} a_{mnkl}\left[M|\Delta x^q_{k,l}-\Delta x^t_{k,l}|\right]\to 0. \tag{1}$$

Thus for each fixed k and l as $q,t\to\infty$, since $A=(a_{mnkl})$ is nonnegative, we are granted that

$$M|\Delta x^q_{k,l}-\Delta x^t_{k,l}|\to 0$$

and by continuity of M, $(x^q_{k,l})$ is a Cauchy sequence in $\mathbb{C}$ for each fixed k and l.

Since $\mathbb{C}$ is complete as $t\to\infty$, we have $x^q_{k,l}\to x_{k,l}$ for each (k,l) Now from equation (1), we have for $\varepsilon>0$, there exists a natural number N such that

$$\sum_{k,l=0,0\ q,t>N}^{\infty,\infty} a_{mnkl}\left[M|\Delta x^q_{k,l}-\Delta x^t_{k,l}|\right]<\varepsilon \text{ for all } m,n. \tag{2}$$

Since for any fixed natural number R, from equation (2) we have

$$\sum_{k,l\leqslant R\ q,t>N}^{\infty,\infty} a_{mnkl}\left[M|\Delta x^q_{kl}-\Delta x^t_{k,l}|\right]<\varepsilon \text{ for all } m,n$$

by letting $t\to\infty$ in the above expression we obtain

$$\sum_{k,l\leqslant R\ q>N}^{\infty,\infty} a_{mnkl}\left[M|\Delta x^q_{k,l}-\Delta x_{k,l}|\right]<\varepsilon.$$

Since R is arbitrary, by letting $R\to\infty$ we obtain

$$\sum_{k,l=0,0}^{\infty,\infty} a_{mnkl}\left[M|\Delta x^q_{k,l}-\Delta x_{k,l}|\Big)^{p_{k,l}}\right]<\varepsilon \text{ for all } m,n.$$

Thus $g(x^q-x)\to 0$ as $q\to\infty$. This proves that $(M,A,\Delta)_{OQC}$ is a complete topological linear space. $\square$

THEOREM 7. *Let M be an Orlicz function, $A=(a_{mnkl})$ be a RH-regular matrix such that* $\sup_{mn}\sum_{k,l=0,0}^{\infty,\infty} a_{mnkl}<\infty$ *and* $\beta=\lim_{t\to\infty}\frac{M(t)}{t}>\infty$*. Then*

$$(A,\Delta)_{OQC}=(M,A,\Delta)_{OQC}.$$

Proof. In order to prove that $(A,\Delta)_{OQC} = (M,A,\Delta)_{OQC}$. It is sufficient to show that $(M,A,\Delta)_{OQC} \subseteq (A,\Delta)_{OQC}$. Now, let $\beta > 0$. By definition of β, we have $M(t) \geqslant \beta t$ for all $t \geqslant 0$. Since $\beta > 0$, we have $t \leqslant \frac{1}{\beta} M(t)$ for all $t \geqslant 0$.

Let $x = (x_{kl}) \in (M,A,\Delta)_{OQC}$. Thus, we have

$$\sum_{k,l=0,0}^{\infty,\infty} a_{mnkl} \left[|\Delta x_{k,l}| \right] \leqslant \frac{1}{\beta} \sum_{k,l=0,0}^{\infty,\infty} a_{mnkl} \left[M|\Delta x_{k,l}| \right].$$

This completes the proof. □

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