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SOME PROPERTIES OF A FUNCTION CONNECTING TO EXPONENT OF CONVERGENCE FOR DOUBLE SEQUENCES

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Abstract. In this paper, we extend the notion of exponent of convergence for double sequences and study some properties of a function connecting with a non-decreasing double sequence in a Fréchet metric space.

1. Introduction

Let $A = \{a_k\}_k$ be a non-decreasing sequence of positive real numbers with $\lim\limits_{n\to\infty} a_n = +\infty$. It is well known [5] that there exists a unique number $\lambda = \lambda(A)$, $\lambda(A) \geqslant 0$ such that $\sum\limits_{k=1}^{\infty} {a_k}^{-\sigma} = +\infty$ for each $\sigma > 0$, $\sigma < \lambda$ and $\sum\limits_{k=1}^{\infty} {a_k}^{-\sigma} < +\infty$ for each $\sigma > 0$, $\sigma > \lambda$.

The number $\lambda = \lambda(A)$ is called the exponent of convergence of the sequence A. It is represented by,

$$\lambda(A) = \inf\left\{\sigma > 0 : \sum_{k=1}^{\infty} {a_k}^{-\sigma} < +\infty\right\}.$$

It is also known [4] that,

$$\lambda(A) = \lim_{k\to\infty} \sup \frac{\log k}{\log a_k}.$$

Several authors (see [1], [2], [4], [5]) devoted their studies on exponent of convergence from different points of view. In this context, Kostyrko and Šalát [5] explored some interesting results on exponent of convergence of real sequences. They investigated the exponent of convergence as a real valued function defined on the set S of all real non-decreasing sequence $\{x_k\}_k$ of real numbers with the property $x_1 > \gamma > 0$ endowed with Fréchet metric defined by,

$$\rho(x,y) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|x_i - y_i|}{1 + |x_i - y_i|},$$

where $x = \{x_k\}_k$ and $y = \{y_k\}_k$ are any two elements of S.

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Our aim in this paper is to extend the concept of exponent of convergence for double sequences of real numbers. During the last century, various problems connecting to the convergence and divergence of double sequences were treated by a number of mathematicians (see [6], [7], [8], [9], [10], [11]). Here first we give formulation for the exponent of convergence of double sequences which is analogous to the case of single sequences and then we prove some properties on exponent of convergence for the double sequences.

DEFINITION 1. [3] A double sequence $\{a_{mn}\}$ is said to be monotonically increasing (decreasing) if $a_{mn} \leqslant a_{pq}$ $(a_{mn} \geqslant a_{pq})$ holds if $(m,n) \leqslant (p,q) \iff m \leqslant p$ and $n \leqslant q$ under the partial ordering "$\leqslant$" on the set $\mathbb{N}\times\mathbb{N}$, where $\mathbb{N}$ is the set of positive integers.

DEFINITION 2. [3] A double sequence $\{a_{mn}\}$ of real numbers is said converge to the number ξ if for each $\varepsilon > 0$ there exists $(p,q) \in \mathbb{N}\times\mathbb{N}$ such that $|a_{mn} - \xi| < \varepsilon$ for $(m,n) \geqslant (p,q)$ and it is written as $\lim\limits_{m,n\to\infty} a_{mn} = \xi$.

DEFINITION 3. [3] A double series $\sum_{m,n=1}^{\infty} a_{mn}$ of real numbers is said to converge to ξ in Pringsheim sense if $\lim\limits_{m,n} s_{mn} = \xi$, where $s_{mn} = \sum\limits_{i=1}^{m} \sum_{j=1}^{n} a_{ij}$ is the partial sum of the double series.

Let S denotes the set of real non-decreasing double sequences $\{x_{mn}\}$ with $x_{11} \geqslant 1$, endowed with Fréchet metric defined by $\rho(x,y) = \sum\limits_{i,j} \frac{1}{2^{ij}} \frac{|x_{ij}-y_{ij}|}{1+|x_{ij}-y_{ij}|}$, where $x = \{x_{ij}\}_{i,j\geqslant 1}$, $y = \{y_{ij}\}_{i,j\geqslant 1}$ are points of S.

It is clear that, the set X of all real double sequences with Fréchet metric is a complete metric space. The convergence in this space is taken as co-ordinate wise convergence in Pringsheim sense.

In the following we establish some properties of the space (S,ρ).

THEOREM 1. *If S_1 is the collection of all non-decreasing sequences $\{x_{mn}\}$ of real numbers diverging to $+\infty$ with $x_{11} > 1$ then the space (S_1,ρ) is not a complete metric space.*

Proof. We consider a sequence $\{x^{(i)}\}_{i=1}^{\infty}$ of elements from S_1, where $x^{(i)} = \{a_{mn}^{(i)}\}_{m,n\geqslant 1}$ and

$$a_{mn}^{(i)} = \begin{cases} 1+\frac{1}{i}; & \text{when} \quad (m,n) = (1,1) \\ m+n; & \text{otherwise.} \end{cases}$$

If $\{x^{(i)}\}_{i=1}^{\infty}$ converges to $x = \{a_{mn}\}$ then $a_{11} = 1$, since the space deals with the point wise convergence. Clearly, $x \notin S_1$ and consequently (S_1,ρ) is not complete. $\square$

THEOREM 2. *If T is the collection of all non-decreasing sequences $\{x_{mn}\}$ of real numbers diverging to $+\infty$ with $x_{11}=1$ then T is nowhere dense in S.*

Proof. Clearly, T is a closed subset of S. It is sufficient to prove that the complement of T in S is dense in S.

Let $a=\{a_{mn}\}\in S$. We consider an open ball $B(a,\delta)$, $\delta>0$ with centre at a and radius δ.

If $a_{11}>1$, then $a\in S-T$ and $(S-T)\cap B(a,\delta)\neq\phi$ and the result is proved.

Let $a_{11}=1$. Since, $\{a_{mn}\}$ is non-decreasing and diverges to $+\infty$ then there exists positive integers k and l for which $l+k$ is least such that $a_{kl}>1$.

For $\delta>0$, we can choose a real number $t_0>0$ such that $\frac{t_0}{1+t_0}<\delta$. Again, for any t with $0<t<t_0$, we have $\frac{t}{1+t}<\frac{t_0}{1+t_0}<\delta$.

We consider a sequence $b=\{b_{mn}\}$ such that

$$b_{mn}=\begin{cases}1+t_0; & \text{when} \quad (m,n)<(k,l)\\ a_{mn}; & \text{otherwise.}\end{cases}$$

Since, $b_{11}>1$ then $b\in S-T$ and

$$\begin{aligned}\rho(a,b) &= \sum_{i,j=1}^{\infty}\frac{1}{2^{ij}}\frac{|a_{ij}-b_{ij}|}{1+|a_{ij}-b_{ij}|}\\ &= \sum_{(i,j)<(k,l)}\frac{1}{2^{ij}}\frac{|a_{ij}-b_{ij}|}{1+|a_{ij}-b_{ij}|}\\ &< \frac{t_0}{1+t_0}<\delta.\end{aligned}$$

Then, clearly $b\in(S-T)\cap B(a,\delta)$ and thus $S-T$ is dense in S. Therefore, T is nowhere dense in S. $\square$

THEOREM 3. *The set (S,ρ) is a complete metric space and has the cardinality of continuum.*

Proof. To prove the theorem we show that the set S is a perfect subset in (X,ρ).

Let $\{x^{(i)}\}_{i\geqslant 1}$ be any sequence in S, where $x^{(i)}=\{x_{mn}^{(i)}\}_{m,n\geqslant 1}$ and $i=1,2,3,\ldots$

If $i\to\infty$, $x^{(i)}\to x=\{x_{mn}\}$ (say) and then, $x\in S$. It readily follows from the pointwise convergence in Pringheim sense for double sequence with respect to the Fréchet metric. Hence, S is closed.

We now show that the set S is dense in it self.

We take any point $x=\{x_{mn}\}\in S$. We have to find a sequence $\{x^{(i)}\}_{i\geqslant 1}$ from S such that $\lim\limits_{i\to\infty}x^{(i)}=x$.

We construct the sequence $\{x^{(i)}\}_{i\geqslant 1}$ as follows:

$$x_{mn}{}^{(i)}=\begin{cases}x_{mn}, & \text{for} \quad (m,n)\leqslant(i,i)\\ x_{mn}+1, & \text{otherwise.}\end{cases}$$

It is clear that $\lim\limits_{i\to\infty} x^{(i)} = x$.

Hence, S is a perfect set in (X,ρ) and thus, S has the cardinality of continuum and consequently the space (S,ρ) is complete having cardinality of continuum. $\square$

2. Exponent of convergence

Let $A = \{a_{mn}\}$, $a_{mn} \geqslant 1$ be a non-decreasing double sequence of real numbers. If $\sigma < \tau$ then the convergence of the series $\sum\limits_{m,n=1}^{\infty} a_{mn}^{-\sigma}$ implies the convergence of the series $\sum\limits_{m,n=1}^{\infty} a_{mn}^{-\tau}$. This simple observation leads us to define the exponent of convergence $\lambda = \lambda(A)$ of the double sequence A as follows:

$$\lambda(A) = \inf\left\{\sigma > 0 : \sum_{m,n=1}^{\infty} a_{mn}^{-\sigma} < \infty\right\}.$$

Then, λ is a function from S to $[0,\infty]$.

The following result gives an alternative form for calculation of the exponent of convergence of double sequences under certain assumptions and it can be verified easily.

RESULT 1. If $\{a_{mn}\}$, $a_{mn} > 1$ be a non-decreasing double sequence of real numbers, then the exponent of convergence of the double sequence $\{a_{mn}\}$ is equal to the number $\limsup\limits_{m,n\to\infty} \frac{\log mn}{\log a_{mn}}$.

We now examine some topological properties of the function λ.

THEOREM 4. *For any non-negative real number t there exists some $x \in S$ such that $\lambda(x) = t$.*

Proof. *Case* 1. Let $t = 0$. We choose $x = \{x_{mn}\}$, where $x_{mn} = (mn)^{mn}$. Then,

$$\lambda(x) = \limsup_{m,n\to\infty} \frac{\log(mn)}{\log(mn)^{mn}} = \limsup_{m,n\to\infty} \frac{1}{mn} = 0.$$

Case 2. Let $t > 0$. Then, there exists a positive integer k such that $((m+k)(n+k))^{\frac{1}{t}} > 1$ for positive integers m and n. Here, we choose $x = \{x_{mn}\}$, where $x_{mn} = ((m+k)(n+k))^{\frac{1}{t}}$. Then,

$$\begin{aligned} \lambda(x) &= \limsup_{m,n\to\infty} \frac{\log(mn)}{\log((m+k)(n+k))^{\frac{1}{t}}} \\ &= t.\limsup_{m,n\to\infty} \frac{\log mn}{\log(m+k)(n+k)} \\ &= t. \quad \square \end{aligned}$$

COROLLARY 1. *The function $\lambda : S \to [0,\infty]$ does not belong to Baire class one.*

Proof. It is well known that, the set of points of discontinuity of a function belonging to the Baire class one is a set of first category ([12]). As the function λ is totally discontinuous on S, it can not belong to Baire class one. $\square$

THEOREM 5. *For each real number d, the sets $A^d = \left\{x \in S; \lambda(x) < d\right\}$ and $A_d = \left\{x \in S; \lambda(x) > d\right\}$ belong to the third additive Borel class.*

Proof. First we investigate for the set A^d.

If $d \leqslant 0$, then clearly $A^d = \phi$ (null set) and the above result is true.

Let $d > 0$. Then, we have $A^d = \left\{x \in S : \lambda(x) < d\right\}$. So, there exists σ with $0 < \sigma < d$ for which

$$\begin{aligned} A^d &= \left\{x \in S : \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{mn}^{-\sigma} < \infty\right\} \\ &= \bigcup_{k=k_0}^{\infty} \left\{x : \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{mn}^{-(d-\frac{1}{k})} < \infty\right\}, \end{aligned}$$

where k_0 is the smallest positive integer for which $\sigma = d - \frac{1}{k} > 0$, for all $k \geqslant k_0$.

We set, $H_k = \left\{x \in S : \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{mn}^{-\sigma} < +\infty\right\}$, where $\sigma = d - \frac{1}{k}$, $(k = k_0,\ k_0+1,\ k_0+2, \ldots)$.

Since, $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij}^{-\sigma} < +\infty$, for $\sigma = d - \frac{1}{k}$, $(k = k_0,\ k_0+1,\ k_0+2, \ldots)$, then by Stolz's theorem for convergence of double series of real terms, for each positive integer l, there exists $(m,n) \in \mathbb{N} \times \mathbb{N}$ such that

$$\sum_{j=n+1}^{\infty} \sum_{i=m+1}^{\infty} a_{ij}^{-\sigma} \leqslant \frac{1}{l}, \text{ for } l = 1,2,3,\ldots.$$

Then,

$$\begin{aligned} H_k &= \bigcap_{l=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcap_{p=1}^{\infty} \left\{x \in S : \sum_{j=n+1}^{n+p} \sum_{i=m+1}^{m+p} a_{ij}^{-\sigma} \leqslant \frac{1}{l}\right\} \\ &= \bigcap_{l=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcap_{p=1}^{\infty} H_{klmnp}, \end{aligned}$$

where $H_{klmnp} = \left\{x \in S : \sum_{j=n+1}^{n+p} \sum_{i=m+1}^{m+p} a_{ij}^{-\sigma} \leqslant \frac{1}{l}\right\}$.

For fixed k,l,m,n,p, we prove that H_{klmnp} is a closed set.

This follows immediately from point-wise convergence of double sequence in Pringheim sense.

We take a sequence $\{a^{(s)}\}_{s=1}^{\infty}$ in H_{klmnp}, where $a^{(s)} = \{a_{ij}^{(s)}\}_{i,j \geqslant 1}$, $s = 1,2,\ldots$ and $\lim_{s\to\infty} (a_{ij}{}^{(s)}) = a$, where $a = \{a_{ij}\}_{i,j \geqslant 1}$.

Then, $\lim_{s\to\infty}(a_{ij}{}^{(s)})^{-\sigma} = a_{ij}{}^{-\sigma}$, for all $(i,j)\in\mathbb{N}\times\mathbb{N}$.

Hence, $a\in H_{klmnp}$. Consequently, each of the sets H_{klmnp} is closed.

So, it follows that $A^d = \left\{x\in S:\lambda(x)<d\right\} = \bigcap_{l=1}^{\infty}\bigcup_{m=1}^{\infty}\bigcup_{n=1}^{\infty}\bigcap_{p=1}^{\infty} H_{klmnp}$ is an $F_{\sigma\delta\sigma}$ set in S and therefore, A^d belongs to the third additive Borel class.

We now investigate the set A_d.

If $d<0$, then $A_d = S$ and the result is proved.

Let $d\geqslant 0$, then there exists a positive integer k such that

$$A_d = \left\{x\in S:\lambda(x)>d\right\} = \bigcup_{k=1}^{\infty}\left\{x\in S:\sum_{j=1}^{n}\sum_{i=1}^{m} a_{ij}^{-(d+\frac{1}{k})} = +\infty\right\}.$$

We set, $L_k = \left\{x\in S:\sum_{j=1}^{n}\sum_{i=1}^{m} a_{ij}^{-\sigma} = +\infty\right\}$, where $\sigma = d+\frac{1}{k}, k=1,2,\ldots$.

Thus, for each $p\in\mathbb{N}$ there exists $q\in\mathbb{N}$ such that $\sum_{j=1}^{n+q}\sum_{i=1}^{m+q} a_{ij}\geqslant p$, for all $(m,n)\in\mathbb{N}\times\mathbb{N}$.

We can express the set L_k as

$$L_k = \bigcap_{p=1}^{\infty}\bigcup_{q=1}^{\infty}\bigcap_{m=1}^{\infty}\bigcap_{n=1}^{\infty}\left\{x\in S:\sum_{j=1}^{n+q}\sum_{i=1}^{m+q} a_{ij}^{-\sigma}\geqslant p\right\} = \bigcap_{p=1}^{\infty}\bigcup_{q=1}^{\infty}\bigcap_{m=1}^{\infty}\bigcap_{n=1}^{\infty} L_{kpqmn},$$

where $L_{kpqmn} = \left\{x\in S:\sum_{j=1}^{n+q}\sum_{i=1}^{m+q} a_{ij}^{-\sigma}\geqslant p\right\}$.

Analogously as in the foregoing part of the proof, we can verify that each of the sets L_{kpqmn} is closed.

Hence, the set $A_d = \left\{x\in S:\lambda(x)>d\right\} = \bigcup_{k=1}^{\infty}\bigcap_{p=1}^{\infty}\bigcup_{q=1}^{\infty}\bigcap_{m=1}^{\infty}\bigcap_{n=1}^{\infty} L_{kpqmn}$ is an $F_{\sigma\delta\sigma}$ set and therefore A_d belongs to the third additive Borel class. □

COROLLARY 2. *The function $\lambda: S\to[0,\infty]$ is a measurable function.*

THEOREM 6. *For every real number d, each set $A^d = \left\{x\in S:\lambda(x)<d\right\}$ is of first category in S.*

Proof. By virtue of the above theorem, we can write

$$A^d = \bigcup_{k=k_0}^{\infty}\bigcap_{l=1}^{\infty}\bigcup_{m=1}^{\infty}\bigcup_{n=1}^{\infty}\bigcap_{p=1}^{\infty} H_{klmnp} = \bigcup_{k=k_0}^{\infty}\bigcap_{l=1}^{\infty} H_{kl},$$

where $H_{kl} = \bigcup_{m=1}^{\infty}\bigcup_{n=1}^{\infty}\bigcap_{p=1}^{\infty}\left\{x: x\in S:\sum_{j=n+1}^{n+p}\sum_{i=m+1}^{m+p} a_{ij}^{-\sigma}\leqslant\frac{1}{l}\right\}$.

We first prove that, each of the sets H_{kl} is of first category in S. For this, it is sufficient to show that, H_{kl} is an F_σ set whose complement is dense in S.

Let $z=\{z_{mn}\}_{m,n\geqslant 1}\in S$. For given $\varepsilon>0$, let r be smallest positive integer such that $\sum\limits_{i=r+1}^{\infty}2^{-i}<\varepsilon$. Take $y=\{(mn)^{\alpha}\}_{m,n\geqslant 1}$ where $\alpha=\frac{1}{\sigma}$.

We now construct a sequence $x=\{x_{mn}\}_{m,n\geqslant 1}\in S$ as follows:

Let $x_{mn}=z_{mn}$ for $m=1,2,\ldots,r$; $n=1,2,\ldots,r$.

If $x_{rr}\leqslant(1+r)^{\alpha}$, then put $x_{mn}=(mn)^{\alpha}$ for all (m,n) when $(m,n)\nleq(r,r)$.

If $x_{rr}>(1+r)^{\alpha}$, then put $x_{mn}=x_{rr}$ for $m=r+1,r+2,\ldots,s-1$; $n=r+1,r+2,\ldots,s-1$, where s is the smallest positive integer for which $s^{\alpha}\geqslant x_{rr}$ and $x_{mn}=(mn)^{\alpha}$ for all (m,n) when $(m,n)\nleq(s-1,s-1)$.

It is clear that $\rho(x,z)<\varepsilon$.

Again, there is a positive integer t for which

$$x_{mn}=(mn)^{\alpha}\ \text{for}\ m=t,t+1,\ldots;\ n=t,t+1,\ldots$$

Then, there exist $q\in\mathbb{N}$, such that for all $(m,n)\in\mathbb{N}\times\mathbb{N}$ with $m\geqslant t$ and $n\geqslant t$ we have

$$\begin{aligned}\sum_{j=t}^{n+q}\sum_{i=t}^{m+q}x_{ij}^{-\sigma}&=\sum_{j=t}^{n+q}\sum_{i=t}^{m+q}(mn)^{-\sigma\alpha}\\&=\sum_{j=t}^{n+q}\sum_{i=t}^{m+q}(mn)^{-1}.\end{aligned}$$

Since, the series $\sum\limits_{m=1}^{\infty}\sum\limits_{n=1}^{\infty}(mn)^{-1}$ is divergent, then x belong to the complement of H_{kl}. Therefore, the complement of H_{kl} is dense in S.

Also each of the sets H_{klmnp} is closed. Hence, $H_{kl}=\bigcup\limits_{m=1}^{\infty}\bigcup\limits_{n=1}^{\infty}\bigcap\limits_{p=1}^{\infty}H_{klmnp}$ is an F_{σ} set. Thus, each of the sets H_{kl} is of first category in S and consequently $A^{d}=\bigcup\limits_{k=k_0}^{\infty}\bigcap\limits_{l=1}^{\infty}H_{kl}$ is of first category. $\square$

THEOREM 7. *The set* $\left\{x\in S:\lambda(x)=\infty\right\}$ *is residual in* S.

Proof. By Theorem 6, the set $\left\{x\in S:\lambda(x)<\infty\right\}=\bigcup\limits_{n=1}^{\infty}\left\{x\in S:\lambda(x)<n\right\}$ is of first category in S and also the space S is complete. Hence, the set $\{x\in S:\lambda(x)=\infty\}$ is residual in S. $\square$

THEOREM 8. *For* $t>0$, *the set* $A=\left\{x\in S:\lambda(x)=t\right\}$ *is dense in* S.

Proof. Let $y=\{y_{mn}\}_{m,n\geqslant 1}$ be any point in S. Let us choose $\varepsilon>0$ and $(k,l)\in\mathbb{N}\times\mathbb{N}$ such that $\sum\limits_{i=k}^{\infty}\sum\limits_{j=l}^{\infty}\frac{1}{2^{ij}}<\varepsilon$.

We choose, $(u,v)\in\mathbb{N}\times\mathbb{N}$ such that for every $(m,n)\in\mathbb{N}\times\mathbb{N}$ with $(m,n)>(k,l)$ we have $\{(m+u)(n+v)\}^{\frac{1}{t}}\geqslant\max\{y_{mn}:(m,n)\leqslant(k,l)\}$.

Define a sequence $z = \{z_{mn}\}_{m,n\geqslant 1}$ such that

$$z_{mn} = \begin{cases} y_{mn}; & \text{for} \quad (m,n) \leqslant (k,l), \\ [(m+u)(n+v)]^{\frac{1}{t}}; & \text{for all} \quad (m,n) \quad \text{where} \quad (m,n) \nleq (i,i). \end{cases}$$

Then, clearly $z \in B(y,\varepsilon)$ and

$$\begin{aligned} \lambda(z) &= \limsup_{m,n} \frac{\log mn}{\log z_{mn}} \\ &= \limsup_{m,n} \frac{\log mn}{\log\{(m+u)(n+v)\}^{\frac{1}{t}}} = t \end{aligned}$$

So, $z \in B(y,\varepsilon)\bigcap A$ and consequently $B(y,\varepsilon)\bigcap A \neq \phi$ for arbitrary $\varepsilon > 0$. Hence, A is dense in S. □

THEOREM 9. *The function $\lambda : S \to [0,\infty]$ is totally discontinuous in S.*

Proof. Let $x = \{x_{mn}\}_{m,n\geqslant 1}$ be any point in S. We take a double sequence $y = \{(mn)^{\alpha}\}_{m,n\geqslant 1}$ with $\alpha > 0$ and $\lambda(x) \neq \frac{1}{\alpha}$. Then,

$$\lambda(y) = \limsup_{m,n\to\infty} \frac{\log(mn)}{\log(mn)^{\alpha}} = \frac{1}{\alpha} \quad \text{i.e.,} \quad \lambda(x) \neq \lambda(y).$$

We now construct a sequence $\{x^{(i)}\}_{i=1}^{\infty}$ in S as follows:

$$x_{mn}^{(i)} = x_{mn} \quad \text{for} \quad m = 1,2,\ldots,i;\ n = 1,2,\ldots,i.$$

If $(1+i)^{\alpha} \geqslant x_{mn}$, then we take $x_{mn}^{i} = (mn)^{\alpha}$ for all (m,n) when $(m,n) \nleq (i,i)$.

If $(1+i)^{\alpha} < x_{mn}$, then we take $x_{mn}^{i} = x_{mn}$ for $m = i+1, i+2, \ldots, j-1$; $n = i+1, i+2, \ldots, j-1$, where j is the least positive integer so that $(j)^{\alpha} \geqslant x_{mn}$ and we set $x_{mn}^{i} = (mn)^{\alpha}$ for all (m,n) when $(m,n) \nleq (j-1, j-1)$.

It is clear that, the element of the sequence $\{x^{(i)}\}_{i=1}^{\infty}$ belongs to S and $\lim_{i\to\infty} x^{(i)} = x$.

Then, $\lambda(x^{(i)}) = \frac{1}{\alpha}$ for $i = 1,2,3,\ldots$ and hence $\lim_{i\to\infty} \lambda(x^{(i)}) = \frac{1}{\alpha} \neq \lambda(x)$. □

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