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ON STATISTICAL CONVERGENCE WITH RESPECT TO MEASURE

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Abstract. Several notions of convergence for subsets of metric spaces appear in the literature. In this paper, for real valued measurable functions defined on a measurable space $(X, \mathscr{M}, \mu)$, we obtain a statistical version of Lebesque's bounded convergence theorem (when $\mu(X) < \infty$) and examine the validity of the classical theorems of Measure Theory for statistical convergences.

1. Introduction and background

Let us start with fundamental definitions from the literature. The natural density of a set K of positive integers is defined by

$$\delta(K) := \lim_{n\to\infty} \frac{1}{n}|\{k \leqslant n : k \in K\}|,$$

where $|k \leqslant n : k \in K|$ denotes the number of elements of K not exceeding n.

Statistical convergence of sequences of points was introduced by [6]. Schoenberg [17] established some basic properties of statistical convergence and also studied the concept as a summability method. Later, this concept has been generalized in many directions. More details on this matter and on applications of this concept can be found in [1].

A sequence $x = (x_k)$ is said to be statistically convergent to the number ξ if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n}|\{k \leqslant n : |x_k - \xi| \geqslant \varepsilon\}| = 0.$$

In this case we write $st - \lim x_k = \xi$. Statistical convergence is a natural generalization of ordinary convergence. If $\lim x_k = \xi$, then $st - \lim x_k = \xi$. The converse does not hold in general.

Regarding statistical convergence of numerical sequences we have the following well-known proposition.

PROPOSITION 1. [15] *Let (x_n) be a sequence in $\mathbb{R}$, and $\xi \in \mathbb{R}$. Then*

$$(x_n) \overset{st}{\to} \xi \Leftrightarrow \exists K = \{k_1 < k_2 < \ldots < k_n < \ldots\} \subseteq \mathbb{N} :$$

$$d(K) = 1 \text{ and } \left(x_{n_k}\right) \to \xi$$

(By $\left(x_{n_k}\right) \to \xi$ we denote the usual convergence).

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The corresponding notion of convergence for functions of a real variable was developed in the early 1990's and has recently been independently investigated by Moricz. C. Papachristodoulos [14] introduced the concept on statistical convergence of sequences of measurable functions and studied some properties of this convergence.

DEFINITION 1. [11] Let f_n, f be measurable functions ($n = 1,2,\ldots$) on X. We say that the sequence $(f_n)_{n\in\mathbb{N}}$ converges statistically in measure or asymptotic statistically to f, if and only if

$$(\forall\varepsilon > 0),\quad \mu\left(\{x\in X : |f_n(x) - f(x)| \geqslant \varepsilon\}\right) \overset{st}{\to} 0,\quad n\to\infty.$$

We write

$$f_n \overset{st-\mu}{\to} f,\ n\to\infty.$$

THEOREM 1. [7] *(F. Riesz) Let (f_n) be a sequence of functions which converges in measure to the function f. Then there exists a subsequence*

$$f_{n_1}(x),\ f_{n_2}(x),\ f_{n_3}(x),\ldots \ \textit{where}\ (n_1 < n_2 < n_3 < \ldots)$$

which converges to the function $f(x)$ almost everywhere.

In [9] the concept of $\mathscr{F}$-convergence, generated by some filter was introduced. The concept of a monotone close and a right filter were also defined. Based on these concepts an analogues of classical theorems of real analysis as Lebesgue, Egorov, Riesz and Fatou, with respect to $\mathscr{F}$-convergence were established.

We will obtain a statistical version of Lebesque's bounded convergence theorem (when $\mu(X) < \infty$) in the following section.

2. Statistical version of Lebesque's convergence theorem

The Lebesque's bounded convergence theorem, a classical result of measure theory, has been generalized by many authors in various directions. We are going to present a statistical version of this theorem. Assume that $(X, \mathscr{M}, \mu)$ is a measure space. We consider real valued measurable functions defined on X almost everywhere.

THEOREM 2. *Assume $(X, \mathscr{M}, \mu)$ is a finite measure space, $\mu(X) < \infty$ and let a sequence $f_1(x),\ f_2(x),\ f_3(x),\ldots$ of bounded measurable functions converge statistically in measure to the bounded measurable function $F(x)$;*

$$f_n(x) \overset{st-\mu}{\to} F(x),\quad n\to\infty$$

be defined on X. If there exists a constant M such that for almost all x,

$$d\{\, n : |f_n(x)| \geqslant M\} = 0,$$

then

$$st - \lim_{n\to\infty} \int_X f_n(x)dx = \int_X F(x)dx. \tag{1.1}$$

Proof. First of all, we note that for almost all $x \in X$

$$|F(x)| \leqslant M. \tag{1.2}$$

According to the statistical version of Riesz theorem [8], if $K = \{n_i : n_i \leqslant n_{i+1},\, i \in \mathbb{N}\}$, with $d(K) = 1$, then

$$d\ (m : |f_{n_m}(x) - F(x)| < \varepsilon) = 1,$$

and

$$d\ \{m : |f_{n_m}(x)| \leqslant M\} = 1$$

for each $\varepsilon > 0$ and for each $x \in X$.

Since, for each $m \in \mathbb{N}$,

$$|F(x)| \leqslant |F(x) - f_{n_m}(x)| + |f_{n_m}(x)| < \varepsilon + M,$$

we have $|F(x)| < M + \varepsilon$.

Hence, we get

$$|F(x)| \leqslant M,$$

which leads to (1.2).

Now let $\sigma > 0$ be a positive number. Set

$$A_n(\sigma) = \{x \in X : |f_n(x) - F(x)| \geqslant \sigma\}, \quad B_n(\sigma) = \{x \in X : |f_n(x) - F(x)| < \sigma\}.$$

Then

$$\left\{n \leqslant k : \left| \int_X f_n(x)dx - \int_X F(x)dx \right| \geqslant \sigma \right\}$$

$$\subset \left\{n \leqslant k : \int_X |f_n(x)dx - F(x)|\, dx \geqslant \sigma \right\}$$

$$= \left\{n \leqslant k : \int_{A_n(\sigma)} |f_n(x)dx - F(x)|\, dx \geqslant \frac{\sigma}{2} \right\}$$

$$\cup \left\{n \leqslant k : \int_{B_n(\sigma)} |f_n(x)dx - F(x)|\, dx \geqslant \frac{\sigma}{2} \right\}.$$

It is obvious that for almost all $x \in X$,

$$\{n \leqslant k : |f_n(x) - F(x)| \geqslant 2M\} \subset \{n \leqslant k : |f_n(x)| \geqslant M\},$$

for almost all x of the set $A_n(\sigma)$.

Since

$$d\ (n:|f_n(x)-F(x)|\geqslant 2M)\leqslant d\quad (n:|f_n(x)|\geqslant M)$$

and also

$$d\ (n:|f_n(x)|\geqslant M)=0$$

then, we have

$$d\ (n:|f_n(x)-F(x)|\geqslant 2M)=0.$$

The first law of the mean implies that

$$\int\limits_{A_n(\sigma)} |f_n(x)dx-F(x)|\,dx\leqslant 2M.m\left(A_n\left(\sigma\right)\right) \tag{1.3}$$

Also it is easy to verify

$$\left\{n\leqslant k:\int\limits_{A_n(\sigma)} |f_n(x)dx-F(x)|\,dx\geqslant\frac{\sigma}{2}\right\}\subset\left\{n\leqslant k:mA_n\left(\sigma\right)\geqslant\frac{\sigma}{4M}\right\}$$

for almost all $x\in X$ and $\sigma>0.$ Since

$$d\left\{n:\int\limits_{A_n(\sigma)} |f_n(x)dx-F(x)|\,dx\geqslant\frac{\sigma}{2}\right\}\leqslant d\left\{n:m\left(A_n\left(\sigma\right)\right)\geqslant\frac{\sigma}{4M}\right\}$$

and

$$d\left\{n:m\left(A_n\left(\sigma\right)\right)\geqslant\frac{\sigma}{4M}\right\}=0,$$

then, we have

$$d\left\{n:\int\limits_{A_n(\sigma)} |f_n(x)dx-F(x)|\,dx\geqslant\frac{\sigma}{2}\right\}=0.$$

On the other hand, again by the first law of the mean

$$\int\limits_{B_n(\sigma)} |f_n(x)dx-F(x)|\,dx\leqslant\sigma.m\left(B_n\left(\sigma\right)\right)\leqslant\sigma.m\left(X\right) \tag{1.4}$$

Again it is easy to verify

$$\left\{n\leqslant k:\int\limits_{B_n(\sigma)} |f_n(x)dx-F(x)|\,dx\geqslant\sigma.m\left(X\right)\right\}\subset\left\{n\leqslant k:m\left(B_n\left(\sigma\right)\right)\geqslant m\left(X\right)\right\}$$

for almost all $x\in X$ and $\sigma>0.$ Since

$$d\left\{n:\int\limits_{B_n(\sigma)} |f_n(x)dx-F(x)|\,dx\geqslant\sigma.m\left(X\right)\right\}\leqslant d\left\{n:m\left(B_n\left(\sigma\right)\right)\geqslant m\left(X\right)\right\}=0,$$

then, we have

$$d\left\{n: \int\limits_{B_n(\sigma)} |f_n(x)dx - F(x)|\, dx \geqslant \sigma.m(X)\right\} = 0.$$

Combining the inequality (1.4) with (1.3), we find that

$$d\left\{n: \left|\int\limits_X f_n(x)dx - \int\limits_X F(x)dx\right| \geqslant 2M.m(A_n(\sigma)) + \sigma.m(X)\right\} = 0. \tag{1.5}$$

Now take an arbitrary $\varepsilon > 0$, and select a $\sigma > 0$ so small that

$$\sigma.m(X) < \frac{\varepsilon}{2}.$$

Having fixed this σ, the definition of statistical convergence in measury ensures that we will have

$$st - \lim_{n\to\infty} m(A_n(\sigma)) = 0$$

as $n \to \infty$ as therefore

$$d\left\{n: m(A_n(\sigma)) \geqslant \frac{\varepsilon}{4M}\right\} = 0$$

for $n > N$. For such n, inequality (1.4) assumes the form

$$d\left\{n: \left|\int\limits_X f_n(x)dx - \int\limits_X F(x)dx\right| \geqslant \varepsilon\right\} = 0,$$

this proves the theorem. □

For in measure theory,two measurable functions f, g are considered equal or equivalent, if, $f(x) = g(x)$ $\mu - a.e$ and each equivalence class consists an element of the space $L^0(X)$ of measurable real valued functions. Moreover the space $L^0(X)$ is equipped with the following metric of convergence in measure

$$\rho(f,g) = \inf\{a + \mu[|f-g| > a] : a > 0\}$$

and we have the following well known facts:

$$(f_n) \xrightarrow{\mu} f \Leftrightarrow \rho(f_n, f) \to 0$$

$$(f_n) \xrightarrow{st-\mu} f \Leftrightarrow \rho(f_n, f) \xrightarrow{st} 0$$

$$\Leftrightarrow \exists K = \{k_1 < k_2 < \ldots < k_n < \ldots\},\ \ d(K) = 1: \ \ \rho(f_{k_n}, f) \to 0$$

$$\exists \{a_n\} \subset (0,\infty),\ \ a_n \to \infty: \ \ \text{and} \ \ \mu[|f_{k_n} - f| > a_n] \to 0.$$

Therefore, another proof of theorem 2 can be given as follows:

Proof. From the hypothesis of the theorem 2, the fact that the intersection of two subsets of $\mathbb{N}$ of density 1 and from above condition, we get

$$\exists K=\{k_1<k_2<\ldots<k_n<\ldots\},\quad d(K)=1,\quad \exists\{a_n\}\subset(0,\infty),\quad a_n\to 0:$$

$$|f_{k_n}|\leqslant M\ \mu-a.e \quad\text{and}\quad \mu\left[|f_{k_n}-f|>a_n\right]\to 0.$$

Since $f(x)$ is the pointwise limit $\mu-a.e$ of some subsequence of (f_{k_n}) we take $|f_{k_n}|\leqslant M\ \ \mu-a.e.$ Hence, $|f_{k_n}-f|\leqslant M\ \ \mu-a.e.$ Finally the theorem follows from the inequalites,

$$\begin{aligned}\left|\int_X (f_{k_n}-f)\,d\mu\right| &\leqslant \int_X |(f_{k_n}-f)|\,d\mu\\ &= \int_{|f_{k_n}-f|>a_n} |(f_{k_n}-f)|\,d\mu + \int_{|f_{k_n}-f|\leqslant a_n} |(f_{k_n}-f)|\,d\mu\\ &\leqslant 2M\mu\left[|f_{k_n}-f|>a_n\right]+a_n\mu(x)\end{aligned}$$

The final right hand side above tends to zero. □

If a sequence (f_n) is statistical convergent in measure, then $(f_n)^2$ is not statistical convergent in measure usually.

EXAMPLE 1. Let $f_n(x)=\sqrt{x^4+\dfrac{x}{n}}$, $n\in\mathbb{N}$, $0<x<\infty$ and $f(x)=x^2$, then

(i) The sequence of the functions $f_n(x)$ statistical converges in measury to the function $f(x)$ on interval $(0,\infty)$.

(ii) The sequence of the functions $f_n^2(x)$ does not statistical converges in measury to the function $f^2(x)$ on interval $(0,\infty)$.

(i) We can get the inequality $|f_n(x)-f(x)|\leqslant\dfrac{1}{nx}$. For $n\in\mathbb{N}$,

$$B_n=\{x\in(0,\infty):|f_n(x)-f(x)|\geqslant\varepsilon\}\subset\left\{x\in(0,\infty):\frac{1}{nx}\geqslant\varepsilon\right\}=\left(0,\frac{1}{n\varepsilon}\right).$$

Hence,

$$\mu\left\{x\in(0,\infty):\frac{1}{nx}\geqslant\varepsilon\right\}\xrightarrow{st}0,\quad n\to\infty.$$

Therefore $st-\lim_{n\to\infty}\mu\{x\in(0,\infty):|f_n(x)-f(x)|>\varepsilon\}=0$. The sequence of the functions $f_n(x)$ statistical converges in measury to the function $f(x)$. $\left(f_n\overset{st-\mu}{\to}f\right)$.

(ii) For $n\in\mathbb{N}$,

$$f_n^2(x)=x^4+\frac{x}{n}\quad\text{and}\quad f_n^2(x)-f^2(x)=\frac{x}{n}$$

and we get

$$B_n = \left\{x \in (0,\infty) : \left|f_n^2(x) - f^2(x)\right| > \varepsilon\right\}$$
$$= \left\{x \in (0,\infty) : \frac{x}{n} > \varepsilon\right\} = (n\varepsilon, \infty).$$

Hence, $st - \lim_{n\to\infty} \mu \left\{x \in (0,\infty) : \left|f_n^2(x) - f^2(x)\right| > \varepsilon\right\} \neq 0$ The sequence of the functions $f_n^2(x)$ does not statistical converges in measury to the function $f^2(x)$ ($f_n^2(x) \overset{st-\mu}{\nrightarrow} f_n^2(x)$).

THEOREM 3. *If a sequence of functions $f_n(x)$ converge statistically in measure to the functions $f(x)$ and $g(x)$, then these limit functions are equivalent.*

Proof. Suppose $f_n \overset{st-\mu}{\to} f$ and $f_n \overset{st-\mu}{\to} g$. Then for every $\varepsilon > 0$, we have

$$\begin{cases} st - \lim_{n\to\infty} \mu \{x : |f_n(x) - f(x)| > \varepsilon\} = 0 \\ st - \lim_{n\to\infty} \mu \{x : |f_n(x) - g(x)| > \varepsilon\} = 0. \end{cases} \tag{3.1}$$

To show that $f(x)$ and $g(x)$ are equivalent a.e on X, let us assume the contrary, that is $\mu\{x : f(x) \neq g(x)\} > 0$. Then since $f(x) \neq g(x)$ if and only if $|f(x) - g(x)| > 0$, we have

$$\mu \{x : |f(x) - g(x)| > 0\} > 0.$$

Now since

$$\{x \in X : |f(x) - g(x)| > 0\} \subset \bigcup_{n=1}^{\infty} \left\{x \in X : |f(x) - g(x)| \geqslant \frac{1}{n}\right\} \tag{3.2}$$

we have

$$\mu(\{x \in X : |f(x) - g(x)| > 0\}) \leqslant \sum_{n=1}^{\infty} \mu\left(\left\{x \in X : |f(x) - g(x)| \geqslant \frac{1}{n}\right\}\right). \tag{3.3}$$

By (3.2), the left side of (3.3) is positive. Then not all of terms on the right side are equal to 0. Thus there exists some $n_0 \in \mathbb{N}$ such that

$$\mu\left(\left\{x \in X : |f(x) - g(x)| \geqslant \frac{1}{n_0}\right\}\right) > 0. \tag{3.4}$$

For every $n \in \mathbb{N}$ we have

$$\mu\left(\left\{x \in X : |f(x) - g(x)| \geqslant \tfrac{1}{n_0}\right\}\right) \leqslant \mu\left(\left\{x \in X : |f(x) - f_n(x)| \geqslant \tfrac{1}{2n_0}\right\}\right)$$
$$+\mu\left(\left\{x \in X : |f_n(x) - g(x)| \geqslant \tfrac{1}{2n_0}\right\}\right).$$

Letting $n \to \infty$ on the right side of the last inequality, we have

$$\mu\left(\left\{x \in X : |f(x) - g(x)| \geqslant \frac{1}{n_0}\right\}\right) = 0$$

by (3.1). This contradicts (3.4). $\square$

THEOREM 4. *Let a sequence of functions $f_n(x)$ converge statistically in measure to the function $f(x)$ on X, let Φ be real function that satisfies Lipschitz condition on $\mathbb{R}$. Under these conditions the sequence $(\Phi o f_n)_{n=1}^{\infty}$ is defined on X and the sequence of the functions $(\Phi o f_n)_{n=1}^{\infty}$ converge statistically in measure to the functions $\Phi o f$.*

Proof. Let $\varepsilon > 0$ and the sequence of the functions $f_n(x)$ converge statistically in measure to the function $f(x)$, then

$$\mu\left(\{x \in X : |f_n(x) - f(x)| > \varepsilon\}\right) \xrightarrow{st} 0, \quad n \to \infty$$

and also let $\Phi : \mathbb{R} \to \mathbb{R}$ satisfies Lipschitz condition. Hence there is $L > 0$ such that

$$|\Phi(x) - \Phi(y)| \leqslant L|x - y|$$

for each $x, y \in \mathbb{R}$. Define sets E_n to be,

$$E_n = \{x \in X : |(\Phi o f_n)(x) - (\Phi o f)(x)| > \varepsilon\}$$

Since

$$E_n = \{x \in X : |(\Phi o f_n)(x) - (\Phi o f)(x)| > \varepsilon\} \subset \left\{x \in X : |f_n(x) - f(x)| > \frac{\varepsilon}{L}\right\}$$

and also, the sequence of the functions $f_n(x)$ converge statistically in measure to the function $f(x)$, then it holds that

$$\mu\left(\left\{x \in X : |f_n(x) - f(x)| > \frac{\varepsilon}{L}\right\}\right) \xrightarrow{st} 0, \quad n \to \infty$$

By monotonicity of the measure, $\mu\{x \in X : |(\Phi o f_n)(x) - (\Phi o f)(x)| > \varepsilon\} \xrightarrow{st} 0$. Hence the sequence of the functions $(\Phi o f_n)_{n=1}^{\infty}$ converge statistically in measure to the functions $\Phi o f$. $\square$

REFERENCES

[1] B. T. BILALOV, T. Y. NAZAROVA, *Statistical convergence of functional sequences*, Rocky Mountain J. Math. volume **45**, number 5 (2015), 1413–1423.

[2] J. CONNOR, *Two valued measures and summability*, Analysis **10** (1990), 373–385.

[3] J. CONNOR, *R-type summability methods, Cauchy criteria, P-sets and statistical convergence*, Proc. Amer. Math. Soc. **115**, (1992) 319–327.

[4] S. DOLECKI, G. SALINETTI AND R. J.-B. WETS, *Convergence of functions: Equi-semicontinuity*, International Institute For Applied Systems Analysis, Laxenburg, Austria, (1980).

[5] O. DUMAN, C. ORHAN, *μ-statistically convergent function sequences*, Czech. Math. J., **54**, 129 (2004), 413–422.
[6] H. FAST, *Sur la convergence statistique*, Coll. Math., **2**, (1951), 241–244.
[7] H. FEDERER, *Geometric Measure Theory*, Springer, New York, (1969).
[8] J. A. FRIDY, *On statistical convergence*, *Analysis*, **5** (1985), 301–313.
[9] S. R. SADIGOVA, T. Y. NAZAROVA, *Statistical type Lebesgue and Riesz theorems*, International Journal of Mathematical Analysis, **9**, 34 (2015), 1669–1683.
[10] S. HARTMAN, *Sur une familie singuliére d'ensembles de nombres naturels*, Colloq. Math., **2** (1951), 245–248.
[11] M. BALCERZAK, K. DEMS, A. KOMISARSKI, *Statistical convergence and ideal convergence for sequences of functions*, J. Math. Anal. Appl., **328**, (2007), 715–729.
[12] M. BALCERZAK AND K. MUSIAL, *Vitali Type Convergence Theorems for Banach Space Valued Integrals*, Acta. Math. Sinica, English Ser., **29**, 11 (2013), 2027–2036.
[13] I. P. NATANSON, *Theory of functions of a real variable*, 1955.
[14] C. PAPACHRISTODOULOS, *On convergence of sequences of measurable functions*, PhD thesis, Department of Mathematics, University of Athens, Greece, 2009.
[15] N. PAPANASTASSIOU, *On a new type of convergence of sequences of functions*, Atti Sem. Math. Fis. Univ. Modena, **50**, (2002), 355–368.
[16] T. SALAT, *On statistically convergent sequences of real numbers*, Math. Slovaca, **30** (1980), 139–150.
[17] I. J. SCHOENBERG, *The integrability of certain functions and related summability methods*, Amer. Math. Monthly, **6**, (1959), 361–375.
[18] H. STEINHAUS, *Sur la convergence ordinarie et la convergence asymptotique*, Colloq. Math. **2**, (1951), 73–74.

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