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POSITIVE–DEFINITE FUNCTIONS ON SPHERES AND SIDELNIKOV INEQUALITY

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Abstract. This article is devoted to the new proof of V. M. Sidelnikov inequality (1974). The proof is based on the theory of positive-definite functions on spheres introduced and studied by I. Schoenberg (1942).

1. Introduction

I. Schoenberg [4] introduced a notion of positive-definite functions on metric space M. We shall use spheres S^{n-1} of different dimensionalities as M.

Denote the usual scalar product of vectors $x, y \in \mathbb{R}^n$, $n \geqslant 2$, by $\langle x,y \rangle$ and the norm of vector x by $\|x\| = \sqrt{\langle x,x \rangle}$. Introduce the sphere

$$S^{n-1} = \{x \in \mathbb{R}^n : \ \|x\| = 1\}.$$

It is convenient to give the definition of a positive-definite function on S^{n-1} from [5] as follows: the real-valued function $g(t)$, continuous on $[-1,1]$, is called *positive-definite* (p. d.) on S^{n-1} if

$$\sum_{i,j=1}^{N} g\big(\langle x_i, x_j \rangle\big) w_i w_j \geqslant 0 \tag{1}$$

for any N points $x_1, \ldots, x_N$ from S^{n-1}, for any real numbers $w_1, \ldots, w_N$ and for any positive integer N.

Let $\mathrm{PD}(S^{n-1})$ denote the class of such functions.

EXAMPLE 1. A function $g(t) = t$ belongs to $\mathrm{PD}(S^{n-1})$ since

$$\sum_{i,j=1}^{N} \langle x_i, x_j \rangle w_i w_j = \Big\| \sum_{i=1}^{N} w_i x_i \Big\|^2 \geqslant 0.$$

In Schoenberg's paper [5] the criterion of p. d. functions in terms of Gegenbauer polynomials was given.

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Paper JCA-11-05

2. Gegenbauer polynomials

Let $n \geqslant 2$, $w_n(t) = (1-t^2)^{(n-3)/2}$. The system of Gegenbauer polynomials is defined by equalities $\deg G_k = k$, $G_k(1) = 1$ and orthogonality condition

$$\int_{-1}^{1} G_k(t) G_s(t) w_n(t)\, dt = 0, \qquad k \neq s.$$

The following recurrence relation holds:

$$(k+n-2)G_{k+1}(t) = (2k+n-2)tG_k(t) - kG_{k-1}(t), \quad G_0(t) = 1, \quad G_1(t) = t.$$

Hence,

$$tG_k(t) = \alpha_k G_{k+1}(t) + \beta_k G_{k-1}(t), \tag{2}$$

where

$$\alpha_k = \frac{k+n-2}{2k+n-2} > 0, \quad \beta_k = \frac{k}{2k+n-2} > 0, \quad \alpha_k + \beta_k = 1.$$

3. Schoenberg's result [5]

The following theorem is the important result [5].

THEOREM 1. *A function $g(t)$ is positive-definite on S^{n-1}, if and only if, $g(t)$ can be represented as a Gegenbauer series*

$$g(t) = \sum_{k=0}^{\infty} a_k G_k(t),$$

where all coefficients $a_k \geqslant 0$ and series converges uniformly on $[-1,1]$.

4. The examples of p. d. functions

There are many examples of p. d. functions that can be received by using Schoenberg's theorem. The simplest example is $g(t) = G_k(t)$, i. e. Gegenbauer polynomials are p. d. on S^{n-1}.

Another important example is $g(t) = t^m$, where m is a positive integer. We shall give a new proof of this fact. It's evident that t^m is represented as a finite sum

$$t^m = \sum_{k=0}^{m} a_k^{(m)} G_k(t). \tag{3}$$

Let's show by induction on m that $a_k^{(m)} \geqslant 0$ for all $k \in 0:m$. For $m = 1$ it's evident. Suppose that $m \geqslant 2$ and the following expansion

$$t^{m-1} = \sum_{k=0}^{m-1} a_k^{(m-1)} G_k(t), \quad a_k^{(m-1)} \geqslant 0, \quad k \in 0:m-1,$$

holds. Multiply this equality by t and use formula (2). We get

$$t^m = a_0^{(m-1)} t + \sum_{k=1}^{m-1} a_k^{(m-1)} \left[\alpha_k G_{k+1}(t) + \beta_k G_{k-1}(t)\right],$$

where $\alpha_k, \beta_k > 0$. After collecting like terms we obtain (3), where $a_k^{(m)} \geqslant 0$, $k \in 0:m$.

By Schoenberg's theorem, the function $g(t) = t^m$ is p. d. on S^{m-1}. We come to inequality

$$\sum_{i,j=1}^{N} \langle x_i, x_j \rangle^m w_i w_j \geqslant 0 \tag{4}$$

which holds for any points $x_1, \ldots, x_N \in S^{n-1}$ and any $w_i \in \mathbb{R}$.

For even m it is possible to strengthen inequality (4). The strengthened inequality follows from (3):

$$\sum_{i,j=1}^{N} \langle x_i, x_j \rangle^m w_i w_j = a_0^{(m)} \sum_{i,j=1}^{N} w_i w_j + \sum_{k=1}^{m} a_k^{(m)} \sum_{i,j=1}^{N} G_k\big(\langle x_i, x_j \rangle\big) w_i w_j.$$

Due to described earlier positive definiteness of Gegenbauer polynomials $G_k(t)$, the second sum (for k from 1 to m) is nonnegative, which yields the inequality

$$\sum_{i,j=1}^{N} \langle x_i, x_j \rangle^m w_i w_j \geqslant a_0^{(m)} \left(w_1 + \cdots + w_N\right)^2. \tag{5}$$

For $w_1 = \cdots = w_N = 1$ we get the inequality

$$\sum_{i,j=1}^{N} \langle x_i, x_j \rangle^m \geqslant a_0^{(m)} N^2. \tag{6}$$

Here m is even and $x_1, \cdots, x_N$ are arbitrary points on sphere S^{n-1}.

For even m we shall find coefficients $a_0^{(m)}$ and thus show that $a_0^{(m)} > 0$. Multiply (3) by $w_n(t)$ and integrate over $[-1,1]$:

$$\int_{-1}^{1} t^m w_n(t)\, dt = a_0^{(m)} \int_{-1}^{1} w_n(t)\, dt.$$

Fairly standard integrals are calculated in [2].

Finally, we come to the equality

$$a_0^{(m)} = \frac{(m-1)!!}{n(n+2)\cdots(n+m-2)}. \tag{7}$$

We call inequality (6) for even m with a constant (7) Sidelnikov inequalty because it is a particular case of inequality obtained by V. M. Sidelnikov in [6]. Other proofs of inequality (6) are obtained in works [1]–[3], [7].

Equality in (6) is attained on a system $\{x_1, \ldots, x_N\}$, if and only if, the system $\{x_1, \ldots, x_N\}$ is a spherical semidesign of the order m (see [2], [3]).

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