SOME RESULTS ON POROUS SET RELATING TO RATIO SETS

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Abstract. An attempt has been made in this paper is to show that every Lebesgue measurable linear set with positive measure has a porous subset whose ratio set contains an interval. The category analogue of this result is also established.

1. Introduction

First we recall the definition of porous set [3] as bellow:

DEFINITION 1. ([3]) Let \( A \) be a non-empty subset of real line \( \mathbb{R} \) and \( x \in A \). \( A \) is said to be porous at \( x \), if there exists a constant \( c, \; 0 < c \leq 1 \) and a sequence of intervals \( \{I_n\} \), each containing \( x \), whose length tends to zero as \( n \) tends to infinity, such that each interval \( I_n \) contains an interval \( J_n \) that is disjoint from \( A \) and \( \frac{\lambda(J_n)}{\lambda(I_n)} \geq c \), where \( \lambda(A) \) denotes the Lebesgue measure of \( A \). The set \( A \) is called porous set if it is porous at each of its points.

DEFINITION 2. ([2]) A set \( A \subset \mathbb{R} \) is called \( p \)-porous for a \( p \in (0,1) \) if for every \( x \in \mathbb{R} \), \( \limsup_{y \to 0} \frac{1}{y} \) (the length of the longest interval in \((x-y,x+y)\) which is contiguous to \( A \)) \( \geq p \).

Porous set possesses the following properties:

- Every porous set is of Lebesgue measure zero.
- Every porous set is of first category.

It is to be noted that the converse may not be true. For example, the set of rational number \( \mathbb{Q} \).

DEFINITION 3. ([5]) A set \( A \) is said to have the property of Baire if it can be expressed as symmetric difference of an open set and a set of first category.

H. I. Miller [3] established that every second category set \( A \) having the property of Baire contains a porous subset \( P \) such that difference set of \( P \) written as \( D(P) = \{x-y : x,y \in P\} \) contains an interval. The measure theoretic analogue of this result was shown by Z. Buczolich [2].

In 1962, N. C. Bose Majumder [1] introduced the notion of ratio set in the following way:


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**Definition 4.** The ratio set of a linear set $A$ of non zero abscissa denoted by $R(A)$, is defined by $R(A) = \{ \frac{a}{b} : a, b \in A \}$. Also ratio of two linear sets $A$ and $B$ is defined as $R(A,B) = \{ \frac{a}{b} : a \in A, b \in B \setminus \{0\} \}.$

Bose Majumder [1] established that ratio set $R(A)$ of a linear set $A$ with non zero abscissa having positive Lebesgue measure contains an interval with left hand end point 1.

In this paper we will show that every Lebesgue measurable set $A(\subset \mathbb{R})$ with positive measure contains a porous subset $B$ whose ratio set $R(B)$ contains an interval. Also the category analogue of this result is proved.

## 2. Main results

Before going to establish main results we go through some lemmas.

**Lemma 1.** ([2]) For every closed set $A$ of positive Lebesgue measure, $s \in \mathbb{N}$ and $t \in (0,1)$ there exists a closed set $A_r \subset A$ with the following properties: The set $A_r$ has positive Lebesgue measure and there exists a sequence of natural numbers $n_1, n_2, \ldots, n_j, \ldots$ such that $s | n_j$ for every $j \in \mathbb{N}$ and, letting $d_j = \frac{1}{n_1 \cdots n_j}$, we have either $[kd_j,(k+1)d_j] \cap A_r = \emptyset$ or $\lambda((kd_j,(k+1)d_j) \cap A_r) > t.d_j$ for every $j \in \mathbb{N}$ and $k \in \mathbb{Z}$. Here $\mathbb{N}$ and $\mathbb{Z}$ are the sets of natural numbers and integers respectively.

**Lemma 2.** For every $p \in (0, \frac{1}{2})$ there exists a $t(p) \in (0,1)$ such that if $H_1 \subset (0,1]$, $H_2 \subset (0,1]$ and $\lambda(H_1) > t(p)$, $\lambda(H_2) > t(p)$ then $R(H_1, H_2) = R(G_1, G_2)$, where $G_1 = H_1 \setminus (\frac{1-p}{2}, \frac{1+p}{2})$ and $G_2 = H_2 \setminus (\frac{1-p}{2}, \frac{1+p}{2})$.

**Proof.** Consider $t(p) = 3p$ for $p \in (0, \frac{1}{3})$. Suppose $H_1$ and $H_2$ are two subsets of $(0,1]$ with $\lambda(H_1) > t(p)$ and $\lambda(H_2) > t(p)$. Clearly $R(H_1, H_2)$ is non-empty subset of $(0,\infty)$. If $m$ is an element of $R(H_1, H_2)$ then there exist $y \in H_1$ and $x \in H_2$ such that $y = mx$. From the unit square $S = [0,1] \times [0,1]$, we obtained four squares each of length $\frac{1-p}{2}$ by deleting a horizontal strip and a vertical strip of breath $p \in (0, \frac{1}{3})$. These four squares are of the form

$S_1 = [0, \frac{1-p}{2}] \times [\frac{1+p}{2}, 1]$ which is upper left square,

$S_2 = [0, \frac{1-p}{2}] \times [0, \frac{1-p}{2}]$ which is lower left square,

$S_3 = [\frac{1+p}{2}, 1] \times [\frac{1+p}{2}, 1]$ which is upper right square,

$S_4 = [\frac{1+p}{2}, 1] \times [0, \frac{1-p}{2}]$ which is lower right square.

If we denote by $l_m$ the graph of the line $y = mx$ then $l_m \cap (H_1 \times H_2) \neq \emptyset$. Let $P_x$ (resp. $P_y$) be the projection of the line $y = mx$ on $x$ (resp. $y$) axis. For $m \in (0,\infty)$, clearly

$\lambda(P_x(l_m \cap (S_1 \cup S_2))) \geq \frac{1-3p}{2}$ and $\lambda(P_y(l_m \cap (S_1 \cup S_2))) \geq \frac{1-3p}{2}$.
Since \( \lambda(H_1) > t(p) \), \( \lambda(H_2) > t(p) \) and also \( t(p) = 3p \) we have
\[
\lambda(P_x(l_m \cap (S_1 \cup S_2)) \setminus H_2) < 1 - 3p \text{ and } \lambda(P_y(l_m \cap (S_1 \cup S_2)) \setminus H_1) < 1 - 3p.
\]
Therefore \( l_m \cap (S_1 \cup S_2) \cap (H_2 \times H_1) = l_m \cap (S_1 \cup S_2) \cap (G_2 \times G_1) \neq \emptyset \). Hence for \( m \in R(H_1, H_2) \) implies \( m \in R(G_1, G_2) \). Thus \( R(H_1, H_2) = R(G_1, G_2) \). □

**Theorem 1.** For every set \( A(\subset \mathbb{R}) \) having nonzero abscissa with positive Lebesgue measure and \( p \in (0, \frac{1}{2}) \) there exists a \( p \)-porous set \( B \subset A \) such that \( R(B) = \{ \frac{a}{b} : a, b \in B \} \) contains an interval.

**Proof.** With out loss of generality we consider \( A \) to be a closed set. It is enough to prove the theorem for rational \( p \in (0, \frac{1}{2}) \). Let \( p = \frac{u}{v} \in (0, \frac{1}{2}) \), where \( u, v \in \mathbb{N} \). By Lemma 2 we choose a suitable \( t(p) \) for \( p \in (0, \frac{1}{2}) \) and then applying the Lemma 1 with \( s = 2v \) and \( t = t(p) \) we obtain a closed set \( A_r \subset A \) and the sequences \( n_1, n_2, \ldots \) (of natural numbers) and \( d_1, d_2, \ldots \) \( (d_j = \frac{1}{n_1n_2\ldots n_j}, j \in \mathbb{N}) \) such that either \([kd_j, (k + 1)d_j] \cap A_r = \emptyset \) or \( \lambda((kd_j, (k + 1)d_j) \cap A_r) > td_j \) for every \( j \in \mathbb{N} \) and \( k \in \mathbb{Z} \).

Consider \( B_0 = A_r \). We put
\[
B_j = B_{j-1} \setminus \bigcup_{k \in \mathbb{Z}} \left( (k + \frac{1-p}{2})d_j, (k + \frac{1+p}{2})d_j \right).
\]
Obviously \( B = \bigcap_{j=1}^{\infty} B_j \) is \( p \)-porous. Since \( B_0 \) is of positive Lebesgue measure, according to Bose Majumder’s result [1] the set \( R(B_0) \) contains an interval with 1 as left hand end point. Again since all the sets \( B_j \), \( j = 0, 1, 2, 3, \ldots \) are compact, \( R(B_j) \) are compact and hence closed. By Cantor Baire Stationary theorem [4], we have \( R(B) = R(B_0) \). It is enough to prove \( R(B_0) = R(B_1) = R(B_2) = \ldots \). We have to show that \( R(B_{j-1}) = R(B_j) \) for all \( j \in \mathbb{N} \).

We say that the set \( B_{j-1} \) possesses the property \( P_{j-1} \) if for every integer \( k \) we have either \( (k.d_j, (k + 1).d_j) \cap B_{j-1} = \emptyset \) or \( \lambda((k.d_j, (k + 1).d_j) \cap B_{j-1}) > t.d_j \).

If the set \( B_{j-1} \) possesses the property \( P_{j-1} \), then by \( p = \frac{u}{v} \) and \( s = 2v \), we have \( \frac{pv}{2dj+1} \in \mathbb{N} \). So, in the definition of \( B_j \) which is obtained from \( B_{j-1} \) by deleting a subset of \( B_{j-1} \) which is a union of the intervals of the form \( (k.d_{j+1}, (k + 1).d_{j+1}) \). According to Lemma 1, the set \( B_0 = A_r \) has the property \( P_0 \). Thus by induction \( B_j \) has \( P_j \) property for every \( j \in \mathbb{N} \). Let us take
\[
H_{k,j-1} = [k.d_j, (k + 1).d_j] \cap B_{j-1} \text{ and } G_{k,j-1} = H_{k,j-1} \setminus ((k + \frac{1-p}{2}).d_j, (k + \frac{1+p}{2}).d_j).
\]

Then we have
\[
R(B_{j-1}) = \bigcup_{k,m \in \mathbb{Z}} R(H_{k,j-1}, H_{m,j-1}),
\]
(since the set \( B_{j-1} \) has the property \( P_{j-1} \), the equality holds for those indices \( m, k \in \mathbb{Z} \) for which \( (k.d_j, (k + 1).d_j) \cap B_{j-1} \neq \emptyset \) and \( (m.d_j, (m + 1).d_j) \cap B_{j-1} \neq \emptyset \)).
Also we get
\[ R(B_j) = \bigcup_{k,m \in \mathbb{Z}} R(G_{k,j-1}, G_{m,j-1}). \]

By Lemma 2, \( R(H_{k,j-1}, H_{m,j-1}) = R(G_{k,j-1}, G_{m,j-1}) \) for the indices \( k,m \in \mathbb{Z} \) and \( j \in \mathbb{N} \). Therefore \( R(B_{j-1}) = R(B_j) \) for all \( j \in \mathbb{N} \). Hence the result. □

To establish the category analogue of the Theorem 1 we need following lemmas.

**Lemma 3.** Let \( 0 < a < b \) and \( F = \bigcup_{n=1}^{\infty} F_n \), where \( F_n \)'s are nowhere dense closed subsets of \( \mathbb{R} \). Then the ratio set \( R((a,b) \setminus F) = \left( \frac{a}{b}, \frac{b}{a} \right) = R((a,b)) \), where \( (a,b) \) is an open interval.

**Proof.** Let \( 0 < a < b \). Let \( x,y \in (a,b) \) and \( x < y \). So, \( 0 < a < x < y < b \Rightarrow \frac{a}{y} < \frac{x}{y} < \frac{y}{y} \Rightarrow 0 < \frac{a}{y} < \frac{x}{y} < \frac{a}{x} < \frac{b}{x} < \frac{b}{y} \). Thus \( \frac{x}{y} \) is an interior point of \( \left( \frac{a}{y}, \frac{b}{y} \right) \). Similarly \( \frac{y}{x} \) is an interior point of \( \left( \frac{a}{x}, \frac{b}{x} \right) \). So, \( R((a,b)) = \left( \frac{a}{b}, \frac{b}{a} \right) \), where \( 0 < a < b \).

Since \( F_n \)'s are nowhere dense closed subset of \( \mathbb{R} \), the complement of each \( F_n \) is everywhere dense open subset of \( \mathbb{R} \). By Baire category theorem we have, \( (a,b) \setminus \bigcup_{n=1}^{\infty} F_n = \bigcap_{n=1}^{\infty} [(a,b) \setminus F_n] \) is dense in closed interval \( [a,b] \).

Therefore \( R((a,b) \setminus F) = R((a,b)) \). □

**Lemma 4.** For any two open intervals \( I,J(\subset \mathbb{R}^+) \), the ratio set \( R(I,J) = \{ \frac{x}{y} : x \in I, y \in J \} \) is an open set, where \( \mathbb{R}^+ \) denotes the set of positive real numbers.

**Proof.** Let \( I = (x',y') \) and \( J = (x'',y'') \) be two open intervals in \( \mathbb{R}^+ \). So, \( 0 < x' < y' \) and \( 0 < x'' < y'' \). Let \( v \in R(I,J) \). So, \( v = \frac{p}{q} \), i.e. \( p = vq \), where \( p \in I, q \in J \). Since \( p \) is an interior point of \( I \), there exists \( \delta' > 0 \) such that \( x' < p - \delta' < p < p + \delta' < y' \Rightarrow x' < p - \delta' < vq < p + \delta' < y' \Rightarrow x' < p - \delta' < vq < p + \delta' < y' \Rightarrow x' < v - \frac{q' \delta'}{q} < v < v + \frac{q' \delta'}{q} < y' \Rightarrow \frac{x'}{q} < \frac{q'}{q} \). Since \( q \in J \). This shows that \( v \) is an interior point of \( \left( \frac{x'}{q}, \frac{q'}{q} \right) = R(I,J) \), where \( I = (x',y') \), \( 0 < x' < y' \) and \( J = (x'',y'') \), \( 0 < x'' < y'' \). So, \( R(I,J) \) is an open interval and hence an open set. □

**Theorem 2.** If \( B \subseteq \mathbb{R}^+ \) has the property of Baire and is of second category then there exists a porous set \( P \subseteq B \) such that \( R(P) = \{ \frac{p}{q} : p,q \in P \} \) contains an interval, where \( \mathbb{R}^+ \) denotes the set of positive real numbers.

**Proof.** Suppose \( B \) be a second category subset of positive reals having the property of Baire. So, there exist an open interval \( I = (b,c) \), \( 0 < b < c \) and a sequence \( \{F_n\}_{n=1}^{\infty} \) of closed nowhere dense subsets of \( \mathbb{R}^+ \) such that \( B = (I \setminus F_1) \cup F_2 \). i.e, \( B \supseteq I \setminus F_1 \supseteq I \setminus \bigcup_{n=1}^{\infty} F_n \). Let \( A = I \setminus \bigcup_{n=1}^{\infty} F_n \).
By Lemma 3 we have \( R(A) = (\frac{b}{c}, \frac{c}{b}) \). Clearly \( I \setminus F_1 \) is an open subset of \( \mathbb{R}^+ \).

So, \( I \setminus F_1 = \bigcup_{i=1}^{\infty} I_i \), where \( \{I_i : i \in \mathbb{N}\} \) are pairwise disjoint open subintervals of \( I \).

Therefore \( R(I \setminus F_1) = (\frac{b}{c}, \frac{c}{b}) \). Again since

\[
I \setminus F_1 = \bigcup_{i=1}^{\infty} I_i, \quad R(I \setminus F_1) = R\left( \bigcup_{i=1}^{\infty} I_i \right) = \bigcup_{i,j=1}^{\infty} R(I_i, I_{j1}) = (\frac{b}{c}, \frac{c}{b}),
\]

where

\[
R(I_i, I_{j1}) = \{ \frac{x}{y} : x \in I_i, y \in I_{j1} \}.
\]

By Lemma 4, \( R(I_i, I_{j1}) \) are open sets for each \( i, j \in \mathbb{N} \). So, \( \{R(I_i, I_{j1}) : i, j \in \mathbb{N}\} \) forms an open cover for each closed subinterval of \( (\frac{b}{c}, \frac{c}{b}) \). Let \( r \in (\frac{b}{c}, \frac{c}{b}) \). Then \( \{R(I_i, I_{j1}) : i, j \in \mathbb{N}\} \) is an open cover of \( [\frac{b}{c} + \frac{r}{2}, \frac{c}{b} - \frac{r}{2}] \) and therefore by Heine Borel Covering Theorem, the interval \( [\frac{b}{c} + \frac{r}{2}, \frac{c}{b} - \frac{r}{2}] \) is covered by finitely many of \( \{R(I_i, I_{j1}) : i, j \in \mathbb{N}\} \). So, there exists \( n_1 \in \mathbb{N} \) such that

\[
\bigcup_{i,j=1}^{n_1} R(I_i, I_{j1}) \supseteq [\frac{b}{c} + \frac{r}{2}, \frac{c}{b} - \frac{r}{2}].
\]

For each open interval \( I_i, i = 1, 2, 3, \ldots, n_1 \), there exists a natural number \( k_1 \) and a closed interval \( J_{i1} \) contained in \( I_i, i = 1, 2, 3, \ldots, n_1 \), with end points are of the form \( \frac{k}{3^{k_1}} \) (where \( k \in \mathbb{N} \)) such that

\[
R(int(\bigcup_{i=1}^{n_1} J_{i1})) \supseteq [\frac{b}{c} + \frac{r}{2}, \frac{c}{b} - \frac{r}{2}].
\]

Now each of the intervals \( J_{i1} \) is subdivided into closed sub-intervals of length \( \frac{1}{3^n} \). Remove open middle ninth of each of these intervals of length \( \frac{1}{3^n} \), obtaining intervals of length \( \frac{4}{3^{n_1+2}} \). The union of the interiors of these closed intervals (of length \( \frac{4}{3^{n_1+2}} \)) is an open set, call it \( G_2 \). By argument of Utz [6], we can verify that \( R(G_2) \supseteq [\frac{b}{c} + \frac{r}{2}, \frac{c}{b} - \frac{r}{2}] \). Since \( G_2 \) is a non-empty open subset of \( \mathbb{R}^+ \) and \( F_2 \) is nowhere dense and closed, it follows that \( R(G_2 \setminus F_2) \supseteq [\frac{b}{c} + \frac{r}{2}, \frac{c}{b} - \frac{r}{2}] \). Again \( G_2 \setminus F_2 \) is an open set as \( F_2 \) is closed. So, \( G_2 \setminus F_2 = \bigcup_{i=1}^{\infty} I_i \), where \( \{I_i\}_{i=1}^{\infty} \) are pairwise disjoint countable open intervals. Therefore

\[
R(G_2 \setminus F_2) = R\left( \bigcup_{i=1}^{\infty} I_i \right) = \bigcup_{i,j=1}^{\infty} R(I_{i2}, I_{j2}).
\]

Again by Heine Borel Covering Theorem, \( [\frac{b}{c} + \frac{r}{2}, \frac{c}{b} - \frac{r}{2}] \) is covered by finitely many of \( \{R(I_{i2}, I_{j2}) : i, j \in \mathbb{N}\} \). So, there exists a natural number \( n_2 \), such that
\[ \bigcup_{i,j=1}^{n_2} R(I_{i2}, J_{j2}) \supseteq \left[ \frac{b}{c} + \frac{r}{2}, \frac{c - r}{2} \right]. \]

Thus, there exist a sequence of closed intervals \( \{J_{i2}\}_{i=1}^{n_2} \) and a natural number \( k_2 \) with \( k_2 > k_1 + 2 \) having the following properties.

Each \( J_{i2} \) is contained in \( I_{i2} \), with end points of the form \( \frac{k}{3^{j_2}}, k \in \mathbb{N} \) and such that

\[ R(\text{int}(\bigcup_{i=1}^{n_2} J_{i2})) \supseteq \left[ \frac{b}{c} + \frac{r}{2}, \frac{c - r}{2} \right]. \]

Proceeding as before, we subdivide each of the intervals \( I_{i2} \) into closed subintervals of length \( \frac{1}{3^{j_2}} \). Remove open middle ninth of each of these intervals of length \( \frac{1}{3^{j_2}} \), obtaining intervals of length \( \frac{4}{3^{j_2+2}} \). The union of the interiors of these closed intervals (of length \( \frac{4}{3^{j_2+2}} \)) is an open set, call it \( G_3 \). Again by same argument of Utz [6], we can verify that \( R(G_3) \supseteq \left[ \frac{b}{c} + \frac{r}{2}, \frac{c - r}{2} \right] \). Continuing this process by finite induction, obtain

\[ \{n_i\}_{i=1}^{\infty}, \{k_i\}_{i=1}^{\infty}, \{\{I_{ij}\}_{i=1}^{n_j}\}_{j=1}^{\infty}, \{\{J_{ij}\}_{i=1}^{n_j}\}_{j=1}^{\infty} \text{ and } \{G_i\}_{i=1}^{\infty} \]

that satisfy the following conditions:

\[ k_{i+1} > k_i + 2 \text{ for each } i. \]

For each \( j \), \( \{I_{ij}\}_{i=1}^{n_j} \) is sequence of pair wise disjoint open intervals such that

\[ \bigcup [R(I_{ij}, I_{kj}) : i, k \in \{1, 2, 3, \ldots, n_j\}] \supseteq \left[ \frac{b}{c} + \frac{r}{2}, \frac{c - r}{2} \right]. \]

For each \( j \), \( \{J_{ij}\}_{i=1}^{n_j} \) is a sequence of closed intervals satisfying the following properties:

- The end points of \( J_{ij} \) are of the form \( \frac{k}{3^{j}}, J_{ij} \subseteq I_{ij} \) and

\[ R(\text{int}(\bigcup_{i=1}^{n_j} J_{ij})) \supseteq \left[ \frac{b}{c} + \frac{r}{2}, \frac{c - r}{2} \right]. \]

Additionally, the formation of each \( G_j \) is as follows:

- Each of the closed intervals \( J_{i,j-1} \) is divided into closed sub-intervals of length \( \frac{1}{3^{j-1}} \). Remove the open middle ninth of each of these intervals of length \( \frac{1}{3^{j-1}} \), obtaining closed intervals of length \( \frac{4}{3^{j-1+2}} \). The union of the interiors of these intervals of length \( \frac{4}{3^{j-1+2}} \) is defined as \( G_j \). Clearly \( G_j \) is open for each \( j \) and \( R(G_j) \supseteq \left[ \frac{b}{c} + \frac{r}{2}, \frac{c - r}{2} \right] \).

Finally we have

\[ B \supseteq \bigcup_{i=1}^{n_1} I_{i1} \supseteq \bigcup_{i=1}^{n_1} J_{i1} \supseteq G_2 \supseteq \bigcup_{i=1}^{n_2} I_{i2} \supseteq \bigcup_{i=1}^{n_2} J_{i2} \supseteq \bigcup_{i=1}^{n_2} G_3 \supseteq \bigcup_{i=1}^{n_2} J_{i3} \supseteq \ldots \]
Let $P = \bigcap_{j=1}^{\infty} \left( \bigcup_{i=1}^{n_j} J_{ij} \right)$. Clearly $P$ is a compact subset of $B$. Also, since $P \subset \bigcup_{i=1}^{n_j} J_{ij}$ for each $j \geq 2$, $P$ is porous. Finally, if $s \in \left[ \frac{b}{c} + \frac{r}{2}, \frac{c}{b} - \frac{r}{2} \right]$, for each $j$, there exist $x_j, y_j \in \bigcup_{i=1}^{n_j} J_{ij}$ such that $x_j = sy_j$. By Bolzano-Weierstrass theorem, there exists a sequence $\{j_k\}$ of natural numbers such that $\lim_{k \to \infty} x_{j_k} = x$ and $\lim_{k \to \infty} y_{j_k} = y$. Clearly $x = sy$.

Furthermore by definition of $P$, $x, y \in P$ and therefore $R(P) \supseteq \left[ \frac{b}{c} + \frac{r}{2}, \frac{c}{b} - \frac{r}{2} \right]$. This completes the proof. □

**QUESTION.** It is unknown whether the Theorem 2 is valid without the property of Baire.

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