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ON EULER HARMONIC IDENTITIES FOR MEASURES AND ERROR ESTIMATIONS

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Abstract. Some new approximations of functions are given by using generalized Euler identities involving real Borel measures and harmonic sequences of functions. Also, we estimate those approximations for different classes of functions and different types of measures.

1. Introduction

For every function $f:[a,b]\to\mathbb{R}$ with $n\geqslant 1$ continuous derivatives and for every $x\in[a,b]$ the following formula (*Euler integral identity*), see [14], is valid:

$$f(x)=\frac{1}{b-a}\int_a^b f(t)dt+\tilde{T}_{n-1}(x)+\tilde{R}_n(x),$$

where $\tilde{T}_0(x):=0$ and

$$\tilde{T}_m(x)=\sum_{k=1}^{m}\frac{(b-a)^{k-1}}{k!}B_k\left(\frac{x-a}{b-a}\right)\left[f^{(k-1)}(b)-f^{(k-1)}(a)\right],$$

for $1\leqslant m\leqslant n$, while

$$\tilde{R}_n(x)=-\frac{(b-a)^{n-1}}{n!}\int_a^b\left[B_n^*\left(\frac{x-t}{b-a}\right)-B_n^*\left(\frac{x-a}{b-a}\right)\right]f^{(n)}(t)d(t).$$

Here, $B_k(x),\ k\geqslant 0,$ are the Bernoulli polynomials, and $B_k^*(x),\ k\geqslant 0$ are periodic functions of period 1, related to the Bernoulli polynomials as

$$B_k^*(x)=B_k(x),\ 0\leqslant x<1;\qquad B_k^*(x+1)=B_k^*(x),\ x\in\mathbb{R}.$$

In other words, Euler integral identity expresses the expansion of a function in terms of *Bernoulli polynomials***.** It has been generalized in recent years in a number of ways. Let us mention here some of them. First of all, the reader is referred to the paper [10] in which it is assumed that the function $f:[a,b]\to\mathbb{R}$ is such that $(n-1)^{th}$ derivative

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$f^{(n-1)}$ is continuous function of bounded variation on $[a,b]$ for some $n \geqslant 1$. Further natural generalization of such results, see [11], arises by replacing the Bernoulli polynomials with an arbitrary *harmonic sequence of polynomials*, i.e. the sequence of polynomials satisfying

$$P_k'(t) = P_{k-1}(t),\ k \geqslant 1;\ P_0(t) = 1.$$

The next generalization is obtained by replacing harmonic sequence of polynomials with a harmonic sequence of functions generated by some weight function. More precisely:

For $a,b \in \mathbb{R},\ a < b,$ let $w : [a,b] \to [0,\infty)$ be a probability density function i.e. integrable function satisfying $\int_a^b w(t)dt = 1$. For $n \geqslant 0$ and $t \in [a,b]$ let

$$w_n(t) = \frac{1}{(n-1)!}\int_a^t (t-s)^{n-1} w(s)ds,\ \ n \geqslant 1;\ w_0(t) = w(t).$$

It is well known that w_n is equal to the n-th indefinite integral of w, being equal to zero at a, i.e. $w_n^{(n)}(t) = w(t)$ and $w_n(a) = 0$, for every $n \geqslant 1$. A sequence of functions $H_n : [a,b] \to \mathbb{R},\ n \geqslant 0,$ is called *w-harmonic sequence of functions* on $[a,b]$ if

$$H_n'(t) = H_{n-1}(t),\ n \geqslant 1;\ H_0(t) = w(t),\ t \in [a,b].$$

The sequence $(w_n,\ n \geqslant 0)$ is an example of w-harmonic sequence of functions on $[a,b]$.

Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n-1)}$ is a continuous function of bounded variation on $[a,b]$ for some $n \geqslant 1$. In a recent paper [1] two identities named the *weighted Euler harmonic identities* were proved. The first weighted Euler harmonic identity is

$$f(x) = \int_a^b f_x(t)w(t)dt + \tilde{S}_n(x) + \tilde{R}_n^1(x) \tag{1}$$

and the second one is

$$f(x) = \int_a^b f_x(t)w(t)dt + \tilde{S}_{n-1}(x) + [H_n(a) - H_n(b)]\, f^{(n-1)}(x) + \tilde{R}_n^2(x), \tag{2}$$

where $(H_n,\ n \geqslant 1)$ is w-harmonic sequence of functions on $[a,b]$,

$$\tilde{S}_m(x) = \sum_{k=1}^{m} H_k(x)\left[f^{(k-1)}(b) - f^{(k-1)}(a)\right] + \sum_{k=2}^{m}\left[H_k(a) - H_k(b)\right] f^{(k-1)}(x),$$

for $1 \leqslant m \leqslant n,$ with convention

$$\tilde{S}_1(x) = H_1(x)\left[f(b) - f(a)\right],$$

and

$$f_x(t) = \begin{cases} f(a+x-t), & a \leqslant t \leqslant x \\ f(b+x-t), & x < t \leqslant b \end{cases}, \tag{3}$$

while

$$\tilde{R}_n^1(x) = -(b-a)^n \int_{[a,b]} H_n^*\left(\frac{x-t}{b-a}\right) df^{(n-1)}(t)$$

and

$$\tilde{R}_n^2(x) = -(b-a)^n \int_{[a,b]} \left[H_n^*\left(\frac{x-t}{b-a}\right) - \frac{1}{(b-a)^n} H_n(x)\right] df^{(n-1)}(t).$$

Here H_n^* denotes a periodic function of period 1 defined by H_n as

$$H_n^*(t) = \frac{1}{(b-a)^n} H_n(a+(b-a)t),\ 0 \leqslant t < 1,$$
$$H_n^*(t+1) = H_n^*(t),\ t \in \mathbb{R}.$$

Identities (1) and (2) hold for every $x \in [a,b]$. The reader can find further references to some recent results on generalizations and applications of Euler identities in [2], [13], [5], [6], [7] and [4].

The main aim of this paper is to prove analogous formulae, see Theorem 2 and Theorem 1, by replacing the w-harmonic sequence of functions $(H_n, n \geqslant 1)$ with a new, more general sequence of piecewise defined functions, and then to estimate those approximations for different classes of functions and different types of measures. The key technical result is proved in Lemma 3. The proof is based on Weierstrass Approximation Theorem.

2. Generalized Euler harmonic identities

For $a,b \in \mathbb{R}$, $a < b$, let $C[a,b]$ be the Banach space of all continuous functions $f : [a,b] \to \mathbb{R}$ with the max norm

$$\|f\| = \max_{a \leqslant t \leqslant b} |f(t)|$$

and $M[a,b]$ the Banach space of all real Borel measures on $[a,b]$ with the total variation norm

$$\|\mu\| = |\mu|([a,b]) = \sup_{E_i} \sum_i |\mu(E_i)|$$

where the sup being taken over all partitions $\{E_i\}$ of $[a,b]$. It is well known that $M[a,b]$ is dual of $C[a,b]$, i.e. $M[a,b] = C^*[a,b]$. In the rest of the paper we use the notation

$$\int_{[a,b]} F(s) d\mu(s)$$

to denote the Lebesgue integral of F over $[a,b]$ with respect to the measure μ, while for a given function $\varphi : [a,b] \to \mathbb{R}$ of bounded variation

$$\int_{[a,b]} F(s) d\varphi(s)$$

denotes Lebesgue-Stieltjes integral of F over $[a,b]$ with respect to φ. Also, by

$$\int_a^b F(s)ds$$

we denote the usual Lebesgue integral of F over $[a,b]$.

To make reading easier, let us recall here some notations and properties of the μ-harmonic sequence of functions defined and stated in [3].

For $\mu \in M[a,b]$ define the function $\check{\mu}_n : [a,b] \to \mathbb{R}$, $n \geqslant 1$, by

$$\check{\mu}_n(t) = \frac{1}{(n-1)!} \int_{[a,t]} (t-s)^{n-1} d\mu(s).$$

For $n = 1$,

$$\check{\mu}_1(t) = \int_{[a,t]} d\mu(s) = \mu([a,t]),\ a \leqslant t \leqslant b,$$

which means that $\check{\mu}_1$ is equal to the distribution function of μ.

Note that

$$\check{\mu}_{n+1}(t) = \int_a^t \check{\mu}_n(s)ds,\ a \leqslant t \leqslant b,\ n \geqslant 1.$$

This means that for $n \geqslant 1$, $\check{\mu}_{n+1}$ is differentiable at almost all points of $[a,b]$ and $\check{\mu}_{n+1}' = \check{\mu}_n$ almost everywhere on $[a,b]$ with respect to Lebesgue measure.

Using the Fubini theorem we can easily get the following formula

$$\check{\mu}_n(t) = \frac{1}{(n-2)!} \int_a^t (t-s)^{n-2} \check{\mu}_1(s)ds,\ a \leqslant t \leqslant b,\ n \geqslant 2.$$

From this formula we see immediately that $\check{\mu}_n(a) = 0$, $n \geqslant 2$.

If $\mu \geqslant 0$, then $\check{\mu}_n$, $n \geqslant 1$, is nonnegative and nondecreasing function.

Note that function $g(s) = (t-s)^{n-1}$ is nonincreasing on $[a,t]$ so that from the first expression for $\check{\mu}_n(t)$ we get the estimate

$$|\check{\mu}_n(t)| \leqslant \frac{(t-a)^{n-1}}{(n-1)!} \|\mu\|,\ a \leqslant t \leqslant b,\ n \geqslant 1.$$

DEFINITION 1. A sequence of functions $P_n : [a,b] \to \mathbb{R}$, $n \geqslant 1$, is called a μ-*harmonic sequence of functions* on $[a,b]$ if

$$P_1(t) = c + \check{\mu}_1(t),\ a \leqslant t \leqslant b,$$

for some $c \in \mathbb{R}$, and

$$P_{n+1}(t) = P_{n+1}(a) + \int_a^t P_n(s)ds,\ a \leqslant t \leqslant b,\ n \geqslant 1.$$

Since $P_{n+1},\ n \geqslant 1$ is defined as an indefinite Lebesgue integral of P_n, it is well known that $P_{n+1},\ n \geqslant 1$ is an absolutely continuous function, $P_{n+1}^{'} = P_n$, a.e. on $[a,b]$ with respect to Lebesgue measure, and for every $f \in C[a,b]$ we have

$$\int_{[a,b]} f(t)dP_{n+1}(t) = \int_a^b f(t)P_n(t)dt,\ n \geqslant 1.$$

The sequence $(\check{\mu}_n,\ n \geqslant 1)$ is an example of a μ-harmonic sequence of functions on $[a,b]$.

For $\mu \in M[a,b]$ let $(P_n,\ n \geqslant 1)$ be a μ-harmonic sequence of functions on $[a,b]$. For $x \in [a,b]$, define function $K_n : [a,b] \times [a,b] \to \mathbb{R}$, for $n \geqslant 1$, by

$$K_n(x,t) = \begin{cases} P_n(b-x+t),\ a \leqslant t \leqslant x \\ P_n(a-x+t),\ x < t \leqslant b \end{cases} \tag{4}$$

for $a \leqslant x < b$, and

$$K_n(b,t) = \begin{cases} P_n(t),\ a \leqslant t < b \\ P_n(a),\quad t = b \end{cases}. \tag{5}$$

Thus, for $n \geqslant 2$, $K_n(x,\cdot)$ is continuous on $[a,b] \setminus \{x\}$ and has a jump of $P_n(a) - P_n(b)$ at x. Note that $K_n(x,\cdot),\ n \geqslant 1$ is a function of bounded variation and for $n \geqslant 1, K_{n+1}^{'}(x,\cdot) = K_n(x,\cdot)$ a.e. on $[a,b]$ with respect to Lebesgue measure. Also note that

$$K_n(x,a) = K_n(x,b) = P_n(a+b-x),\ n \geqslant 1.$$

For $x \in [a,b]$ and a function $F : [a,b] \to \mathbb{R}$ introduce a notation which will be used hereafter:

$$F_x(t) = \begin{cases} F(x-a+t),\ a \leqslant t \leqslant a+b-x \\ F(x-b+t),\ a+b-x < t \leqslant b \end{cases}. \tag{6}$$

LEMMA 1. *For every μ-harmonic sequence $(P_n,\ n \geqslant 1)$, integrable function $g : [a,b] \to \mathbb{R}$ and $f \in C[a,b]$ we have*

$$\int_a^b g(t)f(K_n(x,t))dt = \int_a^b g_x(t)f(P_n(t))dt,$$

for every $x \in [a,b]$, where $g_x(t)$ is defined by (6).

Proof. Follows from (4) and (5) using simple calculations,

$$\begin{aligned} &\int_a^b g(t)f(K_n(x,t))dt \\ &= \int_a^x g(t)f(P_n(b-x+t))dt + \int_x^b g(t)f(P_n(a-x+t))dt \\ &= \int_{b-x+a}^b g(x-b+t)f(P_n(t))dt + \int_a^{b-x+a} g(x-a+t)f(P_n(t))dt \\ &= \int_a^b g_x(t)f(P_n(t))dt. \quad \square \end{aligned}$$

LEMMA 2. *For every $f \in C[a,b]$ and $n \geqslant 2$ we have*

$$\int_{[a,b]} f(t)dK_n(x,t) = \int_a^b f(t)K_{n-1}(x,t)dt + f(x)\left[P_n(a) - P_n(b)\right].$$

Proof. Follows directly from properties of Lebesgue-Stieltjes integral of continuous function f over $[a,\ b]$ with respect to K_n, and the given properties of the function K_n. Namely, the function $K_n(x,\cdot)$, $n \geqslant 2$ is almost everywhere differentiable on $[a,b]$ and its derivative is equal to $K_{n-1}(x,\cdot)$ a.e. on $[a,b]$ with respect to Lebesgue measure. Further, it has a jump at x of magnitude $P_n(a) - P_n(b)$, which proves our assertion. □

LEMMA 3. *For every $\mu \in M[a,b]$ and $f \in C[a,b]$ we have*

$$\int_{[a,b]} f(t)dK_1(x,t) = \int_{[a,b]} f_x(t)d\mu(t) - f(x)\mu([a,b]), \tag{7}$$

where $f_x(t)$ is defined by (6).

Proof. Introduce the notations:

$$I(f,\mu) = \int_{[a,b]} f(t)dK_1(x,t)$$

and

$$J(f,\mu) = \int_{[a,b]} f_x(t)d\mu(t) - f(x)\mu([a,b]).$$

Then $I,J : C[a,b] \times M[a,b] \to \mathbb{R}$ are continuous bilinear functionals with

$$|I(f,\mu)| \leqslant \|f\|\,\|\mu\|$$

and

$$|J(f,\mu)| \leqslant 2\,\|f\|\,\|\mu\|.$$

Let us prove that $I(f,\mu) = J(f,\mu)$ for every $f \in C[a,b]$ and every $\mu \in M[a,b]$.

Since the set of all polynomial functions is dense in $C[a,b]$ it is sufficient, by continuity and linearity of funtionals I and J, to prove this equality for the power function $f(t) = t^n$, $a \leqslant t \leqslant b$, $n \geqslant 0$. Also, since $P_1(t) = c + \mu([a,t])$, $a \leqslant t \leqslant b$, for some constant c, and obviously the integral on the left hand side of (7) is independent of the choice of the constant c, we may assume that $c = 0$.

Integrating by parts the left hand side of the equality we find

$$\begin{aligned}\int_{[a,b]} t^n dK_1(x,t) &= t^n K_1(x,t)\big|_a^b - \int_a^b K_1(x,t)dt^n \\ &= b^n K_1(x,b) - a^n K_1(x,a) - n\int_a^b t^{n-1}K_1(x,t)dt.\end{aligned}$$

However, the values $K_1(x,b)$ and $K_1(x,a)$ are equal and can be replaced with

$$P_1(a+b-x) = \check{\mu}_1(a+b-x),$$

thus

$$I(f,\mu) = \check{\mu}_1(a+b-x)(b^n-a^n) - n\int_a^b t^{n-1}K_1(x,t)dt. \tag{8}$$

Using Lemma 1 integral

$$A(x) = \int_a^b t^{n-1}K_1(x,t)dt,$$

may be written in the form

$$A(x) = \int_a^b \left(t^{n-1}\right)_x \check{\mu}_1(t)\,dt = A_1(x) + A_2(x), \tag{9}$$

where

$$\begin{aligned} A_1(x) &= \int_a^{a-x+b} (t+x-a)^{n-1}\check{\mu}_1(t)\,dt \\ &= \int_a^{a-x+b} (t+x-a)^{n-1}\left[\int_{[a,t]} d\mu(s)\right] dt \end{aligned}$$

and

$$\begin{aligned} A_2(x) &= \int_{a-x+b}^{b} (t+x-b)^{n-1}\check{\mu}_1(t)\,dt \\ &= \int_{a-x+b}^{b} (t+x-b)^{n-1}\left[\check{\mu}_1(a-x+b) + \int_{(a-x+b,t]} d\mu(s)\right] dt \\ &= \check{\mu}_1(a-x+b)\int_{a-x+b}^{b} (t+x-b)^{n-1}dt \\ &\quad + \int_{a-x+b}^{b} (t+x-b)^{n-1}\left[\int_{(a-x+b,t]} d\mu(s)\right] dt. \end{aligned}$$

Using the Fubini theorem we get

$$\begin{aligned} A_1(x) &= \int_{[a,a-x+b]} \left[\int_s^{a-x+b} (t+x-a)^{n-1}dt\right] d\mu(s) \\ &= \int_{[a,a-x+b]} \frac{b^n-(s+x-a)^n}{n} d\mu(s) \\ &= \frac{b^n}{n}\check{\mu}_1(a-x+b) - \frac{1}{n}\int_{[a,a-x+b]} (s+x-a)^n d\mu(s) \end{aligned}$$

and

$$
\begin{aligned}
A_2(x) &= \check{\mu}_1\,(a-x+b)\,\frac{(x^n-a^n)}{n} + \int_{(a-x+b,b]} \left[\int_s^b (t+x-b)^{n-1}dt\right] d\mu(s) \\
&= \check{\mu}_1\,(a-x+b)\,\frac{(x^n-a^n)}{n} + \int_{(a-x+b,b]} \frac{x^n-(s+x-b)^n}{n} d\mu(s) \\
&= \frac{-a^n}{n}\check{\mu}_1\,(a-x+b) + \frac{x^n}{n}\check{\mu}_1\,(b) - \frac{1}{n}\int_{(a-x+b,b]} (s+x-b)^n d\mu(s).
\end{aligned}
$$

Replacing the obtained expressions in (9) and (8), after simplifying, we have

$$
\begin{aligned}
I(f,\mu) &= \int_{[a,a-x+b]} (s+x-a)^n d\mu(s) \\
&\quad + \int_{(a-x+b,b]} (s+x-b)^n d\mu(s) - \mu([a,b])x^n \\
&= \int_{[a,a-x+b]} f_x(s)d\mu(s) \\
&\quad + \int_{(a-x+b,b]} f_x(s)d\mu(s) - \mu([a,b])f(x) \\
&= \int_{[a,b]} f_x(t)d\mu(t) - \mu([a,b])f(x) = J(f,\mu).
\end{aligned}
$$

This completes the proof. □

THEOREM 1. *For $\mu \in M[a,b]$ let $(P_n,\ n \geqslant 1)$ be a μ-harmonic sequence of functions on $[a,b]$ and $f:[a,b]\to\mathbb{R}$ such that $f^{(n-1)}$ is a continuous function of bounded variation on $[a,b]$ for some $n \geqslant 1$. Then we have*

$$
\int_{[a,b]} f_x(t)d\mu(t) - \mu(\{a\})f(x) + S_n(x) = R_n(x),
$$

for every $x \in [a,b]$, where $f_x(t)$ is defined by (6),

$$
\begin{aligned}
S_n(x) &= \sum_{k=1}^{n-1} (-1)^k P_k(a+b-x)\left[f^{(k-1)}(b) - f^{(k-1)}(a)\right] \\
&\quad + \sum_{k=1}^{n} (-1)^k f^{(k-1)}(x)\,[P_k(b) - P_k(a)]
\end{aligned}
$$

and

$$
R_n(x) = (-1)^n \int_{[a,b]} [K_n(x,t) - K_n(x,a)]\, df^{(n-1)}(t).
$$

Proof. For $1 \leqslant k \leqslant n$ consider the integral

$$
R_k(x) = (-1)^k \int_{[a,b]} (K_k(x,t) - K_k(x,a))\, df^{(k-1)}(t).
$$

Integrating by parts we get

$$R_k(x) = (-1)^{k-1} \int_{[a,b]} f^{(k-1)}(t) dK_k(x,t), \tag{10}$$

since

$$K_k(x,a) = K_k(x,b) = P_k(a+b-x).$$

From (10), for every $k \geqslant 2,$ by Lemma 2, we find

$$R_k(x) = (-1)^{k-1} \int_a^b K_{k-1}(x,t)\, df^{(k-2)}(t) + (-1)^{k-1} [P_k(a) - P_k(b)]\, f^{(k-1)}(x),$$

i.e.

$$\begin{aligned} R_k(x) = R_{k-1}(x) + (-1)^{k-1} P_{k-1}(a+b-x) \left[f^{(k-2)}(b) - f^{(k-2)}(a) \right] \\ + (-1)^{k-1} [P_k(a) - P_k(b)]\, f^{(k-1)}(x). \end{aligned} \tag{11}$$

By Lemma 3, for $k = 1,$ (10) becomes

$$R_1(x) = \int_{[a,b]} f_x(t) d\mu(t) - f(x)\mu([a,b]), \tag{12}$$

where $f_x(t)$ is defined by (6). From (11) and (12) it follows, by iteration,

$$\begin{aligned} R_n(x) = R_1(x) + \sum_{k=2}^{n} (-1)^{k-1} P_{k-1}(a+b-x) \left[f^{(k-2)}(b) - f^{(k-2)}(a) \right] \\ + \sum_{k=2}^{n} (-1)^k [P_k(b) - P_k(a)]\, f^{(k-1)}(x) \end{aligned}$$

or

$$\begin{aligned} R_n(x) = \int_{[a,b]} f_x(t) d\mu(t) - \mu(\{a\}) f(x) \\ + \sum_{k=1}^{n-1} (-1)^k P_k(a+b-x) \left[f^{(k-1)}(b) - f^{(k-1)}(a) \right] \\ + \sum_{k=1}^{n} (-1)^k f^{(k-1)}(x) [P_k(b) - P_k(a)], \end{aligned}$$

since

$$f(x)\mu([a,b]) = f(x) [P_1(b) - P_1(a) + \mu(\{a\})],$$

which proves our assertion. □

THEOREM 2. *For $\mu \in M[a,b]$ let $(P_n,\ n \geqslant 1)$ be a μ-harmonic sequence of functions on $[a,b]$ and $f : [a,b] \to \mathbb{R}$ such that $f^{(n-1)}$ is a continuous function of bounded variation for some $n \geqslant 1$. Then we have*

$$\int_{[a,b]} f_x(t) d\mu(t) - \mu(\{a\}) f(x) + S_n^1(x) = R_n^1(x),$$

for every $x \in [a,b]$, *where* $f_x(t)$ *is defined by (6),*

$$S_n^1(x) = \sum_{k=1}^{n} (-1)^k P_k(a+b-x)\left[f^{(k-1)}(b) - f^{(k-1)}(a)\right] + \sum_{k=1}^{n} (-1)^k f^{(k-1)}(x)\left[P_k(b) - P_k(a)\right]$$

and

$$R_n^1(x) = (-1)^n \int_{[a,b]} K_n(x,t)\, df^{(n-1)}(t).$$

Proof. Note that

$$\begin{aligned} R_n(x) &= R_n^1(x) - (-1)^n K_n(x,a) \int_{[a,b]} df^{(n-1)}(t) \\ &= R_n^1(x) - (-1)^n P_n(a+b-x)\left[f^{(n-1)}(b) - f^{(n-1)}(a)\right]. \end{aligned}$$

Therefore, our assertion follows from Theorem 1. □

REMARK 1. In the special case, when the measure μ is a *probability measure*, $d\mu(t) = w(t)dt$, with the density w, the μ-harmonic sequence of functions $(P_n,\ n \geqslant 1)$ on $[a,b]$ becomes w-harmonic sequence of functions from the Introduction. In this case P_1 is differentiable a.e. where $P_1'(t) = w(t)$, a.e. and for $a \leqslant t \leqslant b,\ \ n \geqslant 1$,

$$\begin{aligned} \check{\mu}_n(t) &= \frac{1}{(n-1)!} \int_{[a,t]} (t-s)^{n-1} d\mu(s) \\ &= \frac{1}{(n-1)!} \int_a^t (t-s)^{n-1} w(s)ds =: w_n(t). \end{aligned}$$

Therefore, Euler identities proved in Theorem 2 and Theorem 1 generalize formulae (1) and (2), respectively, since in this case $\mu([a,b]) = 1$ and

$$\int\limits_{[a,b]} f_x(t)d\mu(t) = \int\limits_a^b f_x(t)w(t)dt,$$

for every $x \in [a,b]$.

REMARK 2. Applying Theorem 1 with $x = a$ we get the identity

$$\int_{[a,b]} f(t)d\mu(t) - \mu(\{a\})f(a) + S_n(a) = R_n(a),$$

where

$$S_n(a) = \sum_{k=1}^{n-1} (-1)^k \left[P_k(b)f^{(k-1)}(b) - P_k(a)f^{(k-1)}(a)\right] + (-1)^n \left[P_n(b) - P_n(a)\right] f^{(n-1)}(a)$$

for $n>1$, and

$$S_1(a)=-(P_1(b)-P_1(a))f(a)=-\mu\left((a,b]\right)f(a),$$

while

$$R_n(a)=(-1)^n\int_{[a,b]}\left[P_n(t)-P_n(b)\right]df^{(n-1)}(t)$$

for $n>1$, and

$$R_1(a)=-\int_{[a,b]}P_1(t)\,df(t)+(c+\mu\left((a,b]\right)(f(b)-f(a))$$

for some real constant c. We can regard this identity as a generalized trapezoid identity since for $n=1$, $c=-\frac{1}{2}$, and probability measure μ, ($\mu[a,b]=1$), it reduces to the *simple trapezoid identity*

$$\frac{1}{2}(f(a)+f(b))=\int_{[a,b]}f(t)d\mu(t)+\int_{[a,b]}P_1(t)df(t),$$

where

$$P_1(t)=-\frac{1}{2}+\mu\left((a,t]\right),\ a\leqslant t\leqslant b.$$

Similarly, applying Theorem 2 with $x=\frac{a+b}{2}$ we get

$$\int_{[a,b]}f_{\frac{a+b}{2}}(t)d\mu(t)-\mu(\{a\})f\left(\frac{a+b}{2}\right)+S_n^1\left(\frac{a+b}{2}\right)=R_n^1\left(\frac{a+b}{2}\right),$$

where

$$\begin{aligned}S_n^1\left(\frac{a+b}{2}\right)&=\sum_{k=1}^{n}(-1)^kP_k\left(\frac{a+b}{2}\right)\left[f^{(k-1)}(b)-f^{(k-1)}(a)\right]\\&\quad+\sum_{k=1}^{n}(-1)^kf^{(k-1)}\left(\frac{a+b}{2}\right)\left[P_k(b)-P_k(a)\right]\end{aligned}$$

and

$$\begin{aligned}R_n^1\left(\frac{a+b}{2}\right)&=(-1)^n\int_{[a,\frac{a+b}{2}]}P_n\left(t+\frac{b-a}{2}\right)df^{(n-1)}(t)\\&\quad+(-1)^n\int_{(\frac{a+b}{2},b]}P_n\left(t-\frac{b-a}{2}\right)df^{(n-1)}(t).\end{aligned}$$

for $n\geqslant 1$. We can regard this identity as generalized midpoint identity since for $n=1$, $c=-\mu\left([a,\frac{a+b}{2}]\right)$, and probability measure μ, it reduces to the *simple midpoint identity*

$$f\left(\frac{a+b}{2}\right)=\int_{[a,b]}f_{\frac{a+b}{2}}(t)d\mu(t)-R_1^1\left(\frac{a+b}{2}\right).$$

In this special case $P_1\left(\frac{a+b}{2}\right)=c+\mu\left([a,\frac{a+b}{2}]\right)=0$.

For some additional modified versions of the Euler trapezoid formula and the Euler midpoint formula, the reader is referred to the papers [9] and [8].

COROLLARY 1. *Let $f:[a,b]\to\mathbb{R}$ be such that $f^{(n-1)}$ is a continuous function of bounded variation for some $n\geqslant 1$. Then we have*

$$\int_{[a,b]} f_x(t)d\mu(t)+\check{S}_n(x)=\check{R}_n(x).$$

for every $x\in[a,b]$, where

$$\begin{aligned}\check{S}_n(x)=&\sum_{k=1}^{n-1}(-1)^k\check{\mu}_k(a+b-x)\left[f^{(k-1)}(b)-f^{(k-1)}(a)\right]\\&+\sum_{k=1}^{n}(-1)^k f^{(k-1)}(x)\check{\mu}_k(b),\end{aligned}$$

$$\check{R}_n(x)=(-1)^n\int_{[a,b]}\left[\check{K}_n(x,t)-\check{K}_n(x,a)\right]df^{(n-1)}(t)$$

and

$$\check{K}_n(x,t)=\begin{cases}\check{\mu}_n(b-x+t),\ a\leqslant t\leqslant x\\ \check{\mu}_n(a-x+t),\ x<t\leqslant b\end{cases}$$

for $a\leqslant x<b$, while

$$\check{K}_n(b,t)=\begin{cases}\check{\mu}_n(t),\ a\leqslant t<b\\ \check{\mu}_n(a),\quad t=b\end{cases}.$$

Proof. Apply the theorem above to the special case $P_n=\check{\mu}_n,\ n\geqslant 1$, and note that $\check{\mu}_n(a)=0$ for $n\geqslant 2$. □

COROLLARY 2. *Let $f:[a,b]\to\mathbb{R}$ be such that $f^{(n-1)}$ is a continuous function of bounded variation for some $n\geqslant 1$. Then we have*

$$\int_a^b f(t)dt+\bar{S}_n(x)=\bar{R}_n(x).$$

for every $x\in[a,b]$, where

$$\begin{aligned}\bar{S}_n(x)=&\sum_{k=1}^{n-1}\frac{(-1)^k}{k!}(b-x)^k\left[f^{(k-1)}(b)-f^{(k-1)}(a)\right]\\&+\sum_{k=1}^{n}\frac{(-1)^k}{k!}(b-a)^k f^{(k-1)}(x),\end{aligned}$$

$$\bar{R}_n(x)=(-1)^n\int_{[a,b]}\left[\bar{K}_n(x,t)-\bar{K}_n(x,a)\right]df^{(n-1)}(t),$$

and

$$\bar{K}_n(x,t) = \begin{cases} \frac{1}{n!}(b-a-x+t)^n, & a \leqslant t \leqslant x \\ \frac{1}{n!}(t-x)^n, & x < t \leqslant b \end{cases}$$

for $a \leqslant x < b$, *while*

$$\bar{K}_n(b,t) = \begin{cases} \frac{1}{n!}(t-a)^n, & a < t < b \\ 0, & (t=a) \text{ or } (t=b) \end{cases}.$$

Proof. Apply Corollary 1 in the special case when μ is the *Lebesgue measure*, $d\mu(t) = dt$, on $[a,b]$. In this case

$$\check{\mu}_k(t) = \frac{(t-a)^k}{k!},\ k \geqslant 1$$

and

$$\int_{[a,b]} f_x(t)d\mu(t) = \int_a^b f_x(t)dt = \int_a^b f(t)dt. \quad \square$$

REMARK 3. Integrating by parts expression $R_n(x)$ and applying Lemma 2 we see that $R_n(x)$ can be rewritten, for $n \geqslant 2$, as

$$\begin{aligned} R_n(x) &= (-1)^n \int_{[a,b]} [K_n(x,t) - K_n(x,a)]\, d\left[f^{(n-1)}(t) - f^{(n-1)}(x)\right] \\ &= (-1)^{n-1} \int_{[a,b]} \left[f^{(n-1)}(t) - f^{(n-1)}(x)\right] dK_n(x,t) \\ &= (-1)^{n-1} \int_a^b \left[f^{(n-1)}(t) - f^{(n-1)}(x)\right] K_{n-1}(x,t)\, dt, \end{aligned}$$

since

$$K_n(x,a) = K_n(x,b) = P_n(a+b-x),\ n \geqslant 1.$$

It can be easily seen that Theorem 1 also holds for functions $f : [a,b] \to \mathbb{R}$ such that $f^{(n-1)}$ is integrable, while $R_n(x)$ is in the form

$$R_n(x) = (-1)^{n-1} \int_a^b \left[f^{(n-1)}(t) - f^{(n-1)}(x)\right] K_{n-1}(x,t)\, dt,$$

for $n \geqslant 2$.

3. Estimate of the remainder of Euler approximations

In this section we estimate the approximation formula obtained in Theorem 1 for a class of functions f whose derivatives $f^{(n-1)}$ are either L-Lipschitzian on $[a,b]$ or continuous and of bounded variation on $[a,b]$. Analogous results are obtained for a class of functions f possessing derivatives $f^{(n)}$ in $L_p[a,b]$, $1 < p < \infty$, and for different types of measures. Similar estimations can be obtained using Theorem 2. Throughout this section we use the same notations as above.

THEOREM 3. *Let* $f:[a,b]\to\mathbb{R}$ *be such that* $f^{(n-1)}$ *is* L*-Lipschitzian on* $[a,b]$ *for some* $n\geqslant 1$. *Then*

$$\begin{aligned}&\left|\int_{[a,b]} f_x(t)d\mu(t)-\mu(\{a\})f(x)+S_n(x)\right|\\&\leqslant L\int_a^b |P_n(t)-P_n(a+b-x)|\,dt,\end{aligned}$$

for every $x\in[a,b]$.

Proof. If $\varphi:[a,b]\to\mathbb{R}$ is L*-Lipschitzian* on $[a,b]$, i.e.

$$|\varphi(x)-\varphi(y)|\leqslant L\cdot|x-y|,\ x,\ y\in[a,b],$$

then for any integrable function $g:[a,b]\to\mathbb{R}$

$$\left|\int_{[a,b]} g(t)d\varphi(t)\right|\leqslant L\int_a^b |g(t)|\,dt.$$

Using this estimate and Lemma 1 we get

$$\begin{aligned}|R_n(x)|&=\left|\int_{[a,b]}\left[K_n(x,t)-K_n(x,a)\right]df^{(n-1)}(t)\right|\\&\leqslant L\int_a^b |K_n(x,t)-K_n(x,a)|\,dt\\&=L\int_a^b |P_n(t)-P_n(a+b-x)|\,dt,\end{aligned}$$

since $K_n(x,a)=P_n(a+b-x)$. Therefore, our assertion follows from Theorem 1. □

COROLLARY 3. *If* f *is* L*-Lipschitzian on* $[a,b]$, *then*

$$\int_{[a,b]} f_x(t)d\mu(t)-f(x)\mu([a,b])=R_1(x),$$

and

$$|R_1(x)|\leqslant L\int_a^b |\check{\mu}_1(t)-\check{\mu}_1(a+b-x)|\,dt,$$

for every $x\in[a,b]$.

Proof. Put $n=1$ in the theorem above and note that

$$\begin{aligned}S_1(x)&=-f(x)\left[P_1(b)-P_1(a)\right]\\&=-f(x)\left[\check{\mu}_1(b)-\check{\mu}_1(a)\right]\\&=f(x)\mu(\{a\})-f(x)\mu([a,b]).\quad\square\end{aligned}$$

COROLLARY 4. *If f is L-Lipschitzian on $[a,b]$ and $\mu \geqslant 0$, then*

$$\left| \int_{[a,b]} f_x(t) d\mu(t) - f(x)\mu([a,b]) \right| \leqslant L\left[(a+b-2x)\check{\mu}_1(a+b-x) - 2\check{\mu}_2(a+b-x) + \check{\mu}_2(b)\right],$$

for every $x \in [a,b]$.

Proof. In this case $\check{\mu}_n$, $n \geqslant 1$, is nonnegative and nondecreasing function. Apply the corollary above and note that

$$\begin{aligned} &\int_a^b |P_1(t) - P_1(a+b-x)|\, dt \\ &= \int_a^{a+b-x} (\check{\mu}_1(a+b-x) - \check{\mu}_1(t))\, dt + \int_{a+b-x}^b (\check{\mu}_1(t) - \check{\mu}_1(a+b-x))\, dt \\ &= (a+b-2x)\check{\mu}_1(a+b-x) - 2\check{\mu}_2(a+b-x) + \check{\mu}_2(b), \end{aligned}$$

since

$$P_1(t) = c + \check{\mu}_1(t),$$

for some real c and

$$\int_{t_1}^{t_2} \check{\mu}_1(s) ds = \check{\mu}_2(t_2) - \check{\mu}_2(t_1),$$

for $a \leqslant t_1 \leqslant t_2 \leqslant b$. □

The *Ostrowski inequality*, see [15], estimates the deviations of a smooth function from its mean value:

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \leqslant \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a)M,\ a \leqslant x \leqslant b,$$

where $f : [a,b] \to \mathbb{R}$ is a differentiable function such that $|f'(x)| \leqslant M$, for every $x \in [a,b]$. The constant $\frac{1}{4}$ is the best possible.

REMARK 4. In the special case when $d\mu(t) = \frac{1}{b-a}dt$ the inequality of Corollary 4 reduces to Ostrowski inequality for L-Lipschitzian functions, since in this case

$$\int_{[a,b]} f_x(t) d\mu t = \frac{1}{b-a}\int_a^b f(t)dt,$$

$$\check{\mu}_1(t) = \frac{t-a}{b-a},$$

$$\check{\mu}_2(t) = \frac{(t-a)^2}{2(b-a)},$$

and

$$(a+b-2x)\check{\mu}_1(a+b-x) - 2\check{\mu}_2(a+b-x) + \check{\mu}_2(b) = (b-a)\left[\frac{1}{4} + \frac{(x-\frac{a+b}{2})^2}{(b-a)^2}\right].$$

COROLLARY 5. *If $f:[a,b]\to\mathbb{R}$ is such that f' is L-Lipschitzian on $[a,b]$, then*

$$R_2(x)=\int_{[a,b]} f_x(t)d\mu(t)-\mu([a,b])f(x)$$
$$-[c+\check{\mu}_1(a+b-x)][f(b)-f(a)]+[c(b-a)+\check{\mu}_2(b)]f'(x)$$

and

$$|R_2(x)|\leqslant L\cdot\int_a^b|c(t-(a+b-x))+\check{\mu}_2(t)-\check{\mu}_2(a+b-x)|dt,$$

for every $x\in[a,b]$ and $c\in\mathbb{R}$.

Proof. Apply Theorem 3 for $n=2$ and note that

$$S_2(x)=-[c+\check{\mu}_1(a+b-x)][f(b)-f(a)]$$
$$+[c(b-a)+\check{\mu}_2(b)]f'(x)-[\mu([a,b]\ )-\mu(\{a\})]\ f(x),$$

since, by Definition 1, the first two terms of a μ-harmonic sequence (P_n) of functions on $[a,b]$ are

$$P_1(t)=c+\check{\mu}_1(t),$$

and

$$P_2(t)=P_2(a)+\int_a^t(c+\check{\mu}_1(s))\,ds,$$

for some real c. □

REMARK 5. If $c\geqslant 0$ and $\mu\geqslant 0$, using the argument which is very similar to the one used in Corollary 4 we get

$$\int_a^b|c(t-(a+b-x))+\check{\mu}_2(t)-\check{\mu}_2(a+b-x)|dt$$
$$=\int_a^{a+b-x}c((a+b-x)-t)+\check{\mu}_2(a+b-x)-\check{\mu}_2(t)dt$$
$$+\int_{a+b-x}^b c(t-(a+b-x))+\check{\mu}_2(t)-\check{\mu}_2(a+b-x)dt$$
$$=\frac{c}{2}\left[(b-x)^2+(x-a)^2\right]+(a+b-2x)\check{\mu}_2(a+b-x)$$
$$-2\check{\mu}_3(a+b-x)+\check{\mu}_3(b),$$

since $\check{\mu}_2$ is nonnegative and nondecreasing function and

$$\int_{t_1}^{t_2}\check{\mu}_2(s)ds=\check{\mu}_3(t_2)-\check{\mu}_3(t_1),$$

for $a\leqslant t_1\leqslant t_2\leqslant b$.

REMARK 6. In the special case, when $c=-\frac{1}{2}$ and $d\mu(t)=\frac{1}{b-a}dt$, the inequality from Corollary 5 reduces to

$$\begin{aligned} &\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt-\left(x-\frac{a+b}{2}\right)\cdot\frac{f(b)-f(a)}{b-a}\right| \\ &\leqslant \frac{L}{2(b-a)}\int_a^b\left|\left(t-\frac{a+b}{2}\right)^2-\left(x-\frac{a+b}{2}\right)^2\right|dt \\ &=\frac{(b-a)^2}{2}\left[\frac{8}{3}\delta^3(x)-\delta^2(x)+\frac{1}{12}\right]\cdot L, \end{aligned}$$

where

$$\delta(x):=\frac{\left|x-\frac{a+b}{2}\right|}{b-a}.$$

This result was proved in [10]. It is much better than the inequality

$$\begin{aligned} &\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt-\left(x-\frac{a+b}{2}\right)\cdot\frac{f(b)-f(a)}{b-a}\right| \\ &\leqslant \frac{(b-a)^2}{2}\left[\left(\delta^2(x)+\frac{1}{4}\right)^2+\frac{1}{12}\right]\cdot M_2, \end{aligned}$$

proved in [12], where

$$L=M_2:=\sup_{t\in[a,b]}\left|f''(t)\right|,$$

since

$$\frac{8}{3}\delta^3(x)-\delta^2(x)+\frac{1}{12}<\left(\delta^2(x)+\frac{1}{4}\right)^2+\frac{1}{12},$$

for every $x\in[a,b]$.

COROLLARY 6. *Let $f:[a,b]\to\mathbb{R}$ be such that $f^{(n-1)}$ is L-Lipschitzian on $[a,b]$ for some $n\geqslant 1$. Then for $\mu\geqslant 0$ we have*

$$\begin{aligned} &\left|\int_{[a,b]} f_x(t)dd(t)+\check{S}_n(x)\right| \\ &\leqslant L\left[(a+b-2x)\check{\mu}_n(a+b-x)-2\check{\mu}_{n+1}(a+b-x)+\check{\mu}_{n+1}(b)\right], \end{aligned}$$

for every $x\in[a,b]$.

Proof. Apply the theorem above to the μ-harmonic sequence $(\check{\mu}_n,\ n\geqslant 1)$. Then $S_n(x)$ becomes $\check{S}_n(x)$ defined in the Corollary 1, while

$$\begin{aligned} &\int_a^b\left|\check{\mu}_n(t)-\check{\mu}_n(a+b-x)\right|dt \\ &=(a+b-2x)\check{\mu}_n(a+b-x)-2\check{\mu}_{n+1}(a+b-x)+\check{\mu}_{n+1}(b), \end{aligned}$$

as in the proof of the Corollary 4. □

COROLLARY 7. *Let $f:[a,b]\to\mathbb{R}$ be such that $f^{(n-1)}$ is L-Lipschitzian on $[a,b]$ for some $n\geqslant 1$. Then*

$$\left|\int_a^b f(t)dt+\bar{S}_n(x)\right|\leqslant L\left[(a+b-2x)\frac{(b-x)^n}{n!}-2\frac{(b-x)^{n+1}}{(n+1)!}+\frac{(b-a)^{n+1}}{(n+1)!}\right],$$

for every $x\in[a,b]$.

Proof. Apply the corollary above to the Lebesgue measure on $[a,b]$. Then $\check{S}_n(x)$ becomes $\bar{S}_n(x)$ which is defined in Corollary 2. □

For $z\in[a,b]$ let $\mu=\delta_z$ be the *Dirac measure* at point z, i.e. the measure defined by

$$\int_{[a,b]} f(t)d\delta_z(t)=f(z).$$

COROLLARY 8. *Let $f:[a,b]\to\mathbb{R}$ be such that $f^{(n-1)}$ is L-Lipschitzian on $[a,b]$ for some $n\geqslant 1$. Then*

$$f(x-a+z)+T_n^1(x,z)=R_n(x),$$

for every $x,z\in[a,b]$, $z\leqslant a+b-x$, and

$$|R_n(x)|\leqslant L\left[(a+b-2x)\frac{(a+b-x-z)^{n-1}}{(n-1)!}-2\frac{(a+b-x-z)^n}{n!}+\frac{(b-z)^n}{n!}\right],$$

where

$$\begin{aligned}T_n^1(x,z)=&\sum_{k=1}^{n-1}(-1)^k\frac{(a+b-x-z)^{k-1}}{(k-1)!}\left[f^{(k-1)}(b)-f^{(k-1)}(a)\right]\\&+\sum_{k=1}^{n}(-1)^k\frac{(b-z)^{k-1}}{(k-1)!}f^{(k-1)}(x).\end{aligned}$$

Proof. Apply Corollary 6 to $\mu=\delta_z$, $a\leqslant z\leqslant a+b-x$. Then

$$\int_{[a,b]} f_x(t)d\mu(t)=\int_{[a,a+b-x]} f(x-a+t)d\delta_z(t)=f(x-a+z),$$

and $\check{S}_n(x)$ becomes $T_n^1(x,z)$ since

$$\check{\mu}_n(t)=\begin{cases}0, & a\leqslant t<z\\ \frac{(t-z)^{n-1}}{(n-1)!} & z\leqslant t\leqslant b\end{cases},\ n\geqslant 1.\quad\square$$

COROLLARY 9. *Let $f:[a,b]\to\mathbb{R}$ be such that $f^{(n-1)}$ is L-Lipschitzian on $[a,b]$ for some $n\geqslant 1$. Then*

$$f(x-b+z)+T_n^2(x,z)=R_n(x),$$

for every $x,z \in [a,b],\ a+b-x<z\leqslant b$, *and*

$$|R_n(x)| \leqslant L\frac{(b-z)^n}{n!},$$

where

$$T_n^2(x,z)=\sum_{k=1}^{n}(-1)^k\frac{(b-z)^{k-1}}{(k-1)!}f^{(k-1)}(x).$$

Proof. Apply Corollary 6 to $\mu=\delta_z,\ a+b-x<z\leqslant b$. Then

$$\int_{[a,b]} f_x(t)d\mu(t)=\int_{(a+b-x,b]} f(x-b+t)d\delta_z(t)=f(x-b+z),$$

while $\check{S}_n(x)$ becomes $T_n^2(x,z)$. □

THEOREM 4. *Let* $f:[a,b]\to\mathbb{R}$ *be such that* $f^{(n-1)}$ *is a continuous function of bounded variation on* $[a,b]$ *for some* $n\geqslant 1$. *Then*

$$\begin{aligned}&\left|\int_{[a,b]} f_x(t)d\mu(t)-\mu(\{a\})f(x)+S_n(x)\right|\\ &\leqslant \sup_{t\in[a,b]}|P_n(t)-P_n(a+b-x)|\,V_a^b(f^{(n-1)})\end{aligned}$$

for every $x\in[a,b]$, *where* $V_a^b(f^{(n-1)})$ *is the total variation of* $f^{(n-1)}$ *on* $[a,b]$.

Proof. If $F:[a,b]\to\mathbb{R}$ is bounded and the Stieltjes integral

$$\int_{[a,b]} F(t)df^{(n-1)}(t)$$

exists, then

$$\left|\int_{[a,b]} F(t)df^{(n-1)}(t)\right|\leqslant \sup_{t\in[a,b]}|F(t)|\cdot V_a^b(f^{(n-1)}).$$

Applying this estimation to $R_n(x)$ we have

$$\begin{aligned}|R_n(x)|&=\left|\int_{[a,b]}[K_n(x,t)-K_n(x,a)]\,df^{(n-1)}(t)\right|\\ &\leqslant \sup_{t\in[a,b]}|K_n(x,t)-K_n(x,a)|\,V_a^b(f^{(n-1)})\\ &\leqslant \sup_{t\in[a,b]}|P_n(t)-P_n(a+b-x)|\,V_a^b(f^{(n-1)}).\end{aligned}$$

Therefore, our assertion follows from Theorem 1. □

COROLLARY 10. *If f is a continuous function of bounded variation on $[a,b]$, then*

$$\int_{[a,b]} f_x(t)d\mu(t) - f(x)\mu([a,b]) = R_1(x),$$

and

$$|R_1(x)| \leqslant \sup_{t\in[a,b]} |P_1(t) - P_1(a+b-x)| V_a^b(f),$$

for every $x \in [a,b]$.

Proof. Put $n = 1$ in the theorem above. □

COROLLARY 11. *If f is a continuous function of bounded variation on $[a,b]$ and $\mu \geqslant 0$, then*

$$\begin{aligned} &\left| \int_{[a,b]} f_x(t)d\mu(t) - f(x)\mu([a,b]) \right| \\ &\leqslant \frac{1}{2}\left[\check{\mu}_1(b) - \check{\mu}_1(a) + |\check{\mu}_1(a) + \check{\mu}_1(b) - 2\check{\mu}_1(a+b-x)|\right] V_a^b(f), \end{aligned}$$

for every $x \in [a,b]$.

Proof. Put $n = 1$ in the corollary above and note that

$$\begin{aligned} &\sup_{t\in[a,b]} |P_1(t) - P_1(a+b-x)| \\ &= \sup_{t\in[a,b]} |\check{\mu}_1(t) - \check{\mu}_1(a+b-x)| \\ &= \max\{\check{\mu}_1(a+b-x) - \check{\mu}_1(a), \check{\mu}_1(b) - \check{\mu}_1(a+b-x)\} \\ &= \frac{1}{2}\left[\check{\mu}_1(b) - \check{\mu}_1(a) + |\check{\mu}_1(a) + \check{\mu}_1(b) - 2\check{\mu}_1(a+b-x)|\right]. \quad \square \end{aligned}$$

COROLLARY 12. *Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n-1)}$ is a continuous function of bounded variation on $[a,b]$ for some $n \geqslant 2$. Then for $\mu \geqslant 0$ we have*

$$\begin{aligned} &\left| \int_{[a,b]} f_x(t)d\mu(t) + \check{S}_n(x) \right| \\ &\leqslant \frac{1}{2}\left[\check{\mu}_n(b) + |\check{\mu}_n(b) - 2\check{\mu}_n(a+b-x)|\right] V_a^b(f^{(n-1)}), \end{aligned}$$

for every $x \in [a,b]$.

Proof. Apply Corollary 1 and the theorem above to the μ-harmonic sequence $(\check{\mu}_n,\ n \geqslant 1)$. Then for $n \geqslant 2$

$$\sup_{t\in[a,b]} |\check{\mu}_n(t) - \check{\mu}_n(a+b-x)| = \frac{1}{2}\left[\check{\mu}_n(b) + |\check{\mu}_n(b) - 2\check{\mu}_n(a+b-x)|\right]. \quad \square$$

COROLLARY 13. *Let $f:[a,b]\to\mathbb{R}$ be such that $f^{(n-1)}$ is a continuous function of bounded variation on $[a,b]$ for some $n\geqslant 1$. Then*

$$\left|\int_a^b f(t)dt+\bar{S}_n(x)\right|\leqslant\frac{(b-a)^n}{2n!}\left[1+\left|1-2\frac{(b-x)^n}{(b-a)^n}\right|\right]V_a^b(f^{(n-1)}),$$

for every $x\in[a,b]$.

Proof. Apply Corollary 2 and the corollary above to the Lebesgue measure on $[a,b]$. □

COROLLARY 14. *Let $f:[a,b]\to\mathbb{R}$ be such that $f^{(n-1)}$ is a continuous function of bounded variation on $[a,b]$ for some $n\geqslant 1$. Then*

$$\begin{aligned}&\left|f(x-a+z)+T_n^1(x,z)\right|\\&\leqslant\frac{(b-z)^{n-1}}{2(n-1)!}\left[1+\left|1-2\frac{(a+b-x-z)^{n-1}}{(b-z)^{n-1}}\right|\right]V_a^b(f^{(n-1)}),\end{aligned}$$

for every $x,z\in[a,b]$, $z\leqslant a+b-x$, where $T_n^1(x,z)$ is from Corollary 8.

Proof. Apply Corollaries 8 and 12 to $\mu=\delta_z$, $a\leqslant z\leqslant a+b-x$. □

COROLLARY 15. *Let $f:[a,b]\to\mathbb{R}$ be such that $f^{(n-1)}$ is a continuous function of bounded variation on $[a,b]$ for some $n\geqslant 1$. Then*

$$\left|f(x-b+z)+T_n^2(x,z)\right|\leqslant\frac{(b-z)^{n-1}}{(n-1)!}V_a^b(f^{(n-1)}),$$

for every $x,z\in[a,b]$, $a+b-x<z\leqslant b$, where $T_n^2(x,z)$ is from Corollary 9.

Proof. Apply Corollaries 9 and 12 to $\mu=\delta_z$, $a+b-x<z\leqslant b$. □

THEOREM 5. *Let $f:[a,b]\to\mathbb{R}$ be such that $f^{(n)}\in L_p[a,b]$ for some $n\geqslant 1$ and $1<p<\infty$. Then*

$$\begin{aligned}&\left|\int_{[a,b]}f_x(t)d\mu(t)-\mu(\{a\})f(x)+S_n(x)\right|\\&\leqslant\|P_n-P_n(a+b-x)\|_q\|f^{(n)}\|_p,\end{aligned}$$

for every $x\in[a,b]$, where $1/p+1/q=1$.

Proof. By applying the Hölder inequality we have

$$\begin{aligned}|R_n(x)| &\leqslant \int_a^b |K_n(x,t)-K_n(x,a)| \left|f^{(n)}(t)\right| dt \\ &\leqslant \left(\int_a^b |K_n(x,t)-K_n(x,a)|^q\, dt\right)^{1/q} \|f^{(n)}\|_p \\ &= \left(\int_a^b |P_n(t)-P_n(a+b-x)|^q\, dt\right)^{1/q} \|f^{(n)}\|_p,\end{aligned}$$

which proves our assertion. □

COROLLARY 16. *If $f' \in L_p[a,b]$, for $1<p<\infty$, then*

$$\left|\int_{[a,b]} f_x(t)d\mu(t) - f(x)\mu([a,b])\right| \leqslant (b-a)^{1/q} \|\mu\| \|f'\|_p$$

for every $x \in [a,b]$, where $1/p+1/q=1$.

Proof. Put $n=1$ in the theorem above and note that

$$\begin{aligned}&|\check{\mu}_1(t)-\check{\mu}_1(a+b-x)| \\ &= \left|\int_{[a,b]} \left[\chi_{[a,t]}(s)-\chi_{[a,a+b-x]}(s)\right] d\mu(s)\right| \\ &\leqslant \int_{[a,b]} \left|\chi_{[a,t]}(s)-\chi_{[a,a+b-x]}(s)\right| d|\mu|(s) \\ &\leqslant \int_{[a,b]} d|\mu|(s) = \|\mu\|,\end{aligned}$$

where χ_A is the indicator function of A. □

COROLLARY 17. *Let $f:[a,b]\to\mathbb{R}$ be such that $f^{(n)} \in L_p[a,b]$ for some $n \geqslant 2$ and $1<p<\infty$. Then*

$$\begin{aligned}&\left|\int_{[a,b]} f_x(t)d\mu(t) + \check{S}_n(x)\right| \\ &\leqslant \frac{(b-a)^{n-2}}{(n-2)!}\left[\frac{(b-x)^{q+1}+(x-a)^{q+1}}{q+1}\right]^{1/q} \|\mu\| \|f^{(n)}\|_p,\end{aligned}$$

for every $x \in [a,b]$, where $1/p+1/q=1$.

Proof. Apply the theorem above to the μ-harmonic sequence $(\check{\mu}_n,\ n \geqslant 1)$ and note that $\check{\mu}_n$ is L-Lipschitzian, for $n \geqslant 2$, with

$$\begin{aligned}L &= \max_{a\leqslant t\leqslant b} \left|\check{\mu}_n'(t)\right| = \max_{a\leqslant t\leqslant b} |\check{\mu}_{n-1}(t)| \\ &\leqslant \frac{(b-a)^{n-2}}{(n-2)!} \|\mu\|.\end{aligned}$$

Therefore,

$$\begin{aligned}
&\int_a^b |\check{\mu}_n(t) - \check{\mu}_n(a+b-x)|^q \, dt \\
&\leqslant \left[\frac{(b-a)^{n-2}}{(n-2)!} \|\mu\| \right]^q \int_a^b |t-(a+b-x)|^q \, dt \\
&= \left[\frac{(b-a)^{n-2}}{(n-2)!} \|\mu\| \right]^q \frac{(b-x)^{q+1} + (x-a)^{q+1}}{q+1},
\end{aligned}$$

which proves our assertion. □

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