MONOTONIC FUNCTIONS RELATED TO THE $q$−GAMMA AND $q$−TRIGAMMA FUNCTIONS WITH APPLICATIONS

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Dedicated to Professor Fozi Dannan

Abstract. In this paper our aim is to investigate necessary and sufficient conditions for the complete monotonicity properties of some functions related to the $q$-gamma and $q$-trigamma functions. As application of this results, some new inequalities are derived. Our results are shown to be as a generalization of results which were obtained by Qi [6].

1. Introduction

Recall that a non-negative function $f$ defined on $(0, \infty)$ is called completely monotonic if it has derivatives of all orders and

$(-1)^n f^{(n)}(x) \geq 0, \quad n \geq 1$

and $x > 0$ [[5], Def. 1.3]. This inequality is known to be strict unless $f$ is a constant. By the celebrated Bernstein theorem, a function is completely monotonic if and only if it is the Laplace transform of a non-negative measure [[5], Th. 1.4]. The above definition implies the following equivalences:

$f$ is completely monotonic on $(0, \infty)$.

$\iff f \geq 0$, 

$-f'$ is completely monotonic on $(0, \infty)$,

$\iff -f'$ is completely monotonic on $(0, \infty)$, and $\lim_{x \to \infty} f(x) \geq 0$.

Euler’s gamma function is defined for positive real numbers $x$ by

$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt,$

which is one of the most important special functions and has many extensive applications in many branches, for example, statistics, physics, engineering and other mathematical sciences.

The logarithmic derivative of $\Gamma(x)$, denoted $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$, is called the psi or digamma function, and $\psi^{(k)}(x)$ for $k \in \mathbb{N}$ are called the polygamma functions. The


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functions $\Gamma(x)$ and $\psi^{(k)}(x)$ for $k \in \mathbb{N}$ are of fundamental importance in mathematics and have been extensively studied by many authors.

The $q$-analogue of $\Gamma$ is defined by
\[
\Gamma_q(x) = (1 - q)^{1-x} \prod_{j=0}^{\infty} \frac{1 - q^{j+1}}{1 - q^{j+x}}, \quad 0 < q < 1, \ x > 0,
\] (1)
and
\[
\Gamma_q(x) = (q - 1)^{1-x} q^{\frac{x(x-1)}{2}} \prod_{j=0}^{\infty} \frac{1 - q^{-(j+1)}}{1 - q^{-(j+x)}}, \quad q > 1, \ x > 0.
\] (2)

The $q$-gamma function $\Gamma_q(z)$ has the following basic properties:
\[
\lim_{q \to 1^-} \Gamma_q(z) = \lim_{q \to 1^+} \Gamma_q(z) = \Gamma(z),
\] (3)
and
\[
\Gamma_q(z) = q^{\frac{(z-1)(z-2)}{2}} \Gamma_{\hat{q}}(z).
\] (4)

The $q$-digamma function $\psi_q$, the $q$-analogue of the psi or digamma function $\psi$ is defined by
\[
\psi_q(x) = \frac{\Gamma'_q(x)}{\Gamma_q(z)} = -\ln(1-q) + \ln \sum_{k=0}^{\infty} \frac{q^{k+x}}{1 - q^{k+x}}
\] (5)
\[
= -\ln(1-q) + \ln \sum_{k=1}^{\infty} \frac{q^{kx}}{1 - q^k},
\]
for $0 < q < 1$.

Using the Euler-Maclaurin formula, Moak [[3], p. 409] obtained the following $q$-analogue of Stirling formula
\[
\log \Gamma_q(x) \sim \left( x - \frac{1}{2} \right) \log \left( \frac{1 - q^x}{1 - q} \right) + \frac{\text{Li}_2(1 - q^x)}{\log q} + \frac{1}{2} H(q - 1) \log q + C_{\hat{q}} + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} \left( \frac{\log \hat{q}}{\hat{q}^x - 1} \right)^{2k-1} \hat{q}^x P_{2k-3}(\hat{q}^x)
\] (6)
as $x \to \infty$ where $H(.)$ denotes the Heaviside step function, $B_k, \ k = 1,2,\ldots$ are the Bernoulli numbers,
\[
\hat{q} = \begin{cases} q & \text{if } 0 < q < 1 \\ 1/q & \text{if } q > 1. \end{cases}
\]

$\text{Li}_2(z)$ is the dilogarithm function defined for complex argument $z$ as
\[
\text{Li}_2(z) = -\int_0^{z} \frac{\log(1-t)}{t} \, dt, \ z \notin [1, \infty).
\] (7)
$P_k$ is a polynomial of degree $k$ satisfying
\[ P_k(z) = (z - z^2)P_{k-1}'(z) + (kz + 1)P_{k-1}(z), \quad P_0 = P_{-1} = 1, \quad k = 1, 2, \ldots \tag{8} \]
and
\[
C_q = \frac{1}{2} \log(2\pi) + \frac{1}{2} \log \left( \frac{q-1}{\log q} \right) - \frac{1}{24} \log q + \frac{1}{\log(q)} \int_0^{-\log(q)} \frac{udu}{e^u - 1}
+ \log \left( \sum_{m=-\infty}^{\infty} r^{m(6m+1)} - r^{(2m+1)(3m+1)} \right),
\]
where $r = \exp(4\pi^2/\log q)$. Simple computation shows that
\[
\left( \frac{\text{Li}_2(1-q^x)}{\log(q)} \right)' = \frac{xq^x \log(q)}{1-q^x}
\tag{9}
\]
On the other hand, we have (see [2])
\[
\lim_{q \to 1} \frac{\text{Li}_2(1-q^x)}{\log q} = -x, \quad \text{and} \quad \lim_{q \to 1} c_q = \frac{1}{2} \log(2\pi).
\tag{10}
\]
The main aim of this paper is to investigate the monotonicity properties of the function
\[
K_a(x; q) = \frac{1}{12} \psi_q'(x+a) - \log \Gamma_q(x) + \left( x - \frac{1}{2} \right) \log \left( \frac{1-q^x}{1-q} \right) + \frac{\text{Li}_2(1-q^x)}{1-q} + C_q + \frac{1}{2} H(q-1) \log(q),
\tag{11}
\]
where $q \in (0, 1)$, $a \geq 0$ and $x > 0$.

It is worth mentioning that Qi [6] considered the function
\[
K_a(x) = \frac{1}{2} \log(2\pi) - x + \left( x - \frac{1}{2} \right) \log(x) - \log \Gamma(x) + \frac{1}{12} \psi'(x+a), \quad x > 0, \quad a \geq 0,
\tag{12}
\]
which is a special case of the function $K_a(x; q)$ on letting $q$ tends to 1 and proved that $-K_a(x)$ is completely monotonic on $(0, \infty)$ if and only if $a \geq 1/2$ and $K_a(x)$ is completely monotonic on $(0, \infty)$ if and only if $a = 0$. As a consequence, some inequalities for the $q$-gamma function and the function $K_a(x; q)$ were established.

### 2. Lemmas

For proofs in this paper we need the following lemmas.

**Lemma 1.** Let $q \in (0, 1)$, $a \geq 0$ and $x > 0$. Then
\[
K_a^{(2)}(x; q) = \sum_{k=1}^{\infty} \frac{q^{kx} \log(q)}{1-q^k} f(a; q^k),
\tag{13}
\]
where
\[ f(a; y) = \frac{1}{12}(y^a \log^3(y) - 6 \log(y) - 6y \log(y) + 12y - 12), \quad y = q^k, \quad k = 1, 2, \ldots \quad (14) \]

Proof. From (5) and (9) we get
\[ K_\alpha'(x; q) = \frac{1}{12}\psi_{q}^{(2)}(a + x) - \psi_{q}(x) + \log \left( 1 - \frac{q}{1 - q} \right) + \frac{q^k \log(q)}{2(1 - q^k)}, \quad (15) \]
for \( q \in (0, 1) \) and \( x > 0 \). Differentiating (15), using the series expansion
\[
\frac{x}{(1-x)^2} = \sum_{k=1}^{\infty} kx^k, \quad x \in (0, 1)
\]
and (5) we obtain
\[
K_{\alpha}^{(2)}(x; q) = \frac{1}{12}\psi_{q}^{(3)}(x + a) - \psi_{q}'(x) - \frac{q^k \log(q)}{1 - q^k} + \frac{q^k \log^2(q)}{(1 - q^k)^2} - \sum_{k=1}^{\infty} kq^k \log^2(q) + \frac{1}{2} \sum_{k=1}^{\infty} kq^k \log^2(q)
\]
\[ = \sum_{k=1}^{\infty} \frac{q^k \log^2(q)}{1 - q^k} f(a; q^k). \]

Lemma 1 is thus proved. \( \square \)

**Lemma 2.** The function \( f(0, y) \) as defined in (14) is negative for all \( y \in (0, 1) \).

Proof. By using the fact \( y^a = \exp(-a \log(1/y)), \ a \in \mathbb{R} \), we can write \( f(0, y) \) as
\[
f(0, y) = \frac{1}{12}(\log^3(y) - 6 \log(y) - 6y \log(y) + 12y - 12)
\]
\[ = \frac{y}{12} \left( - \frac{\log(1/y)}{y} + 6 \frac{\log(1/y)}{y} - \frac{12}{y} + 6 \log(1/y) + 12 \right)
\]
\[ = \frac{y}{12} \left( - \sum_{k=0}^{\infty} \frac{\log^{k+3}(1/y)}{k!} + 6 \sum_{k=0}^{\infty} \frac{\log^{k+1}(1/y)}{k!} - 12 \sum_{k=0}^{\infty} \frac{\log^k(1/y)}{k!} + 6 \log(1/y) + 12 \right)
\]
\[ = \frac{y}{12} \left( - \sum_{k=3}^{\infty} \frac{\log^{k}(1/y)}{(k-3)!} + 6 \sum_{k=2}^{\infty} \frac{\log^{k}(1/y)}{(k-1)!} - 12 \sum_{k=2}^{\infty} \frac{\log^{k}(1/y)}{k!} \right)
\]
\[ = \frac{y}{12} \left( - \sum_{k=3}^{\infty} \frac{\log^{k}(1/y)}{(k-3)!} + 6 \sum_{k=3}^{\infty} \frac{\log^{k}(1/y)}{(k-1)!} - 12 \sum_{k=3}^{\infty} \frac{\log^{k}(1/y)}{k!} \right)
\]
\[ = -\frac{y}{12} \sum_{k=3}^{\infty} \frac{\log^{k}(1/y)(k-2)(k(k-1)+6)}{k!} < 0,
\]
for all \( y \in (0, 1) \). So, the proof of Lemma 2 is complete. \( \square \)
**Lemma 3.** The function $g$ defined by

$$g(y) = \frac{\log(6) + \log(2 - 2y + y\log(y) + \log(y)) - \log(\log^3(y))}{\log(y)}$$

is increasing on $(0, 1)$, such that $\lim_{y \to 0} g(y) = 0$ and $\lim_{y \to 1} g(y) = \frac{1}{2}$.

**Proof.** Let $y \in (0, 1)$, thus $g'(y) = g_1(y)/y\log^2(y)$, where

$$g_1(y) = -\log(6) - 3 + \log(\log^3(y)) - \log(2 - 2y + y\log(y) + \log(y))$$

$$+ \frac{\log(y) - y\log(y) + y\log^2(y)}{2 - 2y + y\log(y) + \log(y)}.$$

Differentiating $g_1(y)$ yields

$$g_1'(y) = \frac{g_2(y)}{y\log(y)(2 - 2y + y\log(y) + \log(y))^2},$$

where

$$g_2(y) = 12 - 24y + 12y^2 + 12\log(y) + 2\log^2(y) - 12y^2\log(y)$$

$$+ 2y^2\log^2(y) + 8y\log^2(y) + y\log^4(y).$$

Now, we can write $g_2(y)$ as

$$g_2(y) = y^2\left(12 - \frac{24}{y} + \frac{12}{y^2} - 12\frac{\log(1/y)}{y^2} + 2\frac{\log^2(1/y)}{y^2} + 12\log(1/y)$$

$$+ 2\log^2(1/y) + 8\frac{\log^2(1/y)}{y} + \frac{\log^4(1/y)}{y}\right)$$

$$= y^2\left(12 - 24\sum_{k=0}^{\infty} \frac{\log^k(1/k)}{k!} + 12\sum_{k=0}^{\infty} \frac{2^k\log^k(1/k)}{k!}$$

$$- 12\log(1/y)\sum_{k=0}^{\infty} \frac{2^k\log^k(1/k)}{k!} + 2\log^2(1/y)\sum_{k=0}^{\infty} \frac{2^k\log^k(1/k)}{k!} + 12\log(1/y)$$

$$+ 2\log^2(1/y) + 8\log^2(1/y)\sum_{k=0}^{\infty} \frac{\log^k(1/k)}{k!} + \log^4(1/y)\sum_{k=0}^{\infty} \frac{\log^k(1/k)}{k!}\right)$$

$$= y^2\left(-24\sum_{k=4}^{\infty} \frac{\log^k(1/k)}{k!} + 12\sum_{k=4}^{\infty} \frac{2^k\log^k(1/k)}{k!} - 12\sum_{k=3}^{\infty} \frac{2^k\log^{k+1}(1/k)}{k!}$$

$$+ \sum_{k=2}^{\infty} \frac{2^{k+1}\log^{k+2}(1/k)}{k!} + 8\sum_{k=2}^{\infty} \frac{\log^{k+2}(1/k)}{k!} + \sum_{k=0}^{\infty} \frac{\log^{k+4}(1/k)}{k!}\right)$$

$$= y^2\sum_{k=4}^{\infty} \frac{\log^k(1/k)d_k}{k!}.$$
where
\[ a_k = -24 + 2^{k-1}(24 - 12k + k(k-1)) + k(k-1)(8 + (k-2)(k-3)), \quad k \geq 4. \]

We note that \( a_4 = a_5 = a_6 = a_7 = 0, \) \( a_8 = 56, \) \( a_9 = 504, \) \( a_{10} = 2664 \) and \( a_k \geq a_{11} > 0. \)
So the sequence \( a_n \geq 0 \) for all \( k \geq 4. \) Thus \( g_2(y) > 0 \) for all \( y \in (0, 1). \) Therefore the function \( g_1(y) \) is decreasing on \((0, 1)\). On the other hand, by using the l’Hospital’s rule we have
\[
\lim_{y \to 1} \frac{2 - 2y + y \log(y) + \log(y)}{\log^3(y)} = \lim_{y \to 1} \frac{y \log(y)}{6 \log(y)} = \frac{1}{6},
\]
and
\[
\lim_{y \to 1} \frac{\log(y) - y \log(y) + y \log^2(y)}{2 - 2y + y \log(y) + \log(y)} = 1 + \lim_{y \to 1} \frac{2 \log(y) + \log^2(y)}{\log(y)} = 3.
\]
Thus implies that \( g_1(y) > \lim_{y \to 1} g_1(y) = 0, \) and consequently the function \( g(y) \) is increasing on \((0, 1)\). Finally, by using the l’Hospital’s rule we get
\[
\lim_{y \to 1} g(y) = \frac{1}{2}
\]
and it is easy to proved that
\[
\lim_{y \to 0} g(y) = 0,
\]
which completes the proof. \( \square \)

3. Completely monotonic functions related to the \( q \)-gamma and \( q \)-trigamma functions

**Theorem 1.** Let \( q \in (0, 1). \) Then the function \( K_a(x; q) \) is completely monotonic on \((0, \infty)\) if and only if \( a = 0. \)

**Proof.** By contradiction. Suppose that the function \( K_a(x; q) \) for \( a > 0 \) is completely monotonic on \((0, \infty), \) thus means that \( K_a(x; q) \) is positive on \((0, \infty). \) But, using the \( q \)-analogue of Stirling formula (6) we gave for \( q \in (0, 1) \)
\[
\lim_{x \to 0^+} K_a(x; q) = \frac{1}{12} \psi_q'(a) - \lim_{x \to 0^+} \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} \left( \log(\hat{q}) \right)^{2k-1} \hat{q}^x P_{2k-3}(\hat{q}^x)
\]
\[
= -\infty,
\]
which leads to a contradiction and \( a = 0. \) Now we proved that the function \( K_0(x; q) \) is completely monotonic on \((0, \infty). \) By again using Lemma 1 and Lemma 2 we conclude that \( K_0^{(2)}(x; q) \) is completely monotonic on \((0, \infty). \) Therefore the function \( K_0^1(x; q) \) is increasing on \((0, \infty). \) Thus
\[
K_0'(x; q) \leq \lim_{x \to \infty} K_0'(x; q)
\]
\[
= \lim_{x \to \infty} \left( \frac{1}{12} \psi_q^{(2)}(x) - \psi_q(x) + \log \left( \frac{1 - q^x}{1 - q} \right) + \frac{q^x \log(q)}{2(1 - q^x)} \right). \tag{17}
\]
On the other hand, from (5) we have
\[
\lim_{x \to \infty} \psi^{(k)}(x + a) = 0, \quad \text{and} \quad \lim_{x \to \infty} \psi(x) = -\log(1 - q), \quad k \geq 1
\]  
(18)

for all \( q \in (0, 1) \) and \( a \geq 0 \) Combining (17) and (18) we conclude that \( K'_0(x; q) \leq 0 \) for all \( q \in (0, 1) \) and \( x > 0 \). Consequently, the function \( K_0(x; q) \) is decreasing on \((0, \infty)\). From the asymptotic formula (6) and (18) we have for \( q \in (0, 1) \)
\[
K_0(x; q) \geq \lim_{x \to \infty} K_0(x; q)
= -\lim_{x \to \infty} \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} \left( \frac{\log(\hat{q})}{\hat{q}^x - 1} \right)^{2k-1} \hat{q}^x P_{2k-3}(\hat{q}^x)
= 0.
\]

So the function \( K_0(x; q) \) is completely monotonic on \((0, \infty)\) for \( q \in (0, 1) \). This ends the proof. \( \square \)

**Theorem 2.** Let \( q \in (0, 1) \). Then the function \(-K_a(x; q)\) is completely monotonic on \((0, \infty)\) if and only if \( a \geq g(q) \).

**Proof.** Assume that the function \(-K_a(x; q)\) is completely monotonic on \((0, \infty)\), thus \(-q^{-x}K_a(x; q) \geq 0 \). In [4] proved that
\[
\lim_{x \to \infty} \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} \left( \frac{\log(\hat{q})}{\hat{q}^x - 1} \right)^{2k-1} P_{2k-3}(\hat{q}^x) = \frac{1}{1 - \hat{q}} + \frac{1}{\log(\hat{q})} - \frac{1}{2}
\]
and using (5) we get
\[
\lim_{x \to \infty} q^{-x} \psi'(x + a) = \frac{q^a \log^2(q)}{1 - q}.
\]

Finally, by (20) and (21) such that \( \lim_{x \to \infty} (-q^{-x}K_a(x; q)) \geq 0 \) we conclude that \( a \geq g(q) \).

Conversely, from Lemma 1, we have
\[
-K_a^{(2)}(x; q) = -\sum_{k=1}^{\infty} q^{kx} \log(q) \frac{f(a; q^k)}{1 - q^k},
\]
where \( f(a; y) = y = q^k \) as defined in (14). Moreover, using the fact that the function \( a \mapsto f(a, y) \) is increasing on \((0, \infty)\), and since
\[
\lim_{x \to \infty} f(a, y) = -6 \log(y) - 6y \log(y) + 12y - 12
= y \left( \sum_{k=3}^{\infty} \frac{\log(1/y)^k(6k - 12)}{k!} \right) > 0
\]
and Lemma 2 and the intermediate value Theorem we conclude that the function \( a \mapsto f(a, y) \) admits a zero depending on the values of \( y \) at \( a = g(y) \). From Lemma 3, the
function \( g(y) \) is increasing on \((0, \infty)\) such that \( 0 \leq g(y) \leq \frac{1}{2} \) for all \( y \in (0, 1) \). Therefore to take \( a \geq g(q) \) to ensure that \( f(a, y) > 0 \) for all \( y = q^x \). Thus implies that the function \(-K^{(2)}_a(x; q)\) is completely monotonic on \((0, \infty)\) for \( a \geq g(q) \). So the function \( K'_a(x; q) \) is decreasing on \((0, \infty)\), in particular \( K'_a(x; q) \geq \lim_{x \to \infty} K'_a(x; q) \). In view of (15) and (18) we see that \( K'_a(x; q) \geq 0 \) for all \( q \in (0, 1) \) and \( x \in (0, \infty) \). In particular, the function \( K_a(x; q) \) is increasing on \((0, \infty)\). Thus \( K_a(x; q) \leq \lim_{x \to \infty} K_a(x; q) \). Consequently the function \(-K_a(x; q)\) is completely monotonic on \((0, \infty)\) for \( q \in (0, 1) \). So the proof of Theorem 2 is complete. \( \Box \)

As application of the complete monotonicity properties of the function (11) which are proved in Theorem 1 and Theorem 2 we can provide the following inequalities for the \( q \)-gamma function.

The next result is a generalization of the inequalities proved by Qi in [[6], Remark 4].

**Corollary 1.** Let \( q \in (0, 1) \) and \( x > 0 \). Then the following inequalities

\[
e^{C_q q \frac{1}{2} H(q^{-1})} \exp \left( \frac{1}{12} \psi_q'(x + \alpha) + \frac{\mathrm{Li}_2(1 - q^x)}{1 - q} \right) \leq \Gamma_q(x)
\]

holds for \( \alpha = 0 \) and \( \beta \geq g(q) \).

**Proof.** From Theorem 1 and Theorem 2, we obtain for \( x > 0 \) and \( q \in (0, 1) \)

\[
K_\beta(x; q) \leq 0 \leq K_\alpha(x; q)
\]

holds if and only if \( \beta \geq g(q) \) and \( \alpha = 0 \). \( \Box \)

**References**


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