POSITIVE–DEFINITE FUNCTIONS ON SPHERES AND SIDELNIKOV INEQUALITY

N. O. KOTELINA AND A. B. PEVNYI

Abstract. This article is devoted to the new proof of V. M. Sidelnikov inequality (1974). The proof is based on the theory of positive-definite functions on spheres introduced and studied by I. Schoenberg (1942).

1. Introduction

I. Schoenberg [4] introduced a notion of positive-definite functions on metric space M. We shall use spheres $S^{n-1}$ of different dimensionalities as M.

Denote the usual scalar product of vectors $x, y \in \mathbb{R}^n$, $n \geq 2$, by $\langle x, y \rangle$ and the norm of vector $x$ by $\|x\| = \sqrt{\langle x, x \rangle}$. Introduce the sphere

$$ S^{n-1} = \{ x \in \mathbb{R}^n : \|x\| = 1 \}. $$

It is convenient to give the definition of a positive-definite function on $S^{n-1}$ from [5] as follows: the real-valued function $g(t)$, continuous on $[-1, 1]$, is called positive-definite (p. d.) on $S^{n-1}$ if

$$ \sum_{i,j=1}^{N} g(\langle x_i, x_j \rangle) w_i w_j \geq 0 $$

for any $N$ points $x_1, \ldots, x_N$ from $S^{n-1}$, for any real numbers $w_1, \ldots, w_N$ and for any positive integer $N$.

Let $\text{PD}(S^{n-1})$ denote the class of such functions.

EXAMPLE 1. A function $g(t) = t$ belongs to $\text{PD}(S^{n-1})$ since

$$ \sum_{i,j=1}^{N} \langle x_i, x_j \rangle w_i w_j = \| \sum_{i=1}^{N} w_i x_i \|^2 \geq 0. $$

In Schoenberg’s paper [5] the criterion of p. d. functions in terms of Gegenbauer polynomials was given.


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2. Gegenbauer polynomials

Let \( n \geq 2 \), \( w_n(t) = (1 - t^2)^{(n-3)/2} \). The system of Gegenbauer polynomials is defined by equalities \( \deg G_k = k \), \( G_k(1) = 1 \) and orthogonality condition

\[
\int_{-1}^{1} G_k(t)G_s(t)w_n(t)\,dt = 0, \quad k \neq s.
\]

The following recurrence relation holds:

\[
(k+n-2)G_{k+1}(t) = (2k+n-2)tG_k(t) - kG_{k-1}(t), \quad G_0(t) = 1, \quad G_1(t) = t.
\]

Hence,

\[
tG_k(t) = \alpha_kG_{k+1}(t) + \beta_kG_{k-1}(t), \quad (2)
\]

where

\[
\alpha_k = \frac{k+n-2}{2k+n-2} > 0, \quad \beta_k = \frac{k}{2k+n-2} > 0, \quad \alpha_k + \beta_k = 1.
\]

3. Schoenberg’s result [5]

The following theorem is the important result [5].

THEOREM 1. A function \( g(t) \) is positive-definite on \( S^{n-1} \), if and only if, \( g(t) \) can be represented as a Gegenbauer series

\[
g(t) = \sum_{k=0}^{\infty} a_kG_k(t),
\]

where all coefficients \( a_k \geq 0 \) and series converges uniformly on \( [-1, 1] \).

4. The examples of p. d. functions

There are many examples of p. d. functions that can be received by using Schoenberg’s theorem. The simplest example is \( g(t) = G_k(t) \), i.e. Gegenbauer polynomials are p. d. on \( S^{n-1} \).

Another important example is \( g(t) = t^m \), where \( m \) is a positive integer. We shall give a new proof of this fact. It’s evident that \( t^m \) is represented as a finite sum

\[
t^m = \sum_{k=0}^{m} a_k^{(m)}G_k(t).
\]

Let’s show by induction on \( m \) that \( a_k^{(m)} \geq 0 \) for all \( k \in 0 : m \). For \( m = 1 \) it’s evident. Suppose that \( m \geq 2 \) and the following expansion

\[
t^{m-1} = \sum_{k=0}^{m-1} a_k^{(m-1)}G_k(t), \quad a_k^{(m-1)} \geq 0, \quad k \in 0 : m - 1,
\]
holds. Multiply this equality by $t$ and use formula (2). We get
\[ t^m = a_0^{(m-1)} t + \sum_{k=1}^{m-1} a_k^{(m-1)} \left[ \alpha_k G_{k+1}(t) + \beta_k G_{k-1}(t) \right], \]
where $\alpha_k, \beta_k > 0$. After collecting like terms we obtain (3), where $a_k^{(m)} \geq 0$, $k \in 0 : m$.

By Schoenberg’s theorem, the function $g(t) = t^m$ is p. d. on $S^{m-1}$. We come to inequality
\[ \sum_{i,j=1}^{N} \langle x_i, x_j \rangle^m w_i w_j \geq 0 \]
which holds for any points $x_1, \ldots, x_N \in S^{n-1}$ and any $w_i \in \mathbb{R}$.

For even $m$ it is possible to strengthen inequality (4). The strengthened inequality follows from (3):
\[ \sum_{i,j=1}^{N} \langle x_i, x_j \rangle^m w_i w_j = a_0^{(m)} \sum_{i,j=1}^{N} w_i w_j + \sum_{k=1}^{m} a_k^{(m)} \sum_{i,j=1}^{N} G_k(\langle x_i, x_j \rangle) w_i w_j. \]

Due to described earlier positive definiteness of Gegenbauer polynomials $G_k(t)$, the second sum (for $k$ from 1 to $m$) is nonnegative, which yields the inequality
\[ \sum_{i,j=1}^{N} \langle x_i, x_j \rangle^m w_i w_j \geq a_0^{(m)} (w_1 + \cdots + w_N)^2. \] 

For $w_1 = \cdots = w_N = 1$ we get the inequality
\[ \sum_{i,j=1}^{N} \langle x_i, x_j \rangle^m \geq a_0^{(m)} N^2. \] 

Here $m$ is even and $x_1, \ldots, x_N$ are arbitrary points on sphere $S^{n-1}$.

For even $m$ we shall find coefficients $a_0^{(m)}$ and thus show that $a_0^{(m)} > 0$. Multiply (3) by $w_n(t)$ and integrate over $[-1, 1]$:
\[ \int_{-1}^{1} t^m w_n(t) \, dt = a_0^{(m)} \int_{-1}^{1} w_n(t) \, dt. \]

Fairly standard integrals are calculated in [2].

Finally, we come to the equality
\[ a_0^{(m)} = \frac{(m-1)!!}{n(n+2) \cdots (n+m-2)}. \] 

We call inequality (6) for even $m$ with a constant (7) Sidelnikov inequality because it is a particular case of inequality obtained by V. M. Sidelnikov in [6]. Other proofs of inequality (6) are obtained in works [1]–[3], [7].

Equality in (6) is attained on a system $\{x_1, \ldots, x_N\}$, if and only if, the system $\{x_1, \ldots, x_N\}$ is a spherical semidesign of the order $m$ (see [2], [3]).
REFERENCES


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N. O. Kotelina
Syktyvkar State University 55
Oktyabr’skii pr., Syktyvkar 167001, Russia
e-mail: nkotelina@gmail.com

A. B. Pevnyi
Syktyvkar State University 55
Oktyabr’skii pr., Syktyvkar 167001, Russia
e-mail: pevnyi@syktsu.ru