VECTOR–STABILITY OF MULTIPLE VECTOR REFINABLE VECTORS

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Abstract. The stability is an expected property for refinable vectors, which is widely considered in the study of refinement equations. There are two types of stability for refinable vectors. One is the ordinary-stability, the other is the vector-stability. The ordinary-stability considers the stability of entries of refinable vectors, but the vector-stability considers the stability of refinable vectors themselves where they are considered as elements of super Hilbert spaces. In this paper, we give a necessary and sufficient condition for refinable vectors to be vector-stable. Our results improve some known ones.

1. Introduction and the main result

In this paper, we study vector refinement equations of the following form

$$\Phi(x) = \sum_{k \in \mathbb{Z}} P_k \Phi(2x - k), \quad x \in \mathbb{R}. \tag{1}$$

Here, $$\Phi = (\Phi_1, \cdots, \Phi_n)$$, where $$\Phi_j \in L^2(\mathbb{R})^r, 1 \leq j \leq n$$ and $$\{P_k : k \in \mathbb{Z}\}$$ is the refinement mask such that each $$P_k$$ is an $$r \times r$$ (complex) matrix. A nonzero solution of (1) is called a refinable vector.

Vector refinement equations are widely studied in the literature. Daubechies and Cohen [4], Heil and Colella [12], and Long, Chen and Yan [26, 27] studied the existence of solutions of (1). And Daubechies, Jia, Jiang, Lau, Micchelli, Shen, Zhou etc. discussed the regularity of refinable functions [1–3, 5–11, 16–19, 25, 28, 30–31]. In particular, the stability of solutions of vector refinement equations was characterized by Shen [29] and Jia [22]. Hogan [14, 15] and Shen, Jiang and Lawton [23, 24] gave some characterizations for the stability of solutions of multiple vector refinement equations.

Recall that a vector $$\Phi = (\phi_1, \cdots, \phi_r)^T \in L^2(\mathbb{R})^r$$ is said to be stable [14] if there exist constants $$0 < \beta_1 \leq \beta_2 < \infty$$ such that for any $$a = \{a_{p,k} : 1 \leq p \leq r, k \in \mathbb{Z}\} \in \ell^2(\mathbb{Z})$$,

$$\beta_1 \|a\|_{\ell^2(\mathbb{Z})}^2 \leq \left\| \sum_{k \in \mathbb{Z}} \sum_{p=1}^{r} a_{p,k} \phi_p(\cdot - k) \right\|^2_{L^2(\mathbb{R})} \leq \beta_2 \|a\|_{\ell^2(\mathbb{Z})}^2.$$

For convenience, we call the vector $$\Phi$$ ordinarily stable whenever the above conditions are satisfied.

In this paper, we study the stability of refinable vectors in the following sense.


Keywords and phrases: Super Hilbert spaces, refinement equation, vector-stability.

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DEFINITION 1. Let $\Phi = (\Phi_1, \ldots, \Phi_n)$, where $\Phi_j \in L^2(\mathbb{R})^r$, $1 \leq j \leq n$. $\Phi$ is said to be vector-stable if there exist constants $0 < \beta_1 \leq \beta_2 < \infty$ such that for any $a = \{a_{j,k} : 1 \leq j \leq n, k \in \mathbb{Z}\} \in \ell^2(\mathbb{Z})$,

$$\beta_1 \|a\|_{\ell^2(\mathbb{Z})}^2 \leq \left\| \sum_{j=1}^{n} \sum_{k \in \mathbb{Z}} a_{j,k} \Phi_j(\cdot - k) \right\|_{L^2(\mathbb{R})^r}^2 \leq \beta_2 \|a\|_{\ell^2(\mathbb{Z})}^2.$$ 

Before going further, we introduce some notations used in this paper. The Fourier transform of a function in $L^1(\mathbb{R})$ is defined by

$$\hat{f}(\omega) = \int_{\mathbb{R}} f(x) e^{-ix\omega} \, dx.$$ 

For $F = (f_1, \ldots, f_r)^T$, $G = (g_1, \ldots, g_r)^T \in L^2(\mathbb{R})^r$, the inner product of $F$ and $G$ is given by

$$\langle F, G \rangle_{L^2(\mathbb{R})^r} = \int_{\mathbb{R}} \sum_{p=1}^{r} f_p(x) \overline{g_p(x)} \, dx,$$

the norm of $F$ is defined by $\|F\|_{L^2(\mathbb{R})^r} = \langle F, F \rangle^{1/2}$ and $[F, G](\xi) = \sum_{j=1}^{r} \sum_{k \in \mathbb{Z}} \hat{f}_j(\xi + 2\pi k) \overline{g_j(\xi + 2\pi k)}$.

In the Fourier domain, the refinement equation (1) can be written as

$$\hat{\Phi}(\omega) = P\left( \frac{\omega}{2} \right) \hat{\Phi}\left( \frac{\omega}{2} \right),$$

where

$$P(\omega) := \frac{1}{2} \sum_{k \in \mathbb{Z}} P_k e^{-ik\omega},$$

and the Fourier transform of the vector-valued function $\Phi$ is defined componentwise.

The symbols $\mathbb{N}$ and $\mathbb{Z}_+$ denote the set of natural numbers and non-negative integers, respectively. We denote by $\mathbb{T}$ the quotient group $\mathbb{R}/2\pi \mathbb{Z}$.

For a given integer $m \geq 2$, we say that a point $\omega \in \mathbb{R}$ is $m$-cyclic in $\mathbb{T}$ if $2^m \omega = \omega \neq 0$ in $\mathbb{T}$. It was shown in [13] that if $\omega$ is $m$-cyclic in $\mathbb{T}$ for some integer $m \geq 2$, then for any $k \in \mathbb{Z}$,

$$\omega + 2k\pi = 2^{m_0} \omega + 2^{m-q} \nu \pi$$

for some $n \in \mathbb{N}$, $q \in \{0, \ldots, m-1\}$ and $\nu \in \mathbb{Z} \setminus 2\mathbb{Z}$. Also, if $\omega$ is cyclic, then $\omega + \pi$ is acyclic, i.e., is not $m$-cyclic in $\mathbb{T}$ for any integer $m$.

Let

$$P_{n,k}(\cdot) := \prod_{n>\ell>k} P\left( 2^{\ell} \cdot \right) = P\left( 2^{n-1} \cdot \right) P\left( 2^{n-2} \cdot \right) \cdots P\left( 2^{k+1} \cdot \right), \quad \forall k, n \in \mathbb{Z}.$$ 

For the vector-stability of refinable vectors, Zhang and Sun gave some necessary and sufficient conditions [32, 33]. However, they considered only the case of single vector refinable vectors. In this paper, we extend their results to multiple vector refinable vectors.
Let $\Phi = (\Phi_1, \cdots, \Phi_n)$, where $\Phi_j \in L^2(\mathbb{R})^r$, $1 \leq j \leq n$. We denote by $S_0(\Phi)$ the linear span of $\{\Phi_j(\cdot - k) : 1 \leq j \leq n, k \in \mathbb{Z}\}$ and by $S(\Phi)$ the closure of $S_0(\Phi)$ in $L^2(\mathbb{R})^r$. Define

$$\text{len} S(\Phi) := \min \{\sharp \Psi : S(\Psi) = S(\Phi)\}.$$ 

Our main result is the following.

**THEOREM 1.** Assume that $\Phi = (\Phi_1, \cdots, \Phi_n)$ is a compactly supported solution of the refinement equation (1) and $\text{len} S(\Phi) = n$. Then $\Phi$ is vector-stable if and only if for every $\lambda \in \mathbb{C}^n \setminus \{0\}$,

(i) if $\lambda \tilde{\Phi}(0) = 0$, then there exists $n \in \mathbb{Z}_+$ so that $\lambda \mathcal{P}^n(0)\mathcal{P}(\pi) \neq 0$;

(ii) if $\lambda \mathcal{P}(\omega) = 0$ for some $\omega \in \mathbb{R}$, then $\lambda \mathcal{P}(\omega + \pi) \neq 0$;

(iii) for any integer $m \geq 2$ and any $\omega \in \mathbb{R}$ which is $m$-cyclic in $\mathbb{T}$, there exist $n \in \mathbb{N}$ and $q \in \{0, \cdots, m-1\}$ so that

$$\lambda \mathcal{P}_{mn,q}(\omega)\mathcal{P}(2^q \omega + \pi) \neq 0.$$ 

**REMARK 1.** Though Theorem 1 looks much like Theorem 1 in [13], they solve different problems. One of them gives a necessary and sufficient condition for refinable vectors to be vector-stable, another one gives a necessary and sufficient condition for refinable vectors to be ordinarily-stable. For the difference between two types of stability see [32, Example 4.1].

2. Proof of the main result

In this section, we give the proof of the main result. We begin with some preliminary results.

Given a vector function $F = (f_1, \cdots, f_r)^T$ on $\mathbb{R}$, we set

$$F^0 := \sum_{k \in \mathbb{Z}} \sum_{p=1}^r |f_p(\cdot - k)|.$$ 

Then $F^0$ is a $1$-periodic function. Define

$$\mathcal{L}^2(\mathbb{R})^r := \left\{ F = (f_1, \cdots, f_r)^T : \|F\|_{\mathcal{L}^2(\mathbb{R})^r} := \|F^0\|_{L^2([0,1])} < \infty \right\}.$$ 

If $r = 1$, $\mathcal{L}^2(\mathbb{R})^{(1)}$ is written as $\mathcal{L}^2(\mathbb{R})$ for an abbreviation.

Given a function $\phi$ and a sequence $a$, the semi-convolution $a *_{sd} \phi$ is the sum

$$\sum_{k \in \mathbb{Z}} a(k)\phi(\cdot - k).$$ 

Next we give a necessary and sufficient condition on the vector-stability of single vector in $\mathcal{L}^2(\mathbb{R})^r$. 
PROPOSITION 1. [32, Theorem 3.3] Let $\Psi = (\psi_1, \cdots, \psi_r)^T \in \mathcal{L}^2(\mathbb{R})^r$. Then $\Psi$ is vector-stable if and only if
\[
\sum_{k \in \mathbb{Z}} \sum_{p=1}^r |\hat{\psi}_p(\omega + 2k\pi)|^2 > 0, \quad \text{for all} \quad \omega \in \mathbb{R}.
\]
The following proposition shows that $\Psi_{sd}$ maps $\ell^1(\mathbb{Z})$ to $\mathcal{L}^2(\mathbb{R})$ and maps $\ell^2(\mathbb{Z})$ to $L^2(\mathbb{R})$.

PROPOSITION 2. [16, Theorem 2.1] If $\psi \in \mathcal{L}^2(\mathbb{R})$, then
\[
\|\psi_{sd} c\|_{\mathcal{L}^2(\mathbb{R})} \leq \|\psi\|_{\mathcal{L}^2(\mathbb{R})} \|c\|_{\ell^1(\mathbb{Z})}
\]
and
\[
\|\psi_{sd} c\|_{L^2(\mathbb{R})} \leq \|\psi\|_{\mathcal{L}^2(\mathbb{R})} \|c\|_{\ell^2(\mathbb{Z})}.
\]
The following lemma is the generalized form of Proposition 2.

LEMMA 1. If $\Psi = (\psi_1, \cdots, \psi_r)^T \in \mathcal{L}^2(\mathbb{R})^r$, then
\[
\|\Psi_{sd} c\|_{\mathcal{L}^2(\mathbb{R})^r} \leq \sqrt{r} \|\Psi\|_{\mathcal{L}^2(\mathbb{R})^r} \|c\|_{\ell^1(\mathbb{Z})} \quad (3)
\]
and
\[
\|\Psi_{sd} c\|_{L^2(\mathbb{R})^r} \leq \|\Psi\|_{\mathcal{L}^2(\mathbb{R})^r} \|c\|_{\ell^2(\mathbb{Z})} \quad (4)
\]

Proof. Since
\[
\|\Psi_{sd} c\|_{\mathcal{L}^2(\mathbb{R})^r} = \| (\Psi_{sd} c)^0 \|_{L^2((0,1))} = \left\| \sum_{k \in \mathbb{Z}} \sum_{p=1}^r (\psi_p \ast_{sd} c)(\cdot - k) \right\|_{L^2((0,1))}
\]
\[
\leq \sum_{p=1}^r \left\| \sum_{k \in \mathbb{Z}} (\psi_p \ast_{sd} c)(\cdot - k) \right\|_{L^2((0,1))} = \sum_{p=1}^r \|\psi_p \ast_{sd} c\|_{\mathcal{L}^2(\mathbb{R})},
\]
by Proposition 2, we have
\[
\|\Psi_{sd} c\|_{\mathcal{L}^2(\mathbb{R})^r} \leq \sum_{p=1}^r \|\psi_p\|_{\mathcal{L}^2(\mathbb{R})} \|c\|_{\ell^1(\mathbb{Z})} = \|c\|_{\ell^1(\mathbb{Z})} \sum_{p=1}^r \|\psi_p\|_{\mathcal{L}^2(\mathbb{R})} \|c\|_{\ell^1(\mathbb{Z})}
\]
\[
\leq \|c\|_{\ell^1(\mathbb{Z})} \sqrt{r} \left\| \sum_{p=1}^r (\psi_p)^0 \right\|_{L^2([0,1])} = \sqrt{r} \|\Psi\|_{\mathcal{L}^2(\mathbb{R})^r} \|c\|_{\ell^1(\mathbb{Z})}.
\]
This proves (3).

(4) follows from [32, Lemma 3.1]. \square

Let $F \in L^2(\mathbb{R})^r$, $G \in \mathcal{L}^2(\mathbb{R})^r$ and $c(F, G)(k) = \langle F, G(\cdot - k) \rangle_{L^2(\mathbb{R})^r}$. We have following lemma.
Lemma 2. The following inequality holds:

$$\|c(F,G)\|_{L^2(\mathbb{Z})} \leq \|F\|_{L^2(\mathbb{R})} \|G\|_{L^2(\mathbb{R})}.$$ \hspace{1cm} (5)

Proof. For two sequences $a$ and $b$, let

$$\langle a, b \rangle_{\ell^2(\mathbb{Z})} = \sum_{k \in \mathbb{Z}} a(k) \overline{b(k)}.$$

For any finitely supported sequence $b$, we obtain

$$\langle c(F,G), b \rangle_{\ell^2(\mathbb{Z})} = \langle F, G *_{sd} b \rangle_{L^2(\mathbb{R})}.$$

By Lemma 1, we have

$$|\langle c(F,G), b \rangle_{\ell^2(\mathbb{Z})}| \leq \|F\|_{L^2(\mathbb{R})} \|G *_{sd} b\|_{L^2(\mathbb{R})} \leq \|F\|_{L^2(\mathbb{R})} \|G\|_{L^2(\mathbb{R})} \|b\|_{\ell^2(\mathbb{Z})}.$$

This proves (5). \hspace{1cm} \Box

Now, we give a necessary and sufficient condition on the vector-stability of multiple vectors in $L^2(\mathbb{R})$.

Theorem 2. Let $\Phi = (\Phi_1, \ldots, \Phi_n)$, where $\Phi_j \in L^2(\mathbb{R})$, $1 \leq j \leq n$. Then $\Phi$ is vector-stable if and only if for any $\xi \in \mathbb{R}$, the sequences $\left\{ \hat{\Phi}_j(\xi + 2\pi k) : k \in \mathbb{Z} \right\}$ ($j = 1, \ldots, n$) are linearly independent.

Proof. ($\Rightarrow$). We prove this by contradiction. If, for some $\xi \in \mathbb{R}$, the sequences $\left\{ \hat{\Phi}_j(\xi + 2\pi k) : k \in \mathbb{Z} \right\}$ ($j = 1, \ldots, n$) are linearly dependent, then there exist constants $r_j$ ($j = 1, \ldots, n$), not all zero, such that

$$\sum_{j=1}^n r_j \hat{\Phi}_j(\xi + 2\pi k) = 0 \quad \text{for all } k \in \mathbb{Z}.$$

Let $\tilde{\Phi} := \sum_{j=1}^n r_j \Phi_j$. Then by Proposition 1, $\tilde{\Phi}$ is not vector-stable, namely, $\Phi_1, \ldots, \Phi_n \in L^2(\mathbb{R})$ are not vector-stable. This proves “($\Rightarrow$)”.

($\Leftarrow$). Given $a = \left\{ a_{j,k} : 1 \leq j \leq n, k \in \mathbb{Z} \right\} \in \ell^2(\mathbb{Z})$, then by Lemma 1

$$\left\| \sum_{j=1}^n \sum_{k \in \mathbb{Z}} a_{j,k} \Phi_j(\cdot - k) \right\|_{L^2(\mathbb{R})}^2 = \left\| \sum_{j=1}^n a_j *_{sd} \Phi_j \right\|_{L^2(\mathbb{R})}^2 \leq n \sum_{j=1}^n \left\| a_j *_{sd} \Phi_j \right\|_{L^2(\mathbb{R})}^2 \leq n \sum_{j=1}^n \left\| \Phi_j \right\|_{L^2(\mathbb{R})}^2 \left\| a_j \right\|_{\ell^2(\mathbb{Z})} \leq nC_2 \sum_{j=1}^n \left\| a_j \right\|_{\ell^2(\mathbb{Z})}^2 = nC_2 \| a \|_{\ell^2(\mathbb{Z})}^2,$$
where $C_2 = \max \left\{ |\Phi_1|^2_{\mathcal{L}^2(\mathbb{R})}, \ldots, |\Phi_n|^2_{\mathcal{L}^2(\mathbb{R})} \right\}$ and $a_j = \{a_j(k) = a_{j,k} : k \in \mathbb{Z}\}$ $(1 \leq j \leq n)$.

Note that $\Phi_1, \ldots, \Phi_n \in \mathcal{L}^2(\mathbb{R})$, we have $\{\hat{\Phi}_j(\xi + 2\pi k) : k \in \mathbb{Z}\} \in \ell^2(\mathbb{Z})$ $(1 \leq j \leq n)$. Since the sequences $\{\hat{\Phi}_j(\xi + 2\pi k) : k \in \mathbb{Z}\}$ $(1 \leq j \leq n)$ are linearly independent, their Gram matrix $([\Phi_j, \Phi_k](\xi))_{1 \leq j, k \leq n}$ is nonsingular for $\xi \in \mathbb{T}$ and has all its entries in $\mathcal{B}$. Here

$$\mathcal{B} = \left\{ \sum_{k \in \mathbb{Z}} a_k e^{-2\pi k i \xi} : a = \{a_k : k \in \mathbb{Z}\} \in \ell^1(\mathbb{Z}) \text{ is a sequence} \right\}.$$  

By Wiener’s lemma, the inverse matrix of $([\Phi_j, \Phi_k](\xi))_{1 \leq j, k \leq n}$ also has all its entries in $\mathcal{B}$. Take $b_{j,k} \in \ell^1(\mathbb{Z})$ $(j, k = 1, \ldots, n)$ such that the matrix $\left(\hat{\Psi}_{j,k}(\xi)\right)_{1 \leq j, k \leq n}$ is the inverse of $([\Phi_j, \Phi_k](\xi))_{1 \leq j, k \leq n}$. For $1 \leq j \leq n$, let

$$\Psi_j := \sum_{k=1}^{n} \Phi_k *_{sd} b_{j,k}.$$  

Then by Lemma 1, $\Psi_j \in \mathcal{L}^2(\mathbb{R})$ and for $1 \leq j, m \leq n$

$$[\Psi_j, \Phi_m](\xi) = \sum_{k=1}^{n} \hat{b}_{j,k}(\xi) [\Phi_k, \Phi_m](\xi) = \delta_{j,m} \quad \text{for all } \xi \in \mathbb{T}.$$  

Hence

$$\langle \Psi_j, \Phi_k(\cdot - \alpha) \rangle = \delta_{j,k} \delta_{0,\alpha}$$  

and

$$a_{j,k} = \left\langle \sum_{m=1}^{n} \sum_{\alpha \in \mathbb{Z}} a_{m,\alpha} \Phi_m(\cdot - \alpha), \Psi_j(\cdot - k) \right\rangle \quad \text{for all } 1 \leq j \leq n, k \in \mathbb{Z}.$$  

Thereby, by Lemma 2

$$\|a\|^2_{\mathcal{L}^2(\mathbb{Z})} = \sum_{j=1}^{n} \|a_j\|^2_{\mathcal{L}^2(\mathbb{Z})} \leq \sum_{j=1}^{n} \left\| \sum_{m=1}^{n} \sum_{\alpha \in \mathbb{Z}} a_{m,\alpha} \Phi_m(\cdot - \alpha) \right\|^2_{\mathcal{L}^2(\mathbb{R})} \|\Psi_j\|^2_{\mathcal{L}^2(\mathbb{R})} = \left\| \sum_{m=1}^{n} \sum_{\alpha \in \mathbb{Z}} a_{m,\alpha} \Phi_m(\cdot - \alpha) \right\|^2_{\mathcal{L}^2(\mathbb{R})} \sum_{j=1}^{n} \|\Psi_j\|^2_{\mathcal{L}^2(\mathbb{R})},$$

where $a_j = \{a_j(k) = a_{j,k} : k \in \mathbb{Z}\}$ $(1 \leq j \leq n)$. Let $C_1 = 1/(\sum_{j=1}^{n} \|\Psi_j\|^2_{\mathcal{L}^2(\mathbb{R})})$. Then $C_1 \|a\|^2_{\mathcal{L}^2(\mathbb{Z})} \leq \|\sum_{m=1}^{n} \sum_{\alpha \in \mathbb{Z}} a_{m,\alpha} \Phi_m(\cdot - \alpha)\|^2_{\mathcal{L}^2(\mathbb{R})}$. From the above argument, $\Phi$ is vector-stable. □
Denote by $\Pi(\mathbb{C})$ the ring of polynomials over $\mathbb{C}$. Let $S$ be the linear space of all sequences $h : \mathbb{Z} \to \mathbb{C}$. For $\theta \in \mathbb{C} \setminus \{0\}$, the sequence given by

$$h_\theta : k \to \theta^k, \quad k \in \mathbb{Z},$$

is an element of $S$, which we shall denote by $h_\theta$. We define $\tau h = h(\cdot + 1)$. If $p \in \Pi(\mathbb{C})$, $p(x) = \sum_{k \geq 0} a_k x^k$, then $p$ induces the linear partial difference operator $p(\tau) := \sum_{k \geq 0} a_k \tau^k$. Let $P = (p_{ij})_{1 \leq i \leq m, 1 \leq j \leq n}$ be a matrix with all its entries $p_{ij} \in \Pi(\mathbb{C})$. For $(h_1, \ldots, h_n) \in S^n$, consider the system of linear homogeneous partial difference equations

$$\sum_{j=1}^{n} p_{ij}(\tau)h_j = 0, \quad i = 1, \ldots, m.$$ 

All the solutions to this system of equations form a subspace of $S^n$ which we shall denote by $\tau(P)$. We have the following proposition.

**Proposition 3.** [20, Theorem 2.1] Let $P$ be an $m \times n$ matrix whose entries are elements of $\Pi(\mathbb{C})$. Then the following conditions are equivalent.

(i) $\tau(P) \neq 0$.

(ii) There exist some $\theta \in \mathbb{C} \setminus \{0\}$ and $(a_1, \ldots, a_n) \in \mathbb{C}^n \setminus \{0\}$ such that

$$(a_1 h_\theta, \ldots, a_n h_\theta) \in \tau(P).$$

Let $\Upsilon_1, \ldots, \Upsilon_n \in L^2(\mathbb{R})$ or $\Upsilon_1, \ldots, \Upsilon_n \in L^2(\mathbb{R})^r$. Assumption that $\Upsilon_1, \ldots, \Upsilon_n$ are compactly supported. We define

$$K(\Upsilon_1, \ldots, \Upsilon_n) := \left\{ (h_1, \ldots, h_n) \in S^n : \sum_{j=1}^{n} [\Upsilon_j, h_j] = 0 \right\}$$

and

$$H(\Upsilon_1, \ldots, \Upsilon_n) := \left\{ \sum_{j=1}^{n} [\Upsilon_j, h_j] : (h_1, \ldots, h_n) \in S^n \right\}.$$ 

Here, $[\Upsilon_j, h_j] := \sum_{k \in \mathbb{Z}} \Upsilon_j(\cdot - k)h_j(k)$.

Let $\Phi_1, \ldots, \Phi_n \in L^2(\mathbb{R})^r$ be compactly supported. The following lemma gives a characterization for $K(\Phi_1, \ldots, \Phi_n)$.

**Lemma 3.** There exists a matrix $P$ with $n$ columns whose entries are elements of $\Pi(\mathbb{C})$ such that $K(\Phi_1, \ldots, \Phi_n) = \tau(P)$.

**Proof.** Let $G = (-1, 1)$. Since $\mathbb{R} = \cup_{k \in \mathbb{Z}} (G + k)$, we have that $(h_1, \ldots, h_n) \in K(\Phi_1, \ldots, \Phi_n)$ if and only if

$$\tau^\alpha \left( \sum_{j=1}^{n} [\Phi_j, h_j] \right) \bigg|_G = 0, \quad \forall \alpha \in \mathbb{Z}.$$
Observe that $\tau^\alpha[F, h] = [F, \tau^\alpha h]$. Hence, $(h_1, \cdots, h_n) \in K(\Phi_1, \cdots, \Phi_n)$ if and only if

$$\sum_{j=1}^{n} \sum_{k \in \mathbb{Z}} (\tau^\alpha h_j)(k) \Phi_j(\cdot - k) = 0, \quad \forall \alpha \in \mathbb{Z}. \quad (6)$$

Since $\Phi_1, \cdots, \Phi_n$ are compactly supported, there exists a positive integer $N$ such that $|k| > N$ implies

$$\Phi_j(\cdot - k)|_G = 0, \quad 1 \leq j \leq n.$$ 

This shows that the restriction of the linear space $H(\Phi_1, \cdots, \Phi_n)$ to $G$ is finite dimensional. Choose a basis $\Phi_1, \cdots, \Phi_m$ for it. For $k \in \mathbb{Z}$, $1 \leq j \leq n$, $\Phi_j(\cdot - k)|_G$ can be uniquely represented as follows:

$$\Phi_j(\cdot - k)|_G = \sum_{i=1}^{m} a_{ij}(k) \Phi_i, \quad (7)$$

where the coefficients $a_{ij}(k) \in \mathbb{C}$ and are zero for $|k| > N$. In terms of (7), (6) is equivalent to

$$\sum_{i=1}^{m} \left( \sum_{j=1}^{n} \sum_{|k| \leq N} a_{ij}(k) \tau^\alpha h_j(k) \right) \Phi_i = 0, \quad \forall \alpha \in \mathbb{Z}. \quad (8)$$

Note that $\tau^\alpha h(k) = h(k + \alpha) = \tau^k h(\alpha)$. Since $\Phi_1, \cdots, \Phi_m$ are linearly independent, (8) is equivalent to

$$\sum_{j=1}^{n} \left( \sum_{|k| \leq N} a_{ij}(k) \tau^k \right) h_j = 0, \quad i = 1, \cdots, m.$$ 

Let $P = (p_{ij})_{1 \leq i \leq m, 1 \leq j \leq n}$, where

$$p_{ij}(x) = \sum_{|k| \leq N} a_{ij}(k) x^{k+N}, \quad i = 1, \cdots, m, \quad j = 1, \cdots, n.$$ 

Then we have $K(\Phi_1, \cdots, \Phi_n) = \tau(P)$. \qed

The following proposition is a consequence of the Poisson summation formula.

**Proposition 4.** [20, Lemma 3.2] Let $\phi$ be a compactly supported distribution on $\mathbb{R}$. Then for a given $\omega \in \mathbb{C}$, the sequence $\{e^{ik\omega} : k \in \mathbb{Z}\}$ lies in $K(\phi)$ if and only if

$$\hat{\phi}(\omega + 2k\pi) = 0, \quad \forall k \in \mathbb{Z}.$$ 

Based on Proposition 4, we give the following lemma.

**Lemma 4.** Let $\Psi = (\psi_1, \cdots, \psi_r)^T \in L^2(\mathbb{R})^r$ be compactly supported. Then for a given $\omega \in \mathbb{C}$, the sequence $\{e^{ik\omega} : k \in \mathbb{Z}\}$ lies in $K(\Psi)$ if and only if

$$\hat{\Psi}(\omega + 2k\pi) = 0, \quad \forall k \in \mathbb{Z}.$$
Proof. \((\Rightarrow)\) If \(\{e^{ik\omega} : k \in \mathbb{Z}\} \in K(\Psi),\) then \(\{e^{ik\omega} : k \in \mathbb{Z}\} \in K(\psi_j), 1 \leq j \leq r.\) By Proposition 4, we have \(\hat{\psi}_j(\omega + 2k\pi) = 0, 1 \leq j \leq r, k \in \mathbb{Z},\) that is \(\hat{\Psi}(\omega + 2k\pi) = 0, k \in \mathbb{Z}.\)

\((\Leftarrow)\) If \(\hat{\Psi}(\omega + 2k\pi) = 0, k \in \mathbb{Z},\) then \(\hat{\psi}_j(\omega + 2k\pi) = 0, 1 \leq j \leq r, k \in \mathbb{Z}.\) By Proposition 4, we have \(\{e^{ik\omega} : k \in \mathbb{Z}\} \in K(\psi_j), 1 \leq j \leq r,\) that is \(\{e^{ik\omega} : k \in \mathbb{Z}\} \in K(\Psi).\) □

Now, we give two equivalent conditions of \(K(\Phi_1, \ldots, \Phi_n) \neq 0.\)

**Lemma 5.** Let \(\Phi_1, \ldots, \Phi_n \in L^2(\mathbb{R})^r\) be compactly supported. Then the following conditions are equivalent.

(i) \(K(\Phi_1, \ldots, \Phi_n) \neq 0.\)

(ii) There exist some \(\theta \in \mathbb{C}\backslash\{0\}\) and \((a_1, \ldots, a_n) \in \mathbb{C}^n\backslash\{0\}\) such that

\[
(a_1 h_\theta, \ldots, a_n h_\theta) \in K(\Phi_1, \ldots, \Phi_n). \tag{9}
\]

(iii) There exists some \(\omega \in \mathbb{C}\) such that the sequences \(\{\hat{\Phi}_j(\omega + 2k\pi) : k \in \mathbb{Z}\}\) \((j = 1, \ldots, n)\) are linearly dependent.

Proof. By Lemma 3, \(K(\Phi_1, \ldots, \Phi_n) = \tau(P)\) for some matrix \(P\) of polynomials. Hence, the equivalence between (i) and (ii) follows from Proposition 3.

Suppose (9) is true. Choose \(\omega \in \mathbb{C}\) so that \(e^{i\omega} = \theta,\) and set

\[
\tilde{\Phi} := \sum_{j=1}^{n} a_j \Phi_j. \tag{10}
\]

Then (9) and (10) imply

\[
\sum_{k \in \mathbb{Z}} \theta^k \tilde{\Phi}(-k) = \sum_{k \in \mathbb{Z}} \theta^k \sum_{j=1}^{n} a_j \Phi_j(-k) = \sum_{j=1}^{n} [\Phi_j, a_j h_\theta] = 0.
\]

In other words, \(h_\theta \in K(\tilde{\Phi})\). Hence, by Lemma 4,

\[
\hat{\tilde{\Phi}}(\omega + 2k\pi) = 0, \quad \forall k \in \mathbb{Z}.
\]

It follows from (10) that

\[
\sum_{j=1}^{n} a_j \hat{\Phi}_j(\omega + 2k\pi) = 0, \quad \forall k \in \mathbb{Z}. \tag{11}
\]

Since \((a_1, \ldots, a_n) \neq 0,\) this proves that (ii) implies (iii).

Finally, suppose (iii) holds. Then there exists some \((a_1, \ldots, a_n) \in \mathbb{C}^n\backslash\{0\}\) such that (11) is true. With \(\theta = e^{i\omega}\) and \(\tilde{\Phi}\) given by (10), we obtain

\[
\hat{\tilde{\Phi}}(\omega + 2k\pi) = 0, \quad \forall k \in \mathbb{Z}.
\]
Hence, \( h_\theta \in K(\Phi) \). Thus (9) follows. \( \square \)

A compactly supported \( \Phi = (\Phi_1, \cdots, \Phi_n) \) (\( \Phi_j \in L^2(\mathbb{R})^r \), \( 1 \leq j \leq n \)) is said to have linearly independent shifts if the map

\[
(a_1, \cdots, a_n) \to \sum_{j=1}^{n} \sum_{k \in \mathbb{Z}} a_j(k) \Phi_j(\cdot - k)
\]

is one-to-one on \( \mathbb{C}^Z \). \( S_0(\Phi) \) denotes the linear span of \( \{ \Phi_j(\cdot - k) : k \in \mathbb{Z}, 1 \leq j \leq n \} \). Let \([r_\Phi, s_\Phi]\) be the smallest integer-bounded interval containing \( \text{supp}\Phi \). The length of the interval \([r_\Phi, s_\Phi]\) is

\[
\ell(\Phi) = s_\Phi - r_\Phi.
\]

We call \( \ell(\Phi) \) the length of \( \Phi \). Let \( \Phi = (\Phi_1, \cdots, \Phi_n) \) be a finite collection of compactly supported vector functions on \( \mathbb{R} \). The length of \( \Phi \), denoted \( \ell(\Phi) \), is defined by

\[
\ell(\Phi) = \sum_{j=1}^{n} \ell(\Phi_j).
\]

The following lemma shows that a compactly supported vector function is linear combination of a collection of linearly independent compactly supported vector functions.

**Lemma 6.** Let \( \Phi = (\Phi_1, \cdots, \Phi_n) \) (\( \Phi_j \in L^2(\mathbb{R})^r \), \( 1 \leq j \leq n \)) be compactly supported. Then there exists a compactly supported \( \Psi \) with the following properties:

(i) The shifts of the \( \Psi \) are linearly independent;

(ii) \( \Phi \subset S_0(\Psi) \).

**Proof.** If \( K(\Phi_1, \cdots, \Phi_n) = 0 \), then we may take \( \Psi = \Phi \).

Suppose \( K(\Phi_1, \cdots, \Phi_n) \neq 0 \). By Lemma 5, there exist some \( \theta \in \mathbb{C}\setminus\{0\} \) and \((a_1, \cdots, a_n) \in \mathbb{C}^n \setminus \{0\}\) such that

\[
(a_1 h_\theta, \cdots, a_n h_\theta) \in K(\Phi_1, \cdots, \Phi_n),
\]

that is

\[
\sum_{j=1}^{n} \sum_{k \in \mathbb{Z}} a_j \theta^k \Phi_j(\cdot - k) = 0. \tag{12}
\]

After shifting the \( \Phi_j \) appropriately, we may assume that all \( r_\Phi = 0 \). Then \( s_\Phi = \ell(\Phi_j) \).

Let

\[
\ell = \max\{\ell(\Phi_j) : a_j \neq 0\}.
\]

For simplicity, we assume that \( a_1 \neq 0 \) and \( \ell(\Phi_1) = \ell \). Let \( \rho = \sum_{j=1}^{n} a_j \Phi_j \) and \( \Psi = \sum_{k=0}^{\infty} \theta^k \rho(\cdot - k) \). By our choice of \( \rho \), we deduce from (12) that

\[
\sum_{k \in \mathbb{Z}} \theta^k \rho(\cdot - k) = 0.
\]
Let $\Psi = (\Psi, \Phi_2, \ldots, \Phi_n)$. We have
\[
\Psi - \theta \Psi(-1) = \sum_{k=0}^{\infty} \theta^k \rho(-k) - \sum_{k=0}^{\infty} \theta^{k+1} \rho(-k-1) = \rho = \sum_{j=1}^{n} a_j \Phi_j.
\]
Since $a_1 \neq 0$, we have $\Phi_1 \in \mathcal{S}_0(\Psi)$, that is $\Phi \subset \mathcal{S}_0(\Psi)$. Clearly, $\text{supp} \Psi \subseteq [0, \infty)$. Note that
\[
\Psi(x) = \sum_{k=0}^{\infty} \theta^k \rho(x-k) = \sum_{j=1}^{n} \sum_{k=0}^{\infty} a_j \theta^k \Phi_j(x-k) = \sum_{j=1}^{n} \sum_{k \in \mathbb{Z}} a_j \theta^k \Phi_j(x-k)
\]
\[
= \sum_{k \in \mathbb{Z}} \theta^k \rho(x-k) = 0, \quad x > \ell(\Phi) - 1.
\]
Consequently, $\text{supp} \Psi \subseteq [0, \ell - 1]$, that is $\ell(\Psi) < \ell(\Phi)$. Repeat the preceding process until $\ell(\Psi)$ achieves its minimum. The resulting vector function $\Psi$ has the property that the shifts of $\Psi$ are linearly independent. □

**Proof of Theorem 1.** Suppose that $\Phi$ is vector-stable. Note that every element of $2\pi \mathbb{Z} \setminus \{0\}$ has the form $2^{n+1}(2k+1)\pi$ for some $n \in \mathbb{Z}_+$ and $k \in \mathbb{Z}$. Then we have
\[
\hat{\Phi}(2^{n+1}(2k+1)\pi) = P^n(0)\hat{P}(\pi)\hat{\Phi}((2k+1)\pi).
\]
If the condition (i) is false, then \(\{\lambda \hat{\Phi}(2k\pi) : k \in \mathbb{Z}\} = 0\). Also, if the condition (ii) is false, then
\[
\lambda \hat{\Phi}(2\omega + 4k\pi) = \lambda \hat{P}(\omega)\hat{\Phi}(\omega + 2k\pi) = 0, \quad \forall k \in \mathbb{Z}
\]
and
\[
\lambda \hat{\Phi}(2\omega + 2\pi + 4k\pi) = \lambda \hat{P}(\omega + \pi)\hat{\Phi}(\omega + 2k\pi + \pi) = 0, \quad \forall k \in \mathbb{Z}.
\]
Therefore, we have \(\{\lambda \hat{\Phi}(2\omega + 2k\pi) : k \in \mathbb{Z}\} = 0\).

Now, if $P$ does not satisfy condition (iii), then we show that
\[
\lambda \hat{\Phi}(\omega + 2k\pi) = 0, \quad \forall k \in \mathbb{Z}.
\]
Suppose $\omega$ is m-cyclic in $\mathbb{T}$ for some integer $m \geq 2$, and let $k \in \mathbb{Z}$ be given. Then by (2), there exist $n \in \mathbb{N}, q \in \{0, \ldots, m - 1\}$ and $\nu \in \mathbb{Z} \setminus 2\mathbb{Z}$ such that
\[
\lambda \hat{\Phi}(\omega + 2k\pi) = \lambda \hat{\Phi}(2^{mn}\omega + 2^{mn-q}\nu\pi) = \lambda \hat{P}_{mn,q}(\omega)\hat{\Phi}(2^{q+1}\omega + 2\nu\pi)
\]
\[
= \lambda \hat{P}_{mn,q}(\omega)P(2^q\omega + \nu\pi)\hat{\Phi}(2^q\omega + \nu\pi).
\]
So, $\lambda \hat{\Phi}(\omega + 2k\pi)$ is zero if condition (iii) is not satisfied. By arbitrariness of $k$, we have \(\{\lambda \hat{\Phi}(2\omega + 2k\pi) : k \in \mathbb{Z}\} = 0\). This completes the proof of necessity.

To prove sufficiency, suppose that the shifts of $\Phi$ is not vector-stable. Moreover, assume that $P$ satisfies conditions (i) and (ii). Then we show that (iii) is violated.

Since the shifts of $\Phi$ is not vector-stable, by Theorem 2, there exists $\omega_0 \in \mathbb{R}$ such that
\[
\lambda \hat{\Phi}(\omega_0 + 2k\pi) = 0, \quad \forall k \in \mathbb{Z}.
\]
Lemma 6 implies that $\Phi$ is a finite linear combination of the shifts of the $\Psi$. In the Fourier domain, this implies the existence of a trigonometric polynomial matrix $U$ satisfying
\begin{equation}
\hat{\Phi}(\omega) = U(\omega)\hat{\Psi}(\omega).
\end{equation}

Note that the shifts of the $\Psi$ are linearly independent. Then by (13), we get $\lambda U(\omega_0) = 0$. Since that $\text{len}S(\Phi) = n$, we have that $\det U$ is a non-trivial trigonometric polynomial, which in turn implies that the rational trigonometric polynomials matrix $B$ is well-defined by the relation
\begin{equation}
U(2\omega)B(\omega) = P(\omega)U(\omega).
\end{equation}

Therewith, equation (14) implies that
\[
\lambda P\left(\frac{\omega_0}{2}\right) U\left(\frac{\omega_0}{2}\right) = \lambda U(\omega_0)B\left(\frac{\omega_0}{2}\right) = 0
\]
and
\[
\lambda P\left(\frac{\omega_0}{2} + \pi\right) U\left(\frac{\omega_0}{2} + \pi\right) = \lambda U(\omega_0)B\left(\frac{\omega_0}{2} + \pi\right) = 0.
\]

By condition (ii), it follows that $U$ is singular at either $\frac{\omega_0}{2}$ or $\frac{\omega_0}{2} + \pi$. Since $\det U$ is a trigonometric polynomial, it has only finitely many zeros in $\mathbb{T} + i\mathbb{R}$ and the arguments used in the proof of [21, Lemma 1] imply that $2^m\omega_0 - \omega_0 \in 2\pi\mathbb{Z}$ for some $m \geq 2$. If $\omega_0 = 0$, then $\lambda \Phi(0) = 0$ and (i) implies that $\lambda P^n(0)P(\pi)$ is not zero for some $n \in \mathbb{Z}_+$. Evidently,
\[
\lambda P^n(0)P(\pi)U(\pi) = U(0)B^n(0)B(\pi) = 0.
\]

But this is a contradiction, since $\pi$ is acyclic, so $\det U(\pi) \neq 0$.

Accepting the existence of the integer $m \geq 2$ and the $\omega_0$ is m-cyclic, we proceed to prove that this violates condition (iii).

Observe that
\[
2^m\omega_0 - \omega_0 = \frac{2^m - 1}{2^m - 1} (2^m\omega_0 - \omega_0) \in 2\pi\mathbb{Z}.
\]

Then $\lambda U(\omega_0) = 0$ implies that $\lambda U(2^m\omega_0) = 0$. Therefore, by (14) and the $2\pi$ periodicity of $U$
\[
\lambda \mathcal{P}_{mn,q}(\omega_0)P(2^q\omega_0 + \pi)U(2^q\omega_0 + \pi)
\]
\[
= \lambda \mathcal{P}_{mn,q}(\omega_0)U(2^{q+1}\omega_0 + 2\pi)B(2^q\omega_0 + \pi)
\]
\[
= \lambda \mathcal{P}_{mn,q}(\omega_0)U(2^{q+1}\omega_0)B(2^q\omega_0 + \pi)
\]
\[
= \lambda U(2^{mn}\omega_0)\mathcal{P}_{mn,q}(\omega_0)B(2^q\omega_0 + \pi)
\]
\[
= 0,
\]
where
\[
\mathcal{B}_{n,k} := \prod_{n \leq t > k} B\left(2^t\cdot\right) = B\left(2^{n-1}\cdot\right)B\left(2^{n-2}\cdot\right) \cdots B\left(2^{k+1}\cdot\right), \quad \forall \ k, n \in \mathbb{Z}.
\]
Since $2^q \omega_0 + \pi$ is acyclic, $\det U(2^q \omega_0 + \pi) \neq 0$, so

$$\lambda \mathcal{P}_{mn,q}(\omega_0)\mathbf{P}(2^q \omega_0 + \pi) = 0$$

for every $n \in \mathbb{N}$ and $q \in \{0, \ldots, m-1\}$. □

3. Results and discussion

A wavelet system is generally derived from a refinable function via a multiresolution analysis. Stability is an important property of refinable function. In this paper, we discuss the vector-stability of refinable vectors and we give a necessary and sufficient condition for refinable vectors to be vector-stable. Our results improve some known ones. Studying vector-stability of refinable vectors in $L^p(\mathbb{R})^r$ is the goal of future work.

REFERENCES


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