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MONOTONIC FUNCTIONS RELATED TO THE q–GAMMA AND q–TRIGAMMA FUNCTIONS WITH APPLICATIONS

KHALED MEHREZ

Dedicated to Professor Fozi Dannan

Abstract. In this paper our aim is to investigate necessary and sufficient conditions for the complete monotonicity properties of some functions related to the q-gamma and q-trigamma functions. As application of this results, some new inequalities are derived. Our results are shown to be as a generalization of results which were obtained by Qi [6].

1. Introduction

Recall that a non-negative function f defined on $(0,\infty)$ is called completely monotonic if it has derivatives of all orders and

$$(-1)^n f^{(n)}(x) \geqslant 0, \;\; n \geqslant 1$$

and $x>0$ [[5], Def. 1.3]. This inequality is known to be strict unless f is a constant. By the celebrated Bernstein theorem, a function is completely monotonic if and only if it is the Laplace transform of a non-negative measure [[5], Th. 1.4]. The above definition implies the following equivalences:

$$\begin{gathered} f \text{ is completely monotonic on } (0,\infty). \\ \Leftrightarrow f \geqslant 0,\; -f' \text{ is completely monotonic on } (0,\infty), \\ \Leftrightarrow -f' \text{ is completely monotonic on} (0,\infty), \text{ and } \lim_{x\to\infty} f(x) \geqslant 0. \end{gathered}$$

Euler's gamma function is defined for positive real numbers x by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt,$$

which is one of the most important special functions and has many extensive applications in many branches, for example, statistics, physics, engineering and other mathematical sciences.

The logarithmic derivative of $\Gamma(x)$, denoted $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$, is called the psi or digamma function, and $\psi^{(k)}(x)$ for $k \in \mathbb{N}$ are called the polygamma functions. The

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functions $\Gamma(x)$ and $\psi^{(k)}(x)$ for $k \in \mathbb{N}$ are of fundamental importance in mathematics and have been extensively studied by many authors.

The q-analogue of Γ is defined by

$$\Gamma_q(x) = (1-q)^{1-x} \prod_{j=0}^{\infty} \frac{1-q^{j+1}}{1-q^{j+x}}, \ \ 0 < q < 1, \ \ x > 0, \tag{1}$$

and

$$\Gamma_q(x) = (q-1)^{1-x} q^{\frac{x(x-1)}{2}} \prod_{j=0}^{\infty} \frac{1-q^{-(j+1)}}{1-q^{-(j+x)}}, \ \ q > 1, \ \ x > 0. \tag{2}$$

The q-gamma function $\Gamma_q(z)$ has the following basic properties:

$$\lim_{q \to 1^-} \Gamma_q(z) = \lim_{q \to 1^+} \Gamma_q(z) = \Gamma(z), \tag{3}$$

and

$$\Gamma_q(z) = q^{\frac{(x-1)(x-2)}{2}} \Gamma_{\frac{1}{q}}(z). \tag{4}$$

The q-digamma function ψ_q, the q-analogue of the psi or digamma function ψ is defined by

$$\begin{aligned} \psi_q(x) &= \frac{\Gamma_q'(x)}{\Gamma_q(z)} \\ &= -\ln(1-q) + \ln q \sum_{k=0}^{\infty} \frac{q^{k+x}}{1-q^{k+x}} \\ &= -\ln(1-q) + \ln q \sum_{k=1}^{\infty} \frac{q^{kx}}{1-q^k}, \end{aligned} \tag{5}$$

for $0 < q < 1$.

Using the Euler-Maclaurin formula, Moak [[3], p. 409] obtained the following q-analogue of Stirling formula

$$\begin{aligned} \log \Gamma_q(x) \sim & \left(x - \frac{1}{2}\right) \log\left(\frac{1-q^x}{1-q}\right) + \frac{\mathrm{Li}_2(1-q^x)}{\log q} + \frac{1}{2} H(q-1) \log q + C_{\hat{q}} \\ & + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} \left(\frac{\log \hat{q}}{\hat{q}^x - 1}\right)^{2k-1} \hat{q}^x P_{2k-3}(\hat{q}^x) \end{aligned} \tag{6}$$

as $x \to \infty$ where $H(.)$ denotes the Heaviside step function, B_k, $k = 1, 2, \ldots$ are the Bernoulli numbers,

$$\hat{q} = \begin{cases} q & \text{if } 0 < q < 1 \\ 1/q & \text{if } q > 1. \end{cases}$$

$\mathrm{Li}_2(z)$ is the dilogarithm function defined for complex argument z as

$$\mathrm{Li}_2(z) = -\int_0^z \frac{\log(1-t)}{t} dt, \ \ z \notin [1, \infty). \tag{7}$$

P_k is a polynomial of degree k satisfying

$$P_k(z) = (z - z^2)P'_{k-1}(z) + (kz+1)P_{k-1}(z), \quad P_0 = P_{-1} = 1, \quad k = 1, 2, \dots \tag{8}$$

and

$$C_q = \frac{1}{2}\log(2\pi) + \frac{1}{2}\log\left(\frac{q-1}{\log q}\right) - \frac{1}{24}\log q + \frac{1}{\log(q)}\int_0^{-\log(q)} \frac{u\,du}{e^u - 1}$$
$$+ \log\left(\sum_{m=-\infty}^{\infty} r^{m(6m+1)} - r^{(2m+1)(3m+1)}\right),$$

where $r = \exp(4\pi^2/\log q)$. Simple computation shows that

$$\left(\frac{\mathrm{Li}_2(1-q^x)}{\log(q)}\right)' = \frac{xq^x \log(q)}{1-q^x} \tag{9}$$

On the other hand, we have (see [2])

$$\lim_{q\to 1} \frac{\mathrm{Li}_2(1-q^x)}{\log q} = -x, \quad \text{and} \quad \lim_{q\to 1} c_q = \frac{1}{2}\log(2\pi). \tag{10}$$

The main aim of this paper is to investigate the monotonicity properties of the function

$$\begin{aligned} K_a(x;q) = {} & \frac{1}{12}\psi_q'(x+a) - \log\Gamma_q(x) + \left(x - \frac{1}{2}\right)\log\left(\frac{1-q^x}{1-q}\right) + \frac{\mathrm{Li}_2(1-q^x)}{1-q} \\ & + C_q + \frac{1}{2}H(q-1)\log(q), \end{aligned} \tag{11}$$

where $q \in (0,1)$, $a \geqslant 0$ and $x > 0$.

It is worth mentioning that Qi [6] considered the function

$$K_a(x) = \frac{1}{2}\log(2\pi) - x + \left(x - \frac{1}{2}\right)\log(x) - \log\Gamma(x) + \frac{1}{12}\psi'(x+a), \quad x > 0, \quad a \geqslant 0, \tag{12}$$

which is a special case of the function $K_a(x;q)$ on letting q tends to 1 and proved that $-K_a(x)$ is completely monotonic on $(0,\infty)$ if and only if $a \geqslant 1/2$ and $K_a(x)$ is completely monotonic on $(0,\infty)$ if and only if $a = 0$. As a consequence, some inequalities for the q-gamma function and the function $K_a(x;q)$ were established.

2. Lemmas

For proofs in this paper we need the following lemmas.

LEMMA 1. *Let $q \in (0,1)$, $a \geqslant 0$ and $x > 0$. Then*

$$K_a^{(2)}(x;q) = \sum_{k=1}^{\infty} \frac{q^{kx}\log(q)}{1-q^k} f(a;q^k), \tag{13}$$

where

$$f(a;y) = \frac{1}{12}(y^a \log^3(y) - 6\log(y) - 6y\log(y) + 12y - 12), \ \ y = q^k, \ \ k = 1,2,\ldots \quad (14)$$

Proof. From (5) and (9) we get

$$K_a'(x;q) = \frac{1}{12}\psi_q^{(2)}(x+a) - \psi_q(x) + \log\left(\frac{1-q^x}{1-q}\right) + \frac{q^x \log(q)}{2(1-q^x)}, \quad (15)$$

for $q \in (0,1)$ and $x > 0$. Differentiating (15), using the series expansion

$$\frac{x}{(1-x)^2} = \sum_{k=1}^{\infty} kx^k, \ \ x \in (0,1)$$

and (5) we obtain

$$\begin{aligned} K_a^{(2)}(x;q) &= \frac{1}{12}\psi_q^{(3)}(x+a) - \psi_q'(x) - \frac{q^x \log(q)}{1-q^x} + \frac{q^x \log^2(q)}{(1-q^x)^2} \\ &= \frac{1}{12}\sum_{k=1}^{\infty} \frac{k^3 q^{k(x+a)} \log^4(q)}{1-q^k} - \sum_{k=1}^{\infty} \frac{kq^{kx}\log^2(q)}{1-q^k} - \sum_{k=1}^{\infty} q^{kx}\log(q) + \frac{1}{2}\sum_{k=1}^{\infty} kq^{kx}\log^2(q) \\ &= \sum_{k=1}^{\infty} \frac{q^{kx}\log(q)}{1-q^k} f(a;q^k). \end{aligned}$$

Lemma 1 is thus proved. □

LEMMA 2. *The function $f(0,y)$ as defined in (14) is negative for all $y \in (0,1)$.*

Proof. By using the fact $y^a = \exp(-a\log(1/y))$, $a \in \mathbb{R}$, we can write $f(0,y)$ as

$$\begin{aligned} f(0,y) &= \frac{1}{12}(\log^3(y) - 6\log(y) - 6y\log(y) + 12y - 12) \\ &= \frac{y}{12}\left(-\frac{\log^3(1/y)}{y} + 6\frac{\log(1/y)}{y} - \frac{12}{y} + 6\log(1/y) + 12\right) \\ &= \frac{y}{12}\left(-\sum_{k=0}^{\infty}\frac{\log^{k+3}(1/y)}{k!} + 6\sum_{k=0}^{\infty}\frac{\log^{k+1}(1/y)}{k!} - 12\sum_{k=0}^{\infty}\frac{\log^k(1/y)}{k!} + 6\log(1/y) + 12\right) \\ &= \frac{y}{12}\left(-\sum_{k=3}^{\infty}\frac{\log^k(1/y)}{(k-3)!} + 6\sum_{k=2}^{\infty}\frac{\log^k(1/y)}{(k-1)!} - 12\sum_{k=2}^{\infty}\frac{\log^k(1/y)}{k!}\right) \\ &= \frac{y}{12}\left(-\sum_{k=3}^{\infty}\frac{\log^k(1/y)}{(k-3)!} + 6\sum_{k=3}^{\infty}\frac{\log^k(1/y)}{(k-1)!} - 12\sum_{k=3}^{\infty}\frac{\log^k(1/y)}{k!}\right) \\ &= -\frac{y}{12}\sum_{k=3}^{\infty}\frac{\log^k(1/y)(k-2)(k(k-1)+6)}{k!} < 0, \end{aligned}$$

for all $y \in (0,1)$. So, the proof of Lemma 2 is complete. □

LEMMA 3. *The function* g *defined by*

$$g(y)=\frac{\log(6)+\log(2-2y+y\log(y)+\log(y))-\log(\log^3(y))}{\log(y)} \tag{16}$$

is increasing on $(0,1)$, *such that* $\lim_{y\to 0} g(y)=0$ *and* $\lim_{y\to 1} g(y)=\frac{1}{2}$.

Proof. Let $y\in(0,1)$, thus $g'(y)=g_1(y)/y\log^2(y)$, where

$$\begin{aligned} g_1(y) &= -\log(6)-3+\log(\log^3(y))-\log(2-2y+y\log(y)+\log(y)\\ &\quad +\frac{\log(y)-y\log(y)+y\log^2(y)}{2-2y+y\log(y)+\log(y)}. \end{aligned}$$

Differentiating $g_1(y)$ yields

$$g_1'(y)=\frac{g_2(y)}{y\log(y)(2-2y+y\log(y)+\log(y))^2},$$

where

$$\begin{aligned} g_2(y) &= 12-24y+12y^2+12\log(y)+2\log^2(y)-12y^2\log(y)\\ &\quad +2y^2\log^2(y)+8y\log^2(y)+y\log^4(y). \end{aligned}$$

Now, we can write $g_2(y)$ as

$$\begin{aligned} g_2(y) &= y^2\left(12-\frac{24}{y}+\frac{12}{y^2}-12\frac{\log(1/y)}{y^2}+2\frac{\log^2(1/y)}{y^2}+12\log(1/y)\right.\\ &\quad \left.+2\log^2(1/y)+8\frac{\log^2(1/y)}{y}+\frac{\log^4(1/y)}{y}\right)\\ &= y^2\left(12-24\sum_{k=0}^{\infty}\frac{\log^k(1/k)}{k!}+12\sum_{k=0}^{\infty}\frac{2^k\log^k(1/k)}{k!}\right.\\ &\quad -12\log(1/y)\sum_{k=0}^{\infty}\frac{2^k\log^k(1/k)}{k!}+2\log^2(1/y)\sum_{k=0}^{\infty}\frac{2^k\log^k(1/k)}{k!}+12\log(1/y)\\ &\quad \left.+2\log^2(1/y)+8\log^2(1/y)\sum_{k=0}^{\infty}\frac{\log^k(1/k)}{k!}+\log^4(1/y)\sum_{k=0}^{\infty}\frac{\log^k(1/k)}{k!}\right)\\ &= y^2\left(-24\sum_{k=4}^{\infty}\frac{\log^k(1/k)}{k!}+12\sum_{k=4}^{\infty}\frac{2^k\log^k(1/k)}{k!}-12\sum_{k=3}^{\infty}\frac{2^k\log^{k+1}(1/k)}{k!}\right.\\ &\quad \left.+\sum_{k=2}^{\infty}\frac{2^{k+1}\log^{k+2}(1/k)}{k!}+8\sum_{k=2}^{\infty}\frac{\log^{k+2}(1/k)}{k!}+\sum_{k=0}^{\infty}\frac{\log^{k+4}(1/k)}{k!}\right)\\ &= y^2\sum_{k=4}^{\infty}\frac{\log^k(1/k)a_k}{k!} \end{aligned}$$

where

$$a_k = -24 + 2^{k-1}(24 - 12k + k(k-1)) + k(k-1)(8 + (k-2)(k-3)), \;\; k \geqslant 4.$$

We note that $a_4 = a_5 = a_6 = a_7 = 0$, $a_8 = 56$, $a_9 = 504$, $a_{10} = 2664$ and $a_k \geqslant a_{11} > 0$. So the sequence $a_n \geqslant 0$ for all $k \geqslant 4$. Thus $g_2(y) > 0$ for all $y \in (0,1)$. Therefore the function $g_1(y)$ is decreasing on $(0,1)$. On the other hand, by using the l'Hospital's rule we have

$$\lim_{y\to 1} \frac{2 - 2y + y\log(y) + \log(y)}{\log^3(y)} = \lim_{y\to 1} \frac{y\log(y)}{6\log(y)} = \frac{1}{6},$$

and

$$\lim_{y\to 1} \frac{\log(y) - y\log(y) + y\log^2(y)}{2 - 2y + y\log(y) + \log(y)} = 1 + \lim_{y\to 1} \frac{2\log(y) + \log^2(y)}{\log(y)} = 3.$$

Thus implies that $g_1(y) > \lim_{y\to 1} g_1(y) = 0$, and consequently the function $g(y)$ is increasing on $(0,1)$. Finally, by using the l'Hospital's rule we get

$$\lim_{y\to 1} g(y) = \frac{1}{2}$$

and it is easy to proved that

$$\lim_{y\to 0} g(y) = 0,$$

which completes the proof. □

3. Completely monotonic functions related to the q-gamma and q-trigamma functions

THEOREM 1. *Let $q \in (0,1)$. Then the function $K_a(x;q)$ is completely monotonic on $(0,\infty)$ if and only if $a = 0$.*

Proof. By contradiction. Suppose that the function $K_a(x;q)$ for $a > 0$ is completely monotonic on $(0,\infty)$, thus means that $K_a(x;q)$ is positive on $(0,\infty)$. But, using the q-analogue of Stirling formula (6) we gave for $q \in (0,1)$

$$\begin{aligned}\lim_{x\to 0^+} K_a(x;q) &= \frac{1}{12}\psi_q'(a) - \lim_{x\to 0^+} \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!}\left(\frac{\log(\hat{q})}{\hat{q}^x - 1}\right)^{2k-1} \hat{q}^x P_{2k-3}(\hat{q}^x)\\ &= -\infty,\end{aligned}$$

which leads to a contradiction and $a = 0$. Now we proved that the function $K_0(x;q)$ is completely monotonic on $(0,\infty)$. By again using Lemma 1 and Lemma 2 we conclude that $K_0^{(2)}(x;q)$ is completely monotonic on $(0,\infty)$. Therefore the function $K_0'(x;q)$ is increasing on $(0,\infty)$. Thus

$$\begin{aligned}K_0'(x;q) &\leqslant \lim_{x\to\infty} K_0'(x;q)\\ &= \lim_{x\to\infty}\left(\frac{1}{12}\psi_q^{(2)}(x) - \psi_q(x) + \log\left(\frac{1-q^x}{1-q}\right) + \frac{q^x\log(q)}{2(1-q^x)}\right).\end{aligned} \tag{17}$$

On the other hand, from (5) we have

$$\lim_{x\to\infty} \psi^{(k)}(x+a) = 0, \text{ and } \lim_{x\to\infty} \psi(x) = -\log(1-q), \ k \geqslant 1 \tag{18}$$

for all $q \in (0,1)$ and $a \geqslant 0$ Combining (17) and (18) we conclude that $K_0'(x;q) \leqslant 0$ for all $q \in (0,1)$ and $x > 0$. Consequently, the function $K_0(x;q)$ is decreasing on $(0,\infty)$. From the asymptotic formula (6) and (18) we have for $q \in (0,1)$

$$\begin{aligned} K_0(x;q) &\geqslant \lim_{x\to\infty} K_0(x;q) \\ &= -\lim_{x\to\infty} \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} \left(\frac{\log(\hat{q})}{\hat{q}^x - 1}\right)^{2k-1} \hat{q}^x P_{2k-3}(\hat{q}^x) \\ &= 0. \end{aligned} \tag{19}$$

So the function $K_0(x;q)$ is completely monotonic on $(0,\infty)$ for $q \in (0,1)$. This ends the proof. □

THEOREM 2. *Let* $q \in (0,1)$. *Then the function* $-K_a(x;q)$ *is completely monotonic on* $(0,\infty)$ *if and only if* $a \geqslant g(q)$.

Proof. Assume that the function $-K_a(x;q)$ is completely monotonic on $(0,\infty)$, thus $-q^{-x}K_a(x;q) \geqslant 0$. In [4] proved that

$$\lim_{x\to\infty} \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} \left(\frac{\log(\hat{q})}{\hat{q}^x - 1}\right)^{2k-1} P_{2k-3}(\hat{q}^x) = \frac{1}{1-\hat{q}} + \frac{1}{\log(\hat{q})} - \frac{1}{2} \tag{20}$$

and using (5) we get

$$\lim_{x\to\infty} q^{-x} \psi_q'(x+a) = \frac{q^a \log^2(q)}{1-q}. \tag{21}$$

Finally, by (20) and (21) such that $\lim_{x\to\infty}(-q^{-x}K_a(x;q)) \geqslant 0$ we conclude that $a \geqslant g(q)$.

Conversely, from Lemma 1, we have

$$-K_a^{(2)}(x;q) = -\sum_{k=1}^{\infty} \frac{q^{kx}\log(q)}{1-q^k} f(a;q^k),$$

where $f(a;y)$, $y = q^k$ as defined in (14). Moreover, using the fact that the function $a \mapsto f(a,y)$ is increasing on $(0,\infty)$, and since

$$\begin{aligned} \lim_{x\to\infty} f(a,y) &= -6\log(y) - 6y\log(y) + 12y - 12 \\ &= y\left(\sum_{k=3}^{\infty} \frac{\log(1/y)^k(6k-12)}{k!}\right) > 0 \end{aligned}$$

and Lemma 2 and the intermediate value Theorem we conclude that the function $a \mapsto f(a,y)$ admits a zero depending on the values of y at $a = g(y)$. From Lemma 3, the function $g(y)$ is increasing on $(0,\infty)$ such that $0 \leqslant g(y) \leqslant 1/2$ for all $y \in (0,1)$. Therefore to take $a \geqslant g(q)$ to ensure that $f(a,y) > 0$ for all $y = q^k$. Thus implies that the function $-K_a^{(2)}(x;q)$ is completely monotonic on $(0,\infty)$ for $a \geqslant g(q)$. So the function $K_a'(x;q)$ is decreasing on $(0,\infty)$, in particular $K_a'(x;q) \geqslant \lim_{x\to\infty} K_a'(x;q)$. In view of (15) and (18) we see that $K_a'(x;q) \geqslant 0$ for all $q \in (0,1)$ and $x \in (0,\infty)$. In particular, the function $K_a(x;q)$ is increasing on $(0,\infty)$. Thus $K_a(x;q) \leqslant \lim_{x\to\infty} K_a(x;q)$. Finally (18) and the q-analogue of Stirling formula (6) we conclude that $\lim_{x\to\infty} K_a(x;q) = 0$. Consequently the function $-K_a(x;q)$ is completely monotonic on $(0,\infty)$ for $q \in (0,1)$. So the proof of Theorem 2 is complete. □

As application of the complete monotonicity properties of the function (11) which are proved in Theorem 1 and Theorem 2 we can provide the following inequalities for the q-gamma function .

The next result is a generalization of the inequalities proved by Qi in [[6], Remark 4].

COROLLARY 1. *Let $q \in (0,1)$ and $x > 0$. Then the following inequalities*

$$e^{C_q} q^{\frac{1}{12}H(q-1)} \exp\left(\frac{1}{12}\psi_q'(x+\alpha) + \frac{Li_2(1-q^x)}{1-q}\right) \leqslant \Gamma_q(x) \tag{22}$$

$$\leqslant e^{C_q} q^{\frac{1}{12}H(q-1)} \exp\left(\frac{1}{12}\psi_q'(x+\alpha) + \frac{Li_2(1-q^x)}{1-q}\right),$$

holds for $\alpha = 0$ and $\beta \geqslant g(q)$.

Proof. From Theorem 1 and Theorem 2, we obtain for $x > 0$ and $q \in (0,1)$

$$K_\beta(x;q) \leqslant 0 \leqslant K_\alpha(x;q)$$

holds if and only if $\beta \geqslant g(q)$ and $\alpha = 0$. □

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Khaled Mehrez
Département de Mathématiques ISSAT Kasserine
Université de Kairouan, Tunisia
and
Faculté des Sciences de Tunis
Université Tunis el Manar
e-mail: k.mehrez@yahoo.fr