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ON AN INTEGRAL INEQUALITY OF M. A. MALIK

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Abstract. In this paper, we shall prove some L^r inequalities for the polar derivative of a polynomial having zeros in $|z| \leqslant k \leqslant 1$ and thereby obtain generalizations and refinements of an integral inequality due to Malik [16]. Besides, we shall also provide an alternative proof of a result due to Dewan et al. [9] which is independent of Laguerre's theorem.

1. Introduction

Let $P(z)$ be a polynomial of degree n and $P'(z)$ be its derivative. Then according to the well-known Bernstein's inequality [6] on the derivative of a polynomial, we have

$$\max_{|z|=1} |P'(z)| \leqslant n \max_{|z|=1} |P(z)|. \tag{1.1}$$

Equality holds in (1.1) if and only if $P(z)$ has all its zeros at the origin.

For the class of polynomials $P(z)$ having all zeros in $|z| \leqslant 1$, Turán [20] proved that

$$\max_{|z|=1} |P'(z)| \geqslant \frac{n}{2} \max_{|z|=1} |P(z)|. \tag{1.2}$$

Inequality (1.2) was refined by Aziz and Dawood [2] and they proved under the same hypothesis that

$$\max_{|z|=1} |P'(z)| \geqslant \frac{n}{2} \Big\{ \max_{|z|=1} |P(z)| + \min_{|z|=1} |P(z)| \Big\}. \tag{1.3}$$

Both the inequalities (1.2) and (1.3) are best possible and become equality for polynomials $P(z) = \alpha z^n + \beta$ where $|\alpha| = |\beta|$.

As an extension of (1.2), it was shown by Malik [15] that if $P(z)$ has all its zeros in $|z| \leqslant k$, $k \leqslant 1$, then

$$\max_{|z|=1} |P'(z)| \geqslant \frac{n}{1+k} \max_{|z|=1} |P(z)|, \tag{1.4}$$

where as the corresponding extension of (1.3) and a refinement of (1.4) was given by Govil [12], who under the same hypothesis proved that

$$\max_{|z|=1} |P'(z)| \geqslant \frac{n}{1+k} \left\{ \max_{|z|=1} |P(z)| + \frac{1}{k^{n-1}} \min_{|z|=k} |P(z)| \right\}. \tag{1.5}$$

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In the literature, there already exist some refinements and generalizations of all the above inequalities, for example see Aziz and Shah [5], Dewan et al. [8], [10], Govil et al. [13] etc.

As a generalization of (1.5) to Lucanary polynomials, Aziz and Shah [5] (see also Dewan et al. [10]) proved that if $P(z) = a_n z^n + \sum_{\upsilon=\mu}^{n} a_{n-\upsilon} z^{n-\upsilon}$, $1 \leqslant \mu \leqslant n$, is a polynomial of degree n having all zeros in $|z| \leqslant k$, $k \leqslant 1$, then

$$\max_{|z|=1} |P'(z)| \geqslant \frac{n}{1+k^\mu} \left\{ \max_{|z|=1} |P(z)| + \frac{1}{k^{n-\mu}} \min_{|z|=k} |P(z)| \right\}. \tag{1.6}$$

For $\mu = 1$, inequality (1.6) reduces to inequality (1.5).

Let $D_\alpha P(z)$ denotes the polar derivative of the polynomial $P(z)$ of degree n with respect to the point α. Then

$$D_\alpha P(z) = nP(z) + (\alpha - z)P'(z).$$

The polynomial $D_\alpha P(z)$ is of degree at most $n-1$ and it generalizes the ordinary derivative in the sense that

$$\lim_{\alpha \to \infty} \left\{ \frac{D_\alpha P(z)}{\alpha} \right\} = P'(z).$$

Aziz and Rather [4] extended (1.4) to the polar derivative of a polynomial and proved that if all the zeros of $P(z)$ lie in $|z| \leqslant k$, $k \leqslant 1$, then for every complex number α with $|\alpha| \geqslant k$,

$$\max_{|z|=1} |D_\alpha P(z)| \geqslant n \left(\frac{|\alpha| - k}{1+k} \right) \max_{|z|=1} |P(z)|. \tag{1.7}$$

Recently, several papers where devoted by different authors to polynomials with polar derivatives (for example see [7], [9], [11], [17], [21] etc). In fact in 2009, Dewan et al.[9], extended (1.6) to the polar derivative of a polynomial and proved that if $P(z) = a_n z^n + \sum_{\upsilon=\mu}^{n} a_{n-\upsilon} z^{n-\upsilon}$, $1 \leqslant \mu \leqslant n$, is a polynomial of degree n having all its zeros in $|z| \leqslant k$, $k \leqslant 1$, and α is a complex number with $|\alpha| \geqslant k^\mu$, then

$$\begin{aligned} \max_{|z|=1} |D_\alpha P(z)| \geqslant\ & n \left(\frac{|\alpha| - k^\mu}{1+k^\mu} \right) \max_{|z|=1} |P(z)| + n \left(\frac{|\alpha| + 1}{k^{n-\mu}(1+k^\mu)} \right) m \\ & + n \left(\frac{k^\mu - A_\mu}{1+k^\mu} \right) \max_{|z|=1} |P(z)| + n \left(\frac{A_\mu - k^\mu}{k^n(1+k^\mu)} \right) m, \end{aligned} \tag{1.8}$$

where $m = \min_{|z|=k} |P(z)|$ and

$$A_\mu = \frac{n\left(|a_n| - \frac{m}{k^n}\right) k^{2\mu} + \mu |a_{n-\mu}| k^{\mu-1}}{n\left(|a_n| - \frac{m}{k^n}\right) k^{\mu-1} + \mu |a_{n-\mu}|}. \tag{1.9}$$

Malik [16] obtained an L^r analogue of (1.2) in the sense that the right-hand side of (1.2) is replaced by a factor involving the integral mean of $|P(z)|$ on $|z| = 1$. In fact, he proved that if $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leqslant 1$, then for each $r > 0$,

$$n\left\{\int_0^{2\pi}|P(e^{i\theta})|^r d\theta\right\}^{\frac{1}{r}} \leqslant \left\{\int_0^{2\pi}|1+e^{i\theta}|^r d\theta\right\}^{\frac{1}{r}} \max_{|z|=1}|P'(z)|. \tag{1.10}$$

The corresponding extension of (1.4), which is also a generalisation of (1.10) was obtained by Aziz [1] who proved that if $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leqslant k$ where $k \leqslant 1$, then for each $r > 0$,

$$n\left\{\int_0^{2\pi}\left|\frac{P(e^{i\theta})}{P'(e^{i\theta})}\right|^r d\theta\right\}^{\frac{1}{r}} \leqslant \left\{\int_0^{2\pi}|1+ke^{i\theta}|^r d\theta\right\}^{\frac{1}{r}}. \tag{1.11}$$

More recently Rather et al. [19] extended (1.11) to the polar derivative in the sense that the ordinary derivative $P'(z)$ is replaced by the polar derivative $D_\alpha P(z)$ of $P(z)$. More precisely they proved:

THEOREM A. *If $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leqslant k$ where $k \leqslant 1$, then for every $\alpha \in \mathbb{C}$ with $|\alpha| \geqslant k$, and each $r > 0$,*

$$n(|\alpha|-k)\left\{\int_0^{2\pi}\left|\frac{P(e^{i\theta})}{D_\alpha P(e^{i\theta})}\right|^r d\theta\right\}^{\frac{1}{r}} \leqslant \left\{\int_0^{2\pi}|1+ke^{i\theta}|^r d\theta\right\}^{\frac{1}{r}}. \tag{1.12}$$

The result is best possible and equality holds in (1.12) for $P(z) = (z-k)^n$.

As a generalization of Theorem A they also proved the following result.

THEOREM B. *If $P(z) = a_n z^n + \sum\limits_{\upsilon=\mu}^{n} a_{n-\upsilon} z^{n-\upsilon}$, $1 \leqslant \mu \leqslant n$, is a polynomial of degree n having all its zeros in $|z| \leqslant k$, $k \leqslant 1$, then for $\alpha \in \mathbb{C}$ with $|\alpha| \geqslant k^\mu$, and for each $r > 0$,*

$$n(|\alpha|-k^\mu)\left\{\int_0^{2\pi}\left|\frac{P(e^{i\theta})}{D_\alpha P(e^{i\theta})}\right|^r d\theta\right\}^{\frac{1}{r}} \leqslant \left\{\int_0^{2\pi}|1+k^\mu e^{i\theta}|^r d\theta\right\}^{\frac{1}{r}}. \tag{1.13}$$

For $\mu = 1$, Theorem B reduces to Theorem A.

In the same paper Rather et al. [19] proved the following more general result.

THEOREM C. *If $P(z) = a_n z^n + \sum\limits_{\upsilon=\mu}^{n} a_{n-\upsilon} z^{n-\upsilon}$, $1 \leqslant \mu \leqslant n$, is a polynomial of degree n having all its zeros in $|z| \leqslant k$, $k \leqslant 1$, then for α, $\beta \in \mathbb{C}$ with $|\alpha| \geqslant k^\mu$,*

$|\beta| \leqslant 1$ *and for each* $r > 0$,

$$n(|\alpha| - k^{\mu})\left\{\int_0^{2\pi}\left|\frac{P(e^{i\theta}) + \beta\frac{m}{k^{n-\mu}}}{|D_\alpha P(e^{i\theta})| - \frac{mn}{k^{n-\mu}}}\right|^r d\theta\right\}^{\frac{1}{r}} \leqslant \left\{\int_0^{2\pi}|1 + k^{\mu}e^{i\theta}|^r d\theta\right\}^{\frac{1}{r}}, \tag{1.14}$$

where $m = \min_{|z|=k}|P(z)|$.

In this paper, we shall prove some L^r inequalities for polynomials with polar derivative. We shall first prove a result that generalizes as well as refines Theorems A and B. We shall also present a more general result which not only provides an alternative proof of inequality (1.8) independent of Laguerre's theorem but also yields a refinement of it.

2. Main results

Firstly, we shall prove the following generalization and refinement of inequalities (1.12) and (1.13).

THEOREM 1. *If* $P(z) = a_n z^n + \sum_{\upsilon=\mu}^{n} a_{n-\upsilon} z^{n-\upsilon}$, $1 \leqslant \mu \leqslant n$, *is a polynomial of degree* n *having all its zeros in* $|z| \leqslant k$, $k \leqslant 1$, *then for* $\alpha \in \mathbb{C}$ *with* $|\alpha| \geqslant s_\mu$, *and for each* $r > 0$,

$$n(|\alpha| - s_\mu)\left\{\int_0^{2\pi}\left|\frac{P(e^{i\theta})}{D_\alpha P(e^{i\theta})}\right|^r d\theta\right\}^{\frac{1}{r}} \leqslant \left\{\int_0^{2\pi}|1 + s_\mu e^{i\theta}|^r d\theta\right\}^{\frac{1}{r}}, \tag{2.1}$$

where

$$s_\mu = \frac{n|a_n|k^{2\mu} + \mu|a_{n-\mu}|k^{\mu-1}}{n|a_n|k^{\mu-1} + \mu|a_{n-\mu}|}. \tag{2.2}$$

REMARK 1. By Lemma 1, we have $\frac{\mu}{n}\left|\frac{a_{n-\mu}}{a_n}\right| \leqslant k^{\mu}$, which shows $s_\mu \leqslant k^{\mu}$, therefore, Theorem 1 holds for every $\alpha \in \mathbb{C}$ with $|\alpha| \geqslant k^{\mu}$ as well. Using this and the fact that

$$\int_0^{2\pi}|1 + \mu e^{i\theta}|^r d\theta \leqslant \int_0^{2\pi}|1 + v e^{i\theta}|^r d\theta \tag{2.3}$$

for $0 \leqslant \mu \leqslant v$, it easily follows that (2.1) is a refinement of (1.13).

REMARK 2. For $\mu = 1$, Theorem 1 provides a refinement of Theorem A.

Instead of proving Theorem 1, we prove the following more general result.

THEOREM 2. *If* $P(z) = a_n z^n + \sum\limits_{\upsilon=\mu}^{n} a_{n-\upsilon} z^{n-\upsilon}$, $1 \leqslant \mu \leqslant n$, *is a polynomial of degree n having all its zeros in* $|z| \leqslant k$, $k \leqslant 1$, *then for* $\alpha, \beta \in \mathbb{C}$ *with* $|\alpha| \geqslant A_\mu$, $|\beta| \leqslant 1$ *and for each* $r > 0$,

$$n(|\alpha| - A_\mu)\left\{\int_0^{2\pi}\left|\frac{P(e^{i\theta}) + \beta\frac{mA_\mu}{k^n}}{|D_\alpha P(e^{i\theta})| - \frac{mnA_\mu}{k^n}}\right|^r d\theta\right\}^{\frac{1}{r}} \leqslant \left\{\int_0^{2\pi}|1 + A_\mu e^{i\theta}|^r d\theta\right\}^{\frac{1}{r}}, \tag{2.4}$$

where $m = \min_{|z|=k}|P(z)|$ *and* A_μ *is defined by* (1.9).

Since for every $\alpha \in \mathbb{C}$, we have $|D_\alpha P(e^{i\theta})| \leqslant \max_{|z|=1}|D_\alpha P(z)|$, $0 \leqslant \theta < 2\pi$, the following result easily follows from Theorem 2.

COROLLARY 1. *If* $P(z) = a_n z^n + \sum\limits_{\upsilon=\mu}^{n} a_{n-\upsilon} z^{n-\upsilon}$, $1 \leqslant \mu \leqslant n$, *is a polynomial of degree n having all its zeros in* $|z| \leqslant k$, $k \leqslant 1$, *then for* $\alpha, \beta \in \mathbb{C}$ *with* $|\alpha| \geqslant A_\mu$, $|\beta| \leqslant 1$ *and for each* $r > 0$,

$$\begin{aligned} n(|\alpha| - A_\mu)&\left\{\int_0^{2\pi}\left|P(e^{i\theta}) + \beta\frac{mA_\mu}{k^n}\right|^r d\theta\right\}^{\frac{1}{r}} \\ &\leqslant \left\{\int_0^{2\pi}|1 + A_\mu e^{i\theta}|^r d\theta\right\}^{\frac{1}{r}}\left\{\max_{|z|=1}|D_\alpha P(z)| - \frac{mnA_\mu}{k^n}\right\}, \end{aligned} \tag{2.5}$$

where $m = \min_{|z|=k}|P(z)|$ *and* A_μ *is defined by* (1.9).

If we let $r \to \infty$ in (2.5) and choose argument of β with $|\beta| = 1$, we obtain the following refinement of (1.8).

COROLLARY 2. *If* $P(z) = a_n z^n + \sum\limits_{\upsilon=\mu}^{n} a_{n-\upsilon} z^{n-\upsilon}$, $1 \leqslant \mu \leqslant n$, *is a polynomial of degree n having all its zeros in* $|z| \leqslant k$, $k \leqslant 1$, *then for every* $\alpha \in \mathbb{C}$ *with* $|\alpha| \geqslant A_\mu$,

$$\max_{|z|=1}|D_\alpha P(z)| \geqslant n\left(\frac{|\alpha| - A_\mu}{1 + A_\mu}\right)\max_{|z|=1}|P(z)| + \frac{nA_\mu}{k^n}\left(\frac{1 + |\alpha|}{1 + A_\mu}\right)m$$

which is equivalent to

$$\begin{aligned} \max_{|z|=1}|D_\alpha P(z)| \geqslant\ & \frac{n(|\alpha| - k^\mu)}{1 + k^\mu}\max_{|z|=1}|P(z)| + \frac{n(|\alpha| + 1)}{k^{n-\mu}(1 + k^\mu)}m \\ & + n\left(\frac{k^\mu - A_\mu}{1 + k^\mu}\right)\max_{|z|=1}|P(z)| + \frac{n(A_\mu - k^\mu)}{k^n(1 + k^\mu)}m \\ & + \frac{n(k^\mu - A_\mu)(|\alpha| - A_\mu)}{(1 + k^\mu)(1 + A_\mu)}\left\{\max_{|z|=1}|P(z)| - \frac{m}{k^n}\right\}, \end{aligned} \tag{2.6}$$

where $m = \min_{|z|=k}|P(z)|$ *and* A_μ *is defined by* (1.9). *The result is best possible and equality in* (2.6) *holds for* $P(z) = (z^\mu + k^\mu)^{\frac{n}{\mu}}$, *where* n *is a multiple of* μ.

By Lemma 3, $A_\mu \leqslant k^\mu$, therefore, Corollary 2 holds for every $\alpha \in \mathbb{C}$ with $|\alpha| \geqslant k^\mu$ as well.

In fact, excepting the case when $k = 1$ or $\frac{\mu}{n}\left(\frac{|a_{n-\mu}|}{|a_n| - \frac{m}{k^n}}\right) = k^\mu$, the bound obtained in Corollary 2 is always sharper than the bound obtained in (1.8) and for this it needs to show that

$$\frac{n(k^\mu - A_\mu)(|\alpha| - A_\mu)}{(1+k^\mu)(1+A_\mu)}\left\{\max_{|z|=1}|P(z)| - \frac{m}{k^n}\right\} \geqslant 0. \tag{2.7}$$

In view of inequality (3.3) of Lemma 3, the the above inequality becomes equivalent to

$$\max_{|z|=1}|P(z)| \geqslant \frac{m}{k^n}. \tag{2.8}$$

Now using (1.1) in the Lemma 2, we get

$$|Q^{'}(z)| \leqslant A_\mu n \max_{|z|=1}|P(z)| - \frac{mnA_\mu}{k^n} = nA_\mu\left\{\max_{|z|=1}|P(z)| - \frac{m}{k^n}\right\},$$

and hence (2.8) holds.

The following interesting refinement of (1.6) is obtained from Corollary 2 by dividing both sides of (2.6) by $|\alpha|$ and letting $|\alpha| \to \infty$,

COROLLARY 3. *If* $P(z) = a_n z^n + \sum\limits_{\upsilon=\mu}^{n} a_{n-\upsilon} z^{n-\upsilon}$, $1 \leqslant \mu \leqslant n$, *is a polynomial of degree* n *having all its zeros in* $|z| \leqslant k$, $k \leqslant 1$, *then*

$$\max_{|z|=1}|P'(z)| \geqslant \frac{n}{1+k^\mu}\left\{\max_{|z|=1}|P(z)| + \frac{m}{k^{n-\mu}}\right\} + \frac{n(k^\mu - A_\mu)}{(1+k^\mu)(1+A_\mu)}\left\{\max_{|z|=1}|P(z)| - \frac{m}{k^n}\right\}, \tag{2.9}$$

where $m = \min_{|z|=k}|P(z)|$ *and* A_μ *is defined by* (1.9). *The result is best possible and equality in* (2.9) *holds for* $P(z) = (z^\mu + k^\mu)^{\frac{n}{\mu}}$, *where* n *is a multiple of* μ.

Several other interesting results easily follow from Theorem 2. Here, we mention a few of these. If we take $\beta = 0$ in (2.4), we obtain the following result which gives Theorem 1 as a special case.

COROLLARY 4. *If* $P(z) = a_n z^n + \sum\limits_{\upsilon=\mu}^{n} a_{n-\upsilon} z^{n-\upsilon}$, $1 \leqslant \mu \leqslant n$, *is a polynomial of degree* n *having all its zeros in* $|z| \leqslant k$, $k \leqslant 1$, *then for every* $\alpha \in \mathbb{C}$ *with* $|\alpha| \geqslant A_\mu$ *and for each* $r > 0$,

$$n(|\alpha| - A_\mu)\left\{\int_0^{2\pi}\left|\frac{P(e^{i\theta})}{|D_\alpha P(e^{i\theta})| - \frac{mnA_\mu}{k^n}}\right|^r d\theta\right\}^{\frac{1}{r}} \leqslant \left\{\int_0^{2\pi}|1 + A_\mu e^{i\theta}|^r d\theta\right\}^{\frac{1}{r}}, \tag{2.10}$$

where $m=\min_{|z|=k}|P(z)|$ *and* A_μ *is defined by* (1.9).

Taking $\beta=0$ and dividing both sides of (2.5) by $|\alpha|$ then letting $|\alpha|\to\infty$, we obtain the following generalization and refinement of (1.10).

COROLLARY 5. *If* $P(z)=a_nz^n+\sum_{\upsilon=\mu}^{n}a_{n-\upsilon}z^{n-\upsilon}$, $1\leqslant\mu\leqslant n$, *is a polynomial of degree* n *having all its zeros in* $|z|\leqslant k$, $k\leqslant 1$, *then for each* $r>0$,

$$n\left\{\int_0^{2\pi}|P(e^{i\theta})|^r d\theta\right\}^{\frac{1}{r}}\leqslant\left\{\int_0^{2\pi}|1+A_\mu e^{i\theta}|^r d\theta\right\}^{\frac{1}{r}}\max_{|z|=1}|P'(z)|, \tag{2.11}$$

where A_μ *is defined by* (1.9).

The result is best possible and equality in (2.11) holds for $P(z)=(z^\mu+k^\mu)^{\frac{n}{\mu}}$, where n is a multiple of μ.

3. Lemmas

We need the following lemmas to prove the theorems.

LEMMA 1. *If* $P(z)=a_nz^n+\sum_{\upsilon=\mu}^{n}a_{n-\upsilon}z^{n-\upsilon}$, $1\leqslant\mu\leqslant n$, *is a polynomial of degree* n *having all its zeros in* $|z|\leqslant k$, $k\leqslant 1$, *and* $Q(z)=z^n\overline{P(\frac{1}{\bar z})}$, *then on* $|z|=1$

$$|Q'(z)|\leqslant s_\mu|P'(z)|, \tag{3.1}$$

and

$$\frac{\mu}{n}\left|\frac{a_{n-\mu}}{a_n}\right|\leqslant k^\mu,$$

where s_μ *is defined by* (2.2).

The above lemma is due to Aziz and Rather [3].

LEMMA 2. *If* $P(z)=a_nz^n+\sum_{\upsilon=\mu}^{n}a_{n-\upsilon}z^{n-\upsilon}$, $1\leqslant\mu\leqslant n$, *is a polynomial of degree* n *having all its zeros in* $|z|\leqslant k$, $k\leqslant 1$, *and* $Q(z)=z^n\overline{P(\frac{1}{\bar z})}$, *then on* $|z|=1$

$$|Q'(z)|\leqslant A_\mu|P'(z)|-\frac{nmA_\mu}{k^n}, \tag{3.2}$$

where $m=\min_{|z|=k}|P(z)|$ *and* A_μ *is defined by* (1.9).

The above lemma is due to Mir et al. [18].

LEMMA 3. *If* $P(z)=a_nz^n+\sum_{\upsilon=\mu}^{n}a_{n-\upsilon}z^{n-\upsilon}$, $1\leqslant\mu\leqslant n$, *is a polynomial of degree* n *having all its zeros in* $|z|\leqslant k$, $k\leqslant 1$, *then*

$$A_\mu\leqslant k^\mu, \tag{3.3}$$

where A_μ *is defined by* (1.9).

The above Lemma is due to Dewan et al. [9].

4. Proofs of theorems

Proof of Theorem 2. Since $P(z)$ has all its zeros in $|z| \leqslant k$, $k \leqslant 1$, it follows from Lemma 2 that for $|z| = 1$,

$$|Q'(z)| + \frac{nmA_\mu}{k^n} \leqslant A_\mu |P'(z)|. \tag{4.1}$$

Also $Q(z) = z^n \overline{P(\frac{1}{\bar{z}})}$, then $P(z) = z^n \overline{Q(\frac{1}{\bar{z}})}$ and it can be easily verified that for $|z| = 1$,

$$|Q'(z)| = |nP(z) - zP'(z)| \tag{4.2}$$

and

$$|P'(z)| = |nQ(z) - zQ'(z)|. \tag{4.3}$$

Using (4.3) in (4.1), we get for $|z| = 1$

$$\begin{aligned} \left| Q'(z) + \overline{\beta} \frac{nmA_\mu z^{n-1}}{k^n} \right| &\leqslant |Q'(z)| + \frac{nmA_\mu}{k^n} \\ &\leqslant A_\mu |P'(z)| \\ &= A_\mu |nQ(z) - zQ'(z)|. \end{aligned} \tag{4.4}$$

Now for every $\alpha \in \mathbb{C}$ with $|\alpha| \geqslant A_\mu$, we have

$$\begin{aligned} |D_\alpha P(z)| &\geqslant |\alpha||P'(z)| - |Q'(z)| \\ &\geqslant |\alpha||P'(z)| - |nP(z) - zP'(z)|, \end{aligned}$$

which on using (4.2) and Lemma 2 gives for $|z| = 1$,

$$\begin{aligned} |D_\alpha P(z)| &\geqslant |\alpha||P'(z)| - |Q'(z)| \\ &\geqslant (|\alpha| - A_\mu)|P'(z)| + \frac{mnA_\mu}{k^n}, \end{aligned}$$

$$|D_\alpha P(z)| - \frac{mnA_\mu}{k^n} \geqslant (|\alpha| - A_\mu)|P'(z)|. \tag{4.5}$$

Since $P(z)$ has all its zeros in $|z| \leqslant k$, $k \leqslant 1$, it follows by Gauss-Lucas theorem that all the zeros of $P'(z)$ also lie in $|z| \leqslant k$, $k \leqslant 1$. This implies that the polynomial

$$z^{n-1} \overline{P'\left(\frac{1}{\bar{z}}\right)} = nQ(z) - zQ'(z)$$

has all its zeros in $|z| \geqslant \frac{1}{k} \geqslant 1$. Therefore, it follows from (4.4) and the Maximum Modulus Principle that the function

$$W(z) = \frac{z\left(Q'(z) + \overline{\beta} \frac{mnA_\mu z^{n-1}}{k^n}\right)}{A_\mu (nQ(z) - zQ'(z))}$$

is analytic for $|z| \leqslant 1$ and $|W(z)| \leqslant 1$ for $|z| \leqslant 1$. Furthermore, $W(0) = 0$ and so the function $1 + A_\mu W(z)$ is subordinate to the function $1 + A_\mu z$ for $|z| \leqslant 1$. Hence by a well-known property of sub-ordination [14], we have for each $r > 0$,

$$\int_0^{2\pi} \left|1 + A_\mu W(e^{i\theta})\right|^r d\theta \leqslant \int_0^{2\pi} \left|1 + A_\mu e^{i\theta}\right|^r d\theta. \tag{4.6}$$

Now

$$1 + A_\mu W(z) = \frac{n\left(Q(z) + \overline{\beta}\frac{mA_\mu z^n}{k^n}\right)}{nQ(z) - zQ'(z)},$$

and

$$|P'(z)| = \left|z^{n-1}\overline{P'\left(\frac{1}{\overline{z}}\right)}\right| = |nQ(z) - zQ'(z)| \text{ for } |z| = 1.$$

Therefore for $|z| = 1$,

$$n\left|Q(z) + \overline{\beta}\frac{mA_\mu z^n}{k^n}\right| = |1 + A_\mu W(z)||P'(z)|.$$

Equivalently,

$$n\left|z^n\overline{P\left(\frac{1}{\overline{z}}\right)} + \overline{\beta}\frac{mA_\mu z^n}{k^n}\right| = |1 + A_\mu W(z)||P'(z)|.$$

This implies

$$n\left|P(z) + \beta\frac{mA_\mu}{k^n}\right| = |1 + A_\mu W(z)||P'(z)| \text{ for } |z| = 1. \tag{4.7}$$

From (4.5) and (4.7), we deduce that for $|\alpha| \geqslant A_\mu$ and $r > 0$,

$$n^r(|\alpha| - A_\mu)^r \int_0^{2\pi} \left|\frac{P(e^{i\theta}) + \beta\frac{mA_\mu}{k^n}}{|D_\alpha P(e^{i\theta})| - \frac{mnA_\mu}{k^n}}\right|^r d\theta \leqslant \int_0^{2\pi} \left|1 + A_\mu W(e^{i\theta})\right|^r d\theta.$$

which gives by using (4.6) that

$$n(|\alpha| - A_\mu)\left\{\int_0^{2\pi} \left|\frac{P(e^{i\theta}) + \beta\frac{mA_\mu}{k^n}}{|D_\alpha P(e^{i\theta})| - \frac{mnA_\mu}{k^n}}\right|^r d\theta\right\}^r \leqslant \left\{\int_0^{2\pi} |1 + A_\mu e^{i\theta}|^r d\theta\right\}^r.$$

This completes the proof of Theorem 2. □

REMARK 3. The proof of Theorem 1 follows along the lines of the proof of Theorem 2, by applying inequality (3.1) of Lemma 1 instead of Lemma 2.

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