Journal of Classical Analysis
Volume 11, Number 2 (2017), 129–158
doi:10.7153/jca-2017-11-09

A NEW APPROACH TO STEINER SYMMETRIZATION OF COERCIVE CONVEX FUNCTIONS

LIN YOUJIANG

Abstract. A new way of defining Steiner symmetrization of coercive convex functions is proposed which does not use the Steiner symmetrization of level sets. Some fundamental properties of the new Steiner symmetrization are proved.

1. Introduction

The purpose of this paper is to introduce a new way of defining Steiner symmetrization for coercive convex functions, and to explore its applications. Our new definition is motivated by and can be regarded as an improvement of a functional Steiner symmetrization of [1]. In particular, our new definition has a key property: the invariance of integral, which is not true for the definition of [1]. Moreover, our definition provides a new approach to the familiar functional Steiner symmetrization (see [5]), but we do not use geometric Steiner symmetrization and our approach is more suitable for certain functional problems.

Steiner symmetrization was invented by Steiner [16] to prove the isoperimetric inequality. For over 160 years Steiner symmetrization has been a fundamental tool for attacking problems regarding isoperimetry and related geometric inequalities [7, 11, 12, 13, 14, 16, 17]. Steiner symmetrization appears in the titles of dozens of papers (see e.g. [3, 4, 5, 10]) and plays a key role in recent work such as [2, 13].

Steiner symmetrization is a type of rearrangement. In the 1970s, interest in rearrangements was renewed, as mathematicians began to look for geometric proofs of functional inequalities. Rearrangements were generalized from smooth or convex bodies to measurable sets and to functions in Sobolev spaces. Functional Steiner symmetrization, as a kind of important rearrangement of functions, has been studied in [1, 5, 6, 9]. In the important paper [5], Burchard proved that Steiner symmetrization is continuous in $W^{1,p}(\mathbb{R}^{n+1})$, $1 \leqslant p < \infty$, for every dimension $n \geqslant 1$, in the sense that $f_k \to f$ in $W^{1,p}$ implies $Sf_k \to Sf$ in $W^{1,p}$. In the remarkable paper [8], Cianchi, Fusco analyzed the cases of equality in Steiner symmetrization inequalities for Dirichlet-type integrals. In particular, minimal assumptions are determined under which functions attaining equality are necessarily Steiner symmetric. In [9], Fortier gave a thorough

Mathematics subject classification (2010): 26A51, 26B25.

Keywords and phrases: Rearrangements of functions, Steiner symmetrizations, coercive convex functions, Santaló inequality, Prékopa-Leindler inequality.

Research of the author is supported by the funds of Basic and Advanced Research Project of CQCSTC cstc2015jcyjA00009 and Scientific and Technological Research Program of Chongqing Municipal Education Commission KJ1500628, Scientific research funds of Chongqing Technology and Business University 2015-56-02.

review and exposition of results regarding approximating the symmetric decreasing rearrangement by polarizations and Steiner symmetrizations.

For a nonnegative measurable function f, the familiar definition of its Steiner symmetrization (see [5, 6, 9]) is defined as following:

DEFINITION 1.1. For a nonnegative measurable function f on $\mathbb{R}^n$ which vanishes at infinity, its Steiner symmetrization is defined as

$$\bar{S}_u f(x) = \int_0^\infty \mathscr{X}_{\bar{S}_u E(t)}(x)dt, \tag{1.1}$$

where $\bar{S}_u E(t)$ is the Steiner symmetrization of the level set $E(t) := \{x \in \mathbb{R}^n : f(x) > t\}$ about the hyperplane $u^\perp$ and $\mathscr{X}_{\bar{S}_u E(t)}$ denotes the characteristic function of $\bar{S}_u E(t)$.

During the study of the analogy between convex bodies and log-concave functions, Artstein-Klartag-Milman in [1] defined another functional Steiner transformation as follows:

DEFINITION 1.2. For a coercive convex function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ and a hyperplane $H = u^\perp$ $(u \in S^{n-1})$ in $\mathbb{R}^n$, for any $x = x' + tu$, where $x' \in H$ and $t \in \mathbb{R}$, we define the *Steiner symmetrization* $\widetilde{S}_u f$ of f about H by

$$(\widetilde{S}_u f)(x) = \inf_{t_1+t_2=t} \left[\frac{1}{2} f(x' + 2t_1 u) + \frac{1}{2} f(x' - 2t_2 u)\right]. \tag{1.2}$$

In the paper [1] by Artstein, Klartag and Milman the definition of Steiner symmetrization almost plays a very minor role. It appears only in a remark to the main text and is never used in any proof.

In this paper, we introduce a new way of defining the functional Steiner symmetrization for coercive convex functions.

DEFINITION 1.3. For a coercive convex function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ and a hyperplane $H = u^\perp$ $(u \in S^{n-1})$ in $\mathbb{R}^n$, for any $x = x' + tu \in \mathbb{R}^n$, where $x' \in H$ and $t \in \mathbb{R}$, we define the *Steiner symmetrization* $S_u f$ (or $S_H f$) of f about H by

$$(S_u f)(x) = \sup_{\lambda \in [0,1]} \inf_{t_1+t_2=t} [\lambda f(x' + 2t_1 u) + (1-\lambda) f(x' - 2t_2 u)]. \tag{1.3}$$

Our definition $S_u f$ is motivated by and can be regarded as an improvement of $\widetilde{S}_u f$ in Definition 1.2. When compared with $\bar{S}_u f$ in Definition 1.1, our definition symmetrizes a parabola-like (one-dimension) curve once at a time instead of symmetrizing the level set as in $\bar{S}_u f$. We will elaborate on the relation between Definition 1.3 and Definitions 1.1, 1.2.

The rest of the paper is organized as follows. In Section 2, we give some definitions and preliminaries. In Section 3, we explore the analogy between convex bodies and coercive convex functions using our new definition. We shall prove the following seven properties as listed in Table 1.

In Section 4, we will elaborate on the relation between Definition 1.3 and Definitions 1.1, 1.2.

Table 1. A contrast between convex bodies and coercive convex functions on Steiner symmetrization

	Convex bodies	Coercive Convex Functions
1	For a convex body K, S_uK is still a convex body and symmetric about $u^\perp$.	For a coercive convex function f, S_uf is still a coercive convex function and symmetric about $u^\perp$.
2	$Vol_n(S_uK) = Vol_n(K)$.	$\int_{\mathbb{R}^n} \exp(-S_uf) = \int_{\mathbb{R}^n} \exp(-f)$.
3	K can be transformed into an unconditional body using n Steiner symmetrizations.	f can be transformed into an unconditional function using n Steiner symmetrizations.
4	For any convex bodies $K_1 \subset K_2$, then $S_uK_1 \subset S_uK_2$.	For any coercive convex functions $f_1 \leqslant f_2$, then $S_uf_1 \leqslant S_uf_2$.
5	If K is a symmetric about z, then S_uK is symmetric about $z\|u^\perp$.	If f is even about z, then S_uf is even about $z\|u^\perp$.
6	If the sequence $\{K_i\}$ converges in the Hausdorff metric to K, then the sequence $\{S_uK_i\}$ will converge to S_uK.	If the sequence $\{\exp(-f_i)\}$ converges in the L^p distance to $\exp(-f)$, then the sequence $\{\exp(-S_uf_i)\}$ will converge to $\exp(-S_uf)$.
7	There is a sequence of directions $\{u_i\}$ so that the sequence of convex bodies $K_i = S_{u_i} \dots S_{u_1} K$ converges to the ball with the same volume as K.	There is a sequence of directions $\{u_i\}$ so that the sequence of log-concave functions $\exp(-f_i)$, where $f_i = S_{u_i} \dots S_{u_1} f$, converges to a radial function with the same integral as $\exp(-f)$.

2. Definitions and preliminaries

In this section, we give some basic known definitions.

DEFINITION 2.1. Let K be a non-empty convex set in $\mathbb{R}^n$ and let H be a hyperplane in $\mathbb{R}^n$ with unit normal vector u. The *Steiner symmetrization* S_HK of K about H is defined as:

$$S_HK = \left\{x' + \frac{1}{2}(t_1 - t_2)u : \ x' \in P_H(K),\ t_i \in I_K(x')\ for\ i = 1,2\right\}, \tag{2.1}$$

where

$$P_H(K) = \{x' \in H : \ x' + tu \in K \text{ for some } t \in \mathbb{R}\} \tag{2.2}$$

is the projection of K onto the hyperplane H and

$$I_K(x') = \{t \in \mathbb{R} : \ x' + tu \in K\}. \tag{2.3}$$

A function $f : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty, +\infty\}$ is called *proper* if $\{x \in \mathbb{R}^n : f(x) = -\infty\} = \emptyset$ and $\{x \in \mathbb{R}^n : f(x) = +\infty\} \neq \mathbb{R}^n$.

DEFINITION 2.2. A function $f:\mathbb{R}^n\to\mathbb{R}\cup\{+\infty\}$ is called *convex* if f is proper and

$$f(\alpha x+(1-\alpha)y)\leqslant \alpha f(x)+(1-\alpha)f(y), \tag{2.4}$$

for all $x,\ y\in\mathbb{R}^n$ and for $0\leqslant\lambda\leqslant 1$.

DEFINITION 2.3. A convex function f is said to be *coercive* if

$$\lim_{|x|\to+\infty} f(x)=+\infty. \tag{2.5}$$

A convex function $f:\mathbb{R}^n\to\mathbb{R}\cup\{+\infty\}$ is called coercive if for any $M>0$, there exists $r>0$, s.t. $f(x)>M$, as $|x|>r$, where $|\cdot|$ denotes the Euclidean norm.

3. The functional Steiner symmetrization

First, we give the definition of Steiner symmetrization for coercive convex functions.

DEFINITION 3.1. Let $n\geqslant 1$ be an integer and let $f:\mathbb{R}^n\to\mathbb{R}\cup\{+\infty\}$ be a coercive convex function. For $u\in S^{n-1}$, let $H=u^{\perp}\subset\mathbb{R}^n$ be a hyperplane through origin and orthogonal to u. For any $x=x'+tu\in\mathbb{R}^n$, where $x'\in H$ and $t\in\mathbb{R}$, we define the *Steiner symmetrization* $S_u f$ (or $S_H f$) of f about H by

$$(S_u f)(x)=\sup_{\lambda\in[0,1]}\inf_{t_1+t_2=t}[\lambda f(x'+2t_1u)+(1-\lambda)f(x'-2t_2u)]. \tag{3.1}$$

REMARK 3.1. 1) In the above definition, when $n=1$, $S^0=\{-1,1\}$ and $H=\{0\}$, it is clear that $(S_1 f)(x)=(S_{-1}f)(x)$ for any $x\in\mathbb{R}$. Let Sf denote Steiner symmetrization of one-dimensional function, then

$$Sf(x)=\sup_{\lambda\in[0,1]}\inf_{x_1+x_2=x}[\lambda f(2x_1)+(1-\lambda)f(-2x_2)]. \tag{3.2}$$

2) Similarly, we define the Steiner symmetrization $S_{H'}f$ of f about an affine hyperplane $H'=u^{\perp}+t_0u$ by (3.1), where $x'\in H'$. Let $H=u^{\perp}$, we can easily get that

$$(S_{H'}f)(x)=(S_H f)(x-t_0u). \tag{3.3}$$

We first study the one-dimensional case.

THEOREM 3.1. *If $f:\mathbb{R}\to\mathbb{R}\cup\{+\infty\}$ is a coercive convex function, then $Sf(x)$ is a coercive even convex function and for any $s\in\mathbb{R}$,*

$$Vol_1([f\leqslant s])=Vol_1([Sf\leqslant s]), \tag{3.4}$$

where $[f\leqslant s]=\{x\in\mathbb{R}:f(x)\leqslant s\}$ denotes the sublevel set of f.

In order to prove Theorem 3.1, we need the following two lemmas. Because they are obvious, we omit their proofs.

LEMMA 3.1. *Let* $f:\mathbb{R}\to\mathbb{R}$ *be a coercive convex function, then we have*

(i) If $a=\inf f(t)$*, then* $a\in(-\infty,+\infty)$ *and* $f^{-1}(a)=\{x\in\mathbb{R}: f(x)=a\}$ *is a nonempty finite closed interval* $[\mu,\nu]$*, where* μ *may equal to* ν*.*

(ii) $f(t)$ *is strictly decreasing for the interval* $(-\infty,\mu]$ *and strictly increasing for the interval* $[\nu,+\infty)$*.*

(iii) If $f(c)=f(d)$ *and* $c<d$*, then* $\mu<d$ *and* $c<\nu$*.*

(iv) For c *and* d *given in (iii), we have the right derivative* $f_r'(d)\geqslant 0$ *for* f *is increasing on* $[\mu,+\infty)$*, we also have* $f_r'(c)\leqslant 0$ *for* f *is decreasing on* $(-\infty,\nu]$*.*

LEMMA 3.2. *Let* $f:\mathbb{R}\to\mathbb{R}$ *be a coercive convex function, for two intervals* $[a,a+t_0]$ *and* $[b,b+t_0]$ *with the same length* $t_0>0$*, if* $f(a)=f(a+t_0)$*, then we can get that either* $f(b)\geqslant f(a)$ *or* $f(b+t_0)\geqslant f(a+t_0)$*.*

Next, we prove Theorem 3.1.

Proof. We know that the effective domain of convex function $f:\mathbb{R}^n\to\mathbb{R}\cup\{+\infty\}$ is the nonempty set

$$\mathrm{dom} f := \{x\in\mathbb{R}^n : f(x)<+\infty\}. \tag{3.5}$$

To prove Theorem 3.1, we distinguish two cases: $\mathrm{dom} f=\mathbb{R}$ and $\mathrm{dom} f\neq\mathbb{R}$.

Case (1) $\mathrm{dom} f=\mathbb{R}$. There are three steps.

First Step. We prove that Sf is even. For any $x\in\mathbb{R}$, by (3.2), we have

$$\begin{aligned} Sf(-x) &= \sup_{\lambda\in[0,1]}\inf_{x_1+x_2=-x}[\lambda f(2x_1)+(1-\lambda)f(-2x_2)] \\ &= \sup_{\lambda\in[0,1]}\inf_{x_2\in\mathbb{R}}[\lambda f(-2x_2-2x)+(1-\lambda)f(-2x_2)] \\ &= \sup_{\lambda\in[0,1]}\inf_{x_2\in\mathbb{R}}[\lambda f(2x_2-2x)+(1-\lambda)f(2x_2)] \\ &= \sup_{\lambda'\in[0,1]}\inf_{x_2\in\mathbb{R}}[\lambda' f(2x_2)+(1-\lambda')f(2x_2-2x)] \\ &= Sf(x), \end{aligned} \tag{3.6}$$

where the fourth equality is by replacing λ by $\lambda'=1-\lambda$.

Second Step. We prove that $Sf(0)=\inf f$ and for any $x>0$, there exists some $x'\in\mathbb{R}$ such that

$$Sf(x)=f(x')=f(x'-2x). \tag{3.7}$$

For $x=0$, by (3.2), we have

$$\begin{aligned} Sf(0) &= \sup_{\lambda\in[0,1]}\inf_{x_1+x_2=0}[\lambda f(2x_1)+(1-\lambda)f(-2x_2)] \\ &= \inf_{x_1\in\mathbb{R}} f(2x_1)=\inf_{x\in\mathbb{R}} f(x). \end{aligned} \tag{3.8}$$

For $x>0$, since f is coercive and convex, there exists some $x'\in\mathbb{R}$ satisfying

$$f(x')=f(x'-2x). \tag{3.9}$$

Indeed, let $f_x(x_1):=f(x_1)-f(x_1-2x)$, $a=\inf f$ and $f^{-1}(a)=[\mu,\nu]$, by Lemma 3.1(ii), $f_x(x_1)<0$ when $x_1<\mu$ and $f_x(x_1)>0$ when $x_1>\nu$. Since $f(x_1)$ and $f(x_1-2x)$ are convex functions about $x_1\in\mathbb{R}$ and any convex function is continuous on the interior of its effective domain, thus $f_x(x_1)$ is continuous in $\mathbb{R}$. Therefore, there exists some x' such that $f_x(x')=0$.

Now we prove $Sf(x)=f(x')$, where $x>0$ and x' satisfies equality (3.9). Let $G_x(\lambda)$ be a function about $\lambda\in[0,1]$ defined as

$$G_x(\lambda):=\inf_{x_1\in\mathbb{R}}[\lambda f(2x_1)+(1-\lambda)f(2x_1-2x)], \tag{3.10}$$

then

$$Sf(x)=\sup_{\lambda\in[0,1]}G_x(\lambda). \tag{3.11}$$

For any $\lambda\in[0,1]$, choose $x_1=\frac{x'}{2}$, we have

$$\begin{aligned} G_x(\lambda) &= \inf_{x_1\in\mathbb{R}}[\lambda f(2x_1)+(1-\lambda)f(2x_1-2x)] \\ &\leqslant \lambda f(x')+(1-\lambda)f(x'-2x)=f(x'). \end{aligned} \tag{3.12}$$

Thus,

$$Sf(x)=\sup_{\lambda\in[0,1]}G_x(\lambda)\leqslant f(x'). \tag{3.13}$$

On the other hand, we prove that there exists some $\lambda_0\in[0,1]$ such that $G_x(\lambda_0)=f(x')$. Since f is a convex function defined in $\mathbb{R}$ and by Theorem 1.5.2 in [15], both the right derivative f'_r and the left derivative f'_l exist and $f'_l\leqslant f'_r$.

CLAIM 1. There exists some $\lambda_0\in[0,1]$ satisfying

$$\lambda_0 f'_r(x')+(1-\lambda_0)f'_r(x'-2x)=0. \tag{3.14}$$

Proof. Since $f(x')=f(x'-2x)$ and $x>0$, and by Lemma 3.1(iv), we have $f'_r(x')\geqslant 0$ and $f'_r(x'-2x)\leqslant 0$, thus $f'_r(x')-f'_r(x'-2x)\geqslant 0$.

(i) If $f'_r(x')-f'_r(x'-2x)>0$, let

$$\lambda_0=\frac{-f'_r(x'-2x)}{f'_r(x')-f'_r(x'-2x)},$$

then $\lambda_0\in[0,1]$ and

$$\lambda_0 f'_r(x')+(1-\lambda_0)f'_r(x'-2x)=0.$$

(ii) If $f_r'(x') - f_r'(x'-2x) = 0$, then $f_r'(x') = f_r'(x'-2x) = 0$, thus, for any $\lambda_0 \in [0,1]$, we can get (3.14). □

Fix a λ_0 as defined by (3.14), we define

$$\Phi_{\lambda_0}(x_1) = \lambda_0 f(2x_1) + (1-\lambda_0) f(2x_1 - 2x). \tag{3.15}$$

Since f is a convex function, then Φ_{λ_0} is a convex function about x_1. By (3.14), we have that the right derivative and the left derivative of Φ_{λ_0} at $x_1 = \frac{x'}{2}$ satisfy

$$\Phi'_{\lambda_0 r}(x_1)\big|_{x_1=\frac{x'}{2}} = 2\lambda_0 f_r'(x') + 2(1-\lambda_0) f_r'(x'-2x) = 0, \tag{3.16}$$

and

$$\Phi'_{\lambda_0 l}(x_1)\big|_{x_1=\frac{x'}{2}} \leqslant \Phi'_{\lambda_0 r}(x_1)\big|_{x_1=\frac{x'}{2}} = 0. \tag{3.17}$$

By (3.9), (3.15) and the fact that if a convex function $f:\mathbb{R}\to\mathbb{R}$ satisfies $f_r'(x_0) \geqslant 0$ and $f_l'(x_0) \leqslant 0$ then $f(x_0) = \min\{f(x) : x \in \mathbb{R}\}$, we have

$$\inf_{x_1\in\mathbb{R}} \Phi_{\lambda_0}(x_1) = \Phi_{\lambda_0}(\frac{x'}{2}) = f(x'). \tag{3.18}$$

By (3.10), (3.15) and (3.18), we have

$$Sf(x) = \sup_{\lambda\in[0,1]} G_x(\lambda) \geqslant G_x(\lambda_0) = \inf_{x_1\in\mathbb{R}} \Phi_{\lambda_0}(x_1) = f(x'). \tag{3.19}$$

By (3.7), (3.13) and (3.19), we have $Sf(x) = f(x') = f(x'-2x)$.

Third Step. We prove that Sf is coercive and convex, and for any $s \in \mathbb{R}$,

$$Vol_1([Sf \leqslant s]) = Vol_1([f \leqslant s]).$$

First, we prove that Sf is coercive. Suppose that there exists $M_0 > 0$ and a sequence $\{x_n\}$ satisfying $|x_n| > n$ and $Sf(x_n) < M_0$ for any positive integer n, then by (3.7), there exists x_n' such that

$$Sf(x_n) = f(x_n') = f(x_n' - 2x_n) < M_0. \tag{3.20}$$

Since

$$2\max\{|x_n'|, |x_n' - 2x_n|\} \geqslant |x_n'| + |x_n' - 2x_n| \geqslant 2|x_n| > 2n, \tag{3.21}$$

thus there is a sequence $\{y_n\}$, where $y_n = x_n'$ if $|x_n'| \geqslant |x_n' - 2x_n|$ and $y_n = x_n' - 2x_n$ if $|x_n'| \leqslant |x_n' - 2x_n|$, satisfying $\lim_{n\to+\infty}|y_n| = +\infty$ and $f(y_n) < M_0$, which is contradictory with f is coercive.

Next, we prove that Sf is a convex function in $\mathbb{R}$. First, we prove that $Sf(x)$ is increasing on the interval $[0,+\infty)$. In fact, by (3.7), for any $0 < x_1 < x_2$, there exist x_1' and x_2' such that $Sf(x_i) = f(x_i') = f(x_i' - 2x_i)$ $(i = 1,2)$. By Lemma 3.1(iii), for μ and

ν given in Lemma 3.1, we have $x_i' > \mu \ \ (i=1,2)$ and $x_i' - 2x_i < \nu \ \ (i=1,2)$. If $f(x_1') > f(x_2')$, then $x_1' > x_2'$ for f is increasing on the interval $[\mu, +\infty)$, thus $x_1' - 2x_1 > x_2' - 2x_2$ for $0 < x_1 < x_2$, which implies that $f(x_1' - 2x_1) \leqslant f(x_2' - 2x_2)$ for f is decreasing on the interval $(-\infty, \nu]$. The contradiction implies that $f(x_1') \leqslant f(x_2')$, thus Sf is increasing on the interval $[0, +\infty)$. And because Sf is even, thus to prove Sf is convex in $\mathbb{R}$, it suffices to prove that Sf is convex in $[0, +\infty)$, i.e., for any $0 \leqslant x_1 < x_2$ and $0 < \alpha < 1$

$$Sf(\alpha x_1 + (1-\alpha)x_2) \leqslant \alpha Sf(x_1) + (1-\alpha)Sf(x_2). \tag{3.22}$$

By (3.7), let x_1', x_2' and $x_0 \triangleq (\alpha x_1 + (1-\alpha)x_2)'$ be three real numbers such that

$$Sf(x_1) = f(x_1') = f(x_1' - 2x_1), \ \ Sf(x_2) = f(x_2') = f(x_2' - 2x_2) \tag{3.23}$$

and

$$Sf(\alpha x_1 + (1-\alpha)x_2) = f(x_0) = f(x_0 - 2(\alpha x_1 + (1-\alpha)x_2)). \tag{3.24}$$

Since f is a convex function, we have

$$\alpha f(x_1') + (1-\alpha)f(x_2') \geqslant f(\alpha x_1' + (1-\alpha)x_2') \tag{3.25}$$

and

$$\begin{aligned} &\alpha f(x_1' - 2x_1) + (1-\alpha)f(x_2' - 2x_2) \\ &\geqslant f(\alpha x_1' + (1-\alpha)x_2' - 2(\alpha x_1 + (1-\alpha)x_2)). \end{aligned} \tag{3.26}$$

Since $f(x_0) = f(x_0 - 2(\alpha x_1 + (1-\alpha)x_2))$ and both intervals $[x_0 - 2(\alpha x_1 + (1-\alpha)x_2), x_0]$ and $[\alpha x_1' + (1-\alpha)x_2' - 2(\alpha x_1 + (1-\alpha)x_2), \alpha x_1' + (1-\alpha)x_2']$ have the same length $2(\alpha x_1 + (1-\alpha)x_2) > 0$, by Lemma 3.2, thus either

$$f(\alpha x_1' + (1-\alpha)x_2') \geqslant f(x_0) \tag{3.27}$$

or

$$f(\alpha x_1' + (1-\alpha)x_2' - 2(\alpha x_1 + (1-\alpha)x_2)) \geqslant f(x_0 - 2(\alpha x_1 + (1-\alpha)x_2)). \tag{3.28}$$

If (3.27) holds, then we use (3.25) and if (3.28) holds, then we use (3.26), thus

$$\alpha f(x_1') + (1-\alpha)f(x_2') \geqslant f(x_0). \tag{3.29}$$

By (3.23), (3.24) and (3.26), Sf is a convex function.

Finally, we prove that $Vol_1([f \leqslant s]) = Vol_1([Sf \leqslant s])$ for any $s \in \mathbb{R}$. By $Sf(x)$ is an even convex function, thus $Sf(0) = \inf Sf$. Since $Sf(0) = \inf f$ by (3.8), thus $\inf Sf = \inf f$. Let $a = \inf Sf = \inf f$ and $(Sf)^{-1}(a) = [-\delta, \delta]$ and $f^{-1}(a) = [\mu, \nu]$.

If $s = a$, then $Vol_1([f \leqslant s]) = \nu - \mu$ and $Vol_1([Sf \leqslant s]) = 2\delta$. Next, we prove $\nu - \mu = 2\delta$. By Lemma 3.1, Sf is strictly decreasing on $(-\infty, -\delta)$ and strictly increasing on $(\delta, +\infty)$, similarly f is strictly decreasing on $(-\infty, \mu)$ and strictly increasing on

$(\nu,+\infty)$. For $\delta \geqslant 0$, if $\nu-\mu > 2\delta$, then let $x_0 = \delta + \frac{\nu-\mu-2\delta}{2} > \delta$, thus $Sf(x_0) > Sf(\delta)$, which is contradictory with

$$\begin{aligned} Sf(x_0) &= \sup_{\lambda\in[0,1]} \inf_{x_1\in\mathbb{R}} [\lambda f(2x_1) + (1-\lambda) f(2x_1 - 2x_0)] \\ &\leqslant \sup_{\lambda\in[0,1]} [\lambda f(\nu) + (1-\lambda) f(\nu - 2x_0)] = a, \end{aligned} \tag{3.30}$$

where inequality is by choosing $x_1 = \frac{\nu}{2}$ and last equality is by $\nu - 2x_0 = \mu$. Thus, $\nu-\mu \leqslant 2\delta$. Thus if $\delta = 0$, then $\mu = \nu$. For $\delta > 0$, by (3.7), there exists δ' such that

$$Sf(\delta) = f(\delta') = f(\delta' - 2\delta) = a,$$

which implies that $\nu - \mu \geqslant 2\delta$. Thus, $\nu - \mu = 2\delta$.

If $s > a$, by Lemma 3.1 and (3.7) and Sf is even, there is a unique $x > 0$ and a unique $x' \in \mathbb{R}$ such that

$$Sf(-x) = Sf(x) = s = f(x') = f(x' - 2x),$$

thus we have

$$Vol_1([f \leqslant s]) = Vol_1([Sf \leqslant s]) = 2x.$$

If $s < a$, then $[Sf \leqslant s] = [f \leqslant s] = \emptyset$, thus $Vol_1([f \leqslant s]) = Vol_1([Sf \leqslant s]) = 0$.

Case (2) dom$f \neq \mathbb{R}$. There exist eight cases for dom$f \neq \mathbb{R}$: 1). $[\alpha,\beta]$; 2). (α,β); 3). $(\alpha,\beta]$; 4). $[\alpha,\beta)$; 5). $(-\infty,\beta]$; 6). $(-\infty,\beta)$; 7). $[\alpha,+\infty)$; 8). $(\alpha,+\infty)$. Here we just prove Theorem 3.1 for dom$f = (\alpha,\beta)$. For other cases, we can prove Theorem 3.1 by the same method. For dom$f = (\alpha,\beta)$, there exist three cases: (i). f is decreasing on (α,β); (ii). f is increasing on (α,β); (iii). f is decreasing on $(\alpha,\gamma]$ and increasing on $[\gamma,\beta)$ for some $\gamma \in (\alpha,\beta)$. Note that Sf is even for all cases by the same proof in Case (1).

(i) f is decreasing on (α,β).

First, we prove that there is some $s_0 \in \mathbb{R}$ such that

$$\lim_{x\to\beta, x<\beta} f(x) = s_0. \tag{3.31}$$

Choose $t > 0$ small enough such that $\beta - t > \frac{\alpha+\beta}{2}$. Let $x_0 = \frac{\alpha+\beta}{2}$, since f is convex and decreasing in (α,β), we have

$$\frac{1}{2}f(x_0) + \frac{1}{2}f(\beta - t) \geqslant f\left(\frac{1}{2}x_0 + \frac{1}{2}(\beta - t)\right) \geqslant f\left(\frac{1}{2}x_0 + \frac{1}{2}\beta\right). \tag{3.32}$$

Thus, we have

$$f(\beta - t) \geqslant 2\left[f(\frac{1}{2}x_0 + \frac{1}{2}\beta) - \frac{1}{2}f(x_0)\right], \tag{3.33}$$

which implies that f is bounded below. For a monotone decreasing function, if it is bounded below, then it has infimum. Thus we have equality (3.31).

On the other hand, we have either

$$\lim_{x\to\alpha,x>\alpha} f(x) < +\infty \quad \text{or} \quad \lim_{x\to\alpha,x>\alpha} f(x) = +\infty.$$

For the two cases, we can prove our theorem by the same method.

Step 1. Prove that Sf is coercive.

When $|x| \geqslant \frac{\beta-\alpha}{2}$, by $\mathrm{dom} f = (\alpha,\beta)$, we have

$$Sf(x) = \sup_{\lambda\in[0,1]} \inf_{x_1\in\mathbb{R}} [\lambda f(2x_1) + (1-\lambda)f(2x_1 - 2x)] = +\infty. \tag{3.34}$$

Thus, Sf is coercive.

Step 2. Prove that Sf is convex. We first prove the following claim.

CLAIM 2. For $0 < |x| < \frac{\beta-\alpha}{2}$, we have

$$Sf(x) = f(\beta - 2|x|). \tag{3.35}$$

For $|x| = 0$, we have

$$Sf(0) = \lim_{x\to\beta,x<\beta} f(x). \tag{3.36}$$

Proof. For $0 < |x| < \frac{\beta-\alpha}{2}$, since Sf is even, we may assume that $x > 0$. Since f is decreasing in (α,β), for $\lambda \in [0,1]$,

$$\inf_{x_1\in\mathbb{R}} [\lambda f(2x_1) + (1-\lambda)f(2x_1 - 2x)] = \lambda \lim_{x\to\beta,\, x<\beta} f(x) + (1-\lambda)f(\beta - 2x).$$

Thus, by $f(\beta - 2x) \geqslant \lim_{x\to\beta,x<\beta} f(x)$, we have

$$Sf(x) = \sup_{\lambda\in[0,1]} \left[\lambda \lim_{x\to\beta,\, x<\beta} f(x) + (1-\lambda)f(\beta - 2x)\right] = f(\beta - 2x).$$

For $|x| = 0$, by (3.2), we have

$$Sf(0) = \inf_{x_1\in\mathbb{R}} f(2x_1) = \lim_{x\to\beta,x<\beta} f(x). \tag{3.37}$$

□

By (3.34), (3.35) and (3.36), we know that $\mathrm{dom} Sf = (-\frac{\beta-\alpha}{2}, \frac{\beta-\alpha}{2})$.

Next, we prove that Sf is convex on $\mathbb{R}$. For any $x_1, x_2 \in \mathbb{R}$ and $\lambda \in (0,1)$,

(i) if $|x_1| \geqslant \frac{\beta-\alpha}{2}$ or $|x_2| \geqslant \frac{\beta-\alpha}{2}$, by (3.34), then $Sf(x_1) = +\infty$ or $Sf(x_2) = +\infty$, thus

$$Sf(\lambda x_1 + (1-\lambda)x_2) \leqslant \lambda Sf(x_1) + (1-\lambda)Sf(x_2); \tag{3.38}$$

(ii) if $0<|x_1|<\frac{\beta-\alpha}{2}$ and $0<|x_2|<\frac{\beta-\alpha}{2}$, then $0<|\lambda x_1+(1-\lambda)x_2|<\frac{\beta-\alpha}{2}$, by (3.35) and f is convex and decreasing in (α,β), we have

$$\begin{aligned}\lambda Sf(x_1)+(1-\lambda)Sf(x_2) &= \lambda f(\beta-2|x_1|)+(1-\lambda)f(\beta-2|x_2|)\\ &\geqslant f(\beta-2(\lambda|x_1|+(1-\lambda)|x_2|))\\ &\geqslant f(\beta-2(|\lambda x_1+(1-\lambda)x_2|))\\ &= Sf(\lambda x_1+(1-\lambda)x_2);\end{aligned} \tag{3.39}$$

(iii) if $x_1=0$ and $x_2=0$, equality (3.38) is clearly established by (3.36);

(iv) if $x_1=0$ and $0<|x_2|<\frac{\beta-\alpha}{2}$, by (3.35), (3.36) and the convexity and continuity of f in (α,β), then we have

$$\begin{aligned}Sf(\lambda x_1+(1-\lambda)x_2) &= Sf((1-\lambda)x_2)=f(\beta-2(1-\lambda)|x_2|)\\ &= \lim_{t\to 0,t>0} f(\beta-t-2(1-\lambda)|x_2|)\\ &= \lim_{t\to 0,t>0} f(\lambda(\beta-t)+(1-\lambda)(\beta-t-2|x_2|))\\ &\leqslant \lim_{t\to 0,t>0}[\lambda f(\beta-t)+(1-\lambda)f(\beta-t-2|x_2|)]\\ &= \lambda \lim_{t\to 0,t>0} f(\beta-t)+(1-\lambda)\lim_{t\to 0,t>0} f(\beta-t-2|x_2|)\\ &= \lambda \lim_{x\to\beta,x<\beta} f(x)+(1-\lambda)f(\beta-2|x_2|)\\ &= \lambda Sf(x_1)+(1-\lambda)Sf(x_2);\end{aligned} \tag{3.40}$$

(v) if $0<|x_1|<\frac{\beta-\alpha}{2}$ and $x_2=0$, the proof is same as in the case (iv).

Step 3. We prove that $Vol_1([Sf\leqslant s])=Vol_1([f\leqslant s])$ for any $s\in\mathbb{R}$.

Since Sf is an even convex function, thus $Sf(0)=\inf Sf$. By (3.2), we have $Sf(0)=\inf f$, thus $\inf Sf=\inf f$. Let $a=\inf f=\inf Sf$, then $f^{-1}(a)=\{x\in\mathbb{R}: f(x)=a\}$ has three cases: $\emptyset$, $[\gamma,\beta)$ (where $\gamma\in(\alpha,\beta)$) and (α,β).

(i) If $f^{-1}(a)=\emptyset$, then f is strictly decreasing on (α,β).

If $s=a$, we prove $(Sf)^{-1}(a)=\{0\}$. For any $\delta\in(0,\frac{\beta-\alpha}{2})$, we have $\alpha<\beta-2\delta<\beta$, thus by (3.35) and f is strictly decreasing on (α,β), we have $Sf(\delta)=f(\beta-2\delta)>\lim_{x\to\beta,x<\beta} f(x)=a$, which implies that $(Sf)^{-1}(a)=\{0\}$. Thus $Vol_1([Sf\leqslant s])=Vol_1([f\leqslant s])=0$.

If $a<s<b$, where $b=\lim_{x\to\alpha,x>\alpha} f(x)$. Since $(Sf)^{-1}(a)=\{0\}$, Sf is strictly decreasing on $(-\frac{\beta-\alpha}{2},0)$ and strictly increasing on $(0,\frac{\beta-\alpha}{2})$. If $s\in(a,b)$, by f is strictly decreasing in (α,β), then there exists a unique $x'\in(\alpha,\beta)$ such that $f(x')=s$, thus $[f\leqslant s]=[x',\beta)$. By (3.35) and $0<\frac{\beta-x'}{2}<\frac{\beta-\alpha}{2}$, we have $Sf(\frac{\beta-x'}{2})=f(x')=s$, thus $[Sf\leqslant s]=[-\frac{\beta-x'}{2},\frac{\beta-x'}{2}]$. Thus for $s\in(a,b)$, $Vol_1(Sf\leqslant s)=Vol_1([f\leqslant s])=\beta-x'$.

If $s\geqslant b$, then $b<+\infty$ for $s\in\mathbb{R}$, since $f(x)<b$ for $x\in(\alpha,\beta)$ and $Sf(x)<b$ for $x\in(-\frac{\beta-\alpha}{2},\frac{\beta-\alpha}{2})$, then $Vol_1(Sf\leqslant s)=Vol_1([f\leqslant s])=\beta-\alpha$.

If $s<a$, since $[Sf\leqslant s]=[f\leqslant s]=\emptyset$, then $Vol_1([Sf\leqslant s])=Vol_1([f\leqslant s])=0$.

(ii) If $f^{-1}(a)=[\gamma,\beta)$, where $\gamma\in(\alpha,\beta)$, then f is strictly decreasing on $(\alpha,\gamma]$.

If $s=a$, we prove that $(Sf)^{-1}(a)=[-\frac{\beta-\gamma}{2},\frac{\beta-\gamma}{2}]$. Indeed, for $x\in[-\frac{\beta-\gamma}{2},\frac{\beta-\gamma}{2}]$, if $0<|x|\leqslant\frac{\beta-\gamma}{2}$, then $\gamma\leqslant\beta-2|x|<\beta$, thus by (3.35) we have $Sf(x)=f(\beta-2|x|)=a$, if $|x|=0$, then by (3.36) we have $Sf(0)=a$. Thus $[-\frac{\beta-\gamma}{2},\frac{\beta-\gamma}{2}]\subset(Sf)^{-1}(a)$. On the other hand, if $|x|>\frac{\beta-\gamma}{2}$ and $|x|<\frac{\beta-\alpha}{2}$, then $\alpha<\beta-2|x|<\gamma$, thus by f is strictly decreasing in $(\alpha,\gamma]$, we have $Sf(x)=f(\beta-2|x|)>f(\gamma)=a$, which implies that $x\notin(Sf)^{-1}(a)$ for $|x|>\frac{\beta-\gamma}{2}$. Thus $(Sf)^{-1}(a)=[-\frac{\beta-\gamma}{2},\frac{\beta-\gamma}{2}]$. Thus $Vol_1([Sf\leqslant s])=Vol_1([f\leqslant s])=\beta-\gamma$ for $s=a$.

If $s\in(a,b)$, where $b=\lim_{x\to\alpha,x>\alpha}f(x)$. Since $(Sf)^{-1}(a)=[-\frac{\beta-\gamma}{2},\frac{\beta-\gamma}{2}]$, Sf is strictly decreasing on $(-\frac{\beta-\alpha}{2},-\frac{\beta-\gamma}{2}]$ and strictly increasing on $[\frac{\beta-\gamma}{2},\frac{\beta-\alpha}{2})$. If $s\in(a,b)$, then there exists a unique $x'\in(\alpha,\gamma)$ such that $f(x')=s$, thus $[f\leqslant s]=[x',\beta)$. Since $Sf(\frac{\beta-x'}{2})=f(x')=s$, thus $[Sf\leqslant s]=[-\frac{\beta-x'}{2},\frac{\beta-x'}{2}]$. Thus for $s\in(a,b)$, $Vol_1(Sf\leqslant s)=Vol_1([f\leqslant s])=\beta-x'$.

If $s\geqslant b$, then $b<+\infty$ for $s\in\mathbb{R}$, since $f(x)<b$ for $x\in(\alpha,\beta)$ and $Sf(x)<b$ for $x\in(-\frac{\beta-\alpha}{2},\frac{\beta-\alpha}{2})$, thus $Vol_1(Sf\leqslant s)=Vol_1([f\leqslant s])=\beta-\alpha$.

If $s<a$, since $[Sf\leqslant s]=[f\leqslant s]=\emptyset$, then $Vol_1([Sf\leqslant s])=Vol_1([f\leqslant s])=0$.

(iii) If $f^{-1}(a)=(\alpha,\beta)$, then $f(x)=a$ for $x\in(\alpha,\beta)$, otherwise $f(x)=+\infty$. By (3.2), we have $Sf(x)=a$ for $x\in(-\frac{\beta-\alpha}{2},\frac{\beta-\alpha}{2})$, otherwise $Sf(x)=+\infty$. Thus, we have $Vol_1(Sf\leqslant s)=Vol_1([f\leqslant s])$ for any $s\in\mathbb{R}$.

(ii) f is increasing on (α,β). The proof of this case is the same as f is decreasing on (α,β). Furthermore, we can get the following conclusions.

i) There exists s_0 such that $\lim_{x\to\alpha,x>\alpha}f(x)=s_0$;

ii) If $|x|\geqslant\frac{\beta-\alpha}{2}$, then $Sf(x)=+\infty$;

iii) If $0<|x|<\frac{\beta-\alpha}{2}$, then $Sf(x)=f(\alpha+2|x|)$;

iv) If $|x|=0$, then $Sf(0)=\lim_{x\to\alpha,x>\alpha}f(x)$.

(iii) f is decreasing on $(\alpha,\gamma]$ and increasing on $[\gamma,\beta)$ for some $\gamma\in(\alpha,\beta)$. There exist four cases:

i) $\lim_{x\to\alpha,x>\alpha}f(x)<+\infty$ and $\lim_{x\to\beta,x<\beta}f(x)<+\infty$;

ii) $\lim_{x\to\alpha,x>\alpha}f(x)=+\infty$ and $\lim_{x\to\beta,x<\beta}f(x)<+\infty$;

iii) $\lim_{x\to\alpha,x>\alpha}f(x)<+\infty$ and $\lim_{x\to\beta,x<\beta}f(x)=+\infty$;

iv) $\lim_{x\to\alpha,x>\alpha}f(x)=+\infty$ and $\lim_{x\to\beta,x<\beta}f(x)=+\infty$.

For case iv), when $|x|\geqslant\frac{\beta-\alpha}{2}$, $Sf(x)=+\infty$, when $|x|<\frac{\beta-\alpha}{2}$, we can prove our conclusion by the same method with Case (1) (i.e., the case of $\mathrm{dom} f=\mathbb{R}$).

For cases ii) and iii), we can prove our conclusion by the same method with case i), thus it is sufficient to prove case i).

For case i), we may assume that $\lim_{x\to\alpha,x>\alpha} f(x) > a$ and $\lim_{x\to\beta,x<\beta} f(x) > a$, where $a = \inf f$, otherwise if $\lim_{x\to\alpha,x>\alpha} f(x) = a$ (or $\lim_{x\to\beta,x<\beta} f(x) = a$), then f is increasing in (α,β) (or f is decreasing in (α,β)), the case has been considered in case (i) (or case (ii)).

If $\lim_{x\to\alpha,x>\alpha} f(x) = \lim_{x\to\beta,x<\beta} f(x) = b > a$, then when $|x| < \frac{\beta-\alpha}{2}$, by the entirely same method with that in Case (1) (i.e., the case of $\mathrm{domf} = \mathbb{R}$), we have the following conclusions.

1) Sf is even and convex on $(-\frac{\beta-\alpha}{2}, \frac{\beta-\alpha}{2})$;

2) For any $|x| < \frac{\beta-\alpha}{2}$, there exists $x' \in (\alpha,\beta)$ such that $Sf(x) = f(x') = f(x'-2x)$;

3) For any $s \in (-\infty, b)$, we have $Vol_1([Sf \leqslant s]) = Vol_1([f \leqslant s])$.

By the above 1), 2) and 3), we can easily prove that Sf is convex in $\mathbb{R}$ and for any $s \in \mathbb{R}$, $Vol_1([Sf \leqslant s]) = Vol_1([f \leqslant s])$.

If $\lim_{x\to\alpha,x>\alpha} f(x) \neq \lim_{x\to\beta,x<\beta} f(x)$, we may assume that

$$\lim_{x\to\alpha,x>\alpha} f(x) = b > \lim_{x\to\beta,x<\beta} f(x) = c > a. \tag{3.41}$$

Let $\gamma \in (\alpha,\beta)$ satisfy $f(\gamma) = c$. If $|x| < \frac{\beta-\gamma}{2}$, then by the proof of Case (1) (i.e., the case of $\mathrm{domf} = \mathbb{R}$), there exists $x' \in (\gamma,\beta)$ such that $Sf(x) = f(x') = f(x'-2x)$.

Step 1. We prove that for $|x| \geqslant \frac{\beta-\gamma}{2}$ and $|x| < \frac{\beta-\alpha}{2}$,

$$Sf(x) = f(\beta - 2|x|). \tag{3.42}$$

Since Sf is even, we may assume $\frac{\beta-\gamma}{2} \leqslant x < \frac{\beta-\alpha}{2}$. For any $\lambda \in [0,1]$, we have

$$\begin{aligned}\inf_{x_1\in\mathbb{R}}[\lambda f(2x_1) + (1-\lambda)f(2x_1-2x)] &\leqslant \lambda \lim_{t\to\beta,t<\beta} f(t) + (1-\lambda)f(\beta-2x)\\ &= \lambda c + (1-\lambda)f(\beta-2x).\end{aligned} \tag{3.43}$$

Since $\frac{\beta-\gamma}{2} \leqslant x < \frac{\beta-\alpha}{2}$, then $\alpha < \beta - 2x \leqslant \gamma$. Since f is decreasing on $(\alpha,\gamma]$, thus $f(\beta-2x) \geqslant f(\gamma) = c$. Thus, by (3.43), we have

$$\begin{aligned}Sf(x) &= \sup_{\lambda\in[0,1]}\inf_{x_1\in\mathbb{R}^n}[\lambda f(2x_1) + (1-\lambda)f(2x_1-2x)]\\ &\leqslant \sup_{\lambda\in[0,1]}[\lambda c + (1-\lambda)f(\beta-2x)] = f(\beta-2x).\end{aligned} \tag{3.44}$$

On the other hand, we will prove that $Sf(x) \geqslant f(\beta-2x)$. For $\lambda = 0$ or $\lambda = 1$, we have

$$\inf_{x_1\in\mathbb{R}}[\lambda f(x_1) + (1-\lambda)f(x_1-2x)] = \inf f. \tag{3.45}$$

Thus

$$\begin{aligned}&\sup_{\lambda\in[0,1]}\inf_{x_1\in\mathbb{R}}[\lambda f(x_1) + (1-\lambda)f(x_1-2x)]\\ =\ &\sup_{\lambda\in(0,1)}\inf_{x_1\in\mathbb{R}}[\lambda f(x_1) + (1-\lambda)f(x_1-2x)].\end{aligned} \tag{3.46}$$

Since $\mathrm{dom} f = (\alpha,\beta)$, thus for any $\lambda \in (0,1)$, we have

$$\begin{aligned}&\inf_{x_1\in\mathbb{R}}[\lambda f(x_1)+(1-\lambda)f(x_1-2x)]\\ =&\inf_{x_1\in(\alpha+2x,\beta)}[\lambda f(x_1)+(1-\lambda)f(x_1-2x)].\end{aligned} \tag{3.47}$$

Thus

$$Sf(x)=\sup_{\lambda\in(0,1)}\inf_{x_1\in(\alpha+2x,\beta)}[\lambda f(x_1)+(1-\lambda)f(x_1-2x)]. \tag{3.48}$$

By (3.41), if $f^{-1}(a)=[\mu,\nu]$, then $\alpha<\mu\leqslant\nu<\beta$, thus f is strictly decreasing on $(\alpha,\mu]$ and strictly increasing on $[\nu,\beta)$.

CLAIM 3. For a fixed $\beta'\in(\nu,\beta)\cap(\alpha+2x,\beta)$, there exists $\delta>0$ such that function

$$G_x(x_1):=\lambda f(x_1)+(1-\lambda)f(x_1-2x) \tag{3.49}$$

is decreasing on $(\alpha+2x,\beta']$ for any $0<\lambda<\delta$.

We first use Claim 3 to prove our result, the proof of this claim will be given later. By (3.48) and Claim 3, we have that

$$\begin{aligned}Sf(x)=&\sup_{\lambda\in(0,1)}\inf_{x_1\in(\alpha+2x,\beta)}[\lambda f(x_1)+(1-\lambda)f(x_1-2x)]\\ \geqslant&\sup_{\lambda\in(0,\delta)}\inf_{x_1\in(\alpha+2x,\beta)}[\lambda f(x_1)+(1-\lambda)f(x_1-2x)]\\ =&\sup_{\lambda\in(0,\delta)}\inf_{x_1\in[\beta',\beta)}[\lambda f(x_1)+(1-\lambda)f(x_1-2x)]\\ \geqslant&\sup_{\lambda\in(0,\delta)}[\lambda f(\beta')+(1-\lambda)f(\beta-2x)]\\ =&f(\beta-2x),\end{aligned} \tag{3.50}$$

where the second inequality is by $x_1\in[\beta',\beta)\subset(\nu,\beta)$ and $\beta'-2x\leqslant x_1-2x<\beta-2x\leqslant\gamma$ and f is strictly increasing on (ν,β) and strictly decreasing on $(\alpha,\gamma]$, and the last equality is by $f(\beta-2x)\geqslant f(\beta')$.

Next, we prove Claim 3. For $x_1\in(\alpha+2x,\beta']$, the right derivative of $G_x(x_1)$

$$\begin{aligned}G'_{xr}(x_1)=&\lambda f'_r(x_1)+(1-\lambda)f'_r(x_1-2x)\\ \leqslant&\lambda f'_r(\beta')+(1-\lambda)f'_r(\beta'-2x),\end{aligned} \tag{3.51}$$

where the inequality is by the right derivative of a convex function is increasing on the interior of its effective domain. Since $\beta'\in(\nu,\beta)\cap(\alpha+2x,\beta)$ and $x\in[\frac{\beta-\gamma}{2},\frac{\beta-\alpha}{2})$, then $\beta'-2x\in(\alpha,\gamma+\beta'-\beta)$, thus $f'_r(\beta')>0$ and $f'_r(\beta'-2x)<0$ for f is strictly increasing on (ν,β) and strictly decreasing on $(\alpha,\gamma]$. Thus, by (3.51), we choose

$$\delta=\frac{-f'_r(\beta'-2x)}{f'_r(\beta')-f'_r(\beta'-2x)}, \tag{3.52}$$

then $G'_{xr}(x_1) < 0$ on $(\alpha+2x, \beta']$ for any $\lambda \in (0,\delta)$. Therefore, $G_x(x_1)$ is decreasing on $(\alpha+2x, \beta']$ for any $\lambda \in (0,\delta)$. This completes the proof of Claim 3.

Step 2. We prove that Sf is convex in $\mathbb{R}$. Since Sf is increasing on $[0, \frac{\beta-\alpha}{2})$ and Sf is even on $(-\frac{\beta-\alpha}{2}, \frac{\beta-\alpha}{2})$. Thus, it suffices to prove Sf is convex in $[0, \frac{\beta-\alpha}{2})$. For any $x_1, x_2 \in [\frac{\beta-\gamma}{2}, \frac{\beta-\alpha}{2})$ and $\lambda \in (0,1)$, by (3.42) and f is convex function, we have

$$\begin{aligned} \lambda Sf(x_1) + (1-\lambda)Sf(x_2) &= \lambda f(\beta - 2x_1) + (1-\lambda) f(\beta - 2x_2) \\ &\geqslant f(\beta - 2(\lambda x_1 + (1-\lambda)x_2)) \\ &= Sf(\lambda x_1 + (1-\lambda)x_2), \end{aligned} \tag{3.53}$$

where the last equality is by $\lambda x_1 + (1-\lambda)x_2 \in [\frac{\beta-\gamma}{2}, \frac{\beta-\alpha}{2})$. By (3.53), Sf is convex on $[\frac{\beta-\gamma}{2}, \frac{\beta-\alpha}{2})$. Because that Sf is convex in $[0, \frac{\beta-\gamma}{2}]$ by the proof in Case (1) (i.e., the case of $\mathrm{dom} f = \mathbb{R}$), it suffices to prove that the left derivative of Sf at $x = \frac{\beta-\gamma}{2}$ is less than its right derivative at $x = \frac{\beta-\gamma}{2}$.

By (3.42), we have

$$\begin{aligned} Sf'_r\Big(\frac{\beta-\gamma}{2}\Big) &= \lim_{t\to 0, t>0} \frac{Sf(\frac{\beta-\gamma}{2}+t) - Sf(\frac{\beta-\gamma}{2})}{t} \\ &= \lim_{t\to 0, t>0} \frac{f(\gamma-2t) - f(\gamma)}{t} = -2f'_l(\gamma). \end{aligned} \tag{3.54}$$

For any $t \in (-\frac{\beta-\gamma}{2}, 0)$, we have $\frac{\beta-\gamma}{2} + t \in (0, \frac{\beta-\gamma}{2})$. Thus there exist x', $x'' \in (\gamma, \beta)$ such that $x'' - x' = 2(\frac{\beta-\gamma}{2} + t)$ and

$$Sf\Big(\frac{\beta-\gamma}{2} + t\Big) = f(x') = f(x''). \tag{3.55}$$

Since

$$(x'-\gamma) + 2\Big(\frac{\beta-\gamma}{2} + t\Big) = (x'-\gamma) + (x''-x') = x'' - \gamma < \beta - \gamma, \tag{3.56}$$

we have $x' < \gamma - 2t$. Let $|t|$ be sufficiently small such that $\gamma + 2|t| < \mu$, where μ satisfies $f^{-1}(a) = [\mu, \nu]$, then $f(x') > f(\gamma - 2t)$ for f is strictly decreasing on (γ, μ). Thus we have

$$\begin{aligned} Sf'_l\Big(\frac{\beta-\gamma}{2}\Big) &= \lim_{t\to 0, t<0} \frac{Sf(\frac{\beta-\gamma}{2}+t) - Sf(\frac{\beta-\gamma}{2})}{t} \\ &= \lim_{t\to 0, t<0} \frac{f(x') - f(\gamma)}{t} \\ &\leqslant \lim_{t\to 0, t<0} \frac{f(\gamma-2t) - f(\gamma)}{t} = -2f'_r(\gamma). \end{aligned} \tag{3.57}$$

Since f is convex function, then $f'_l(\gamma) \leqslant f'_r(\gamma)$, by (3.54) and (3.57), we have $Sf'_l(\frac{\beta-\gamma}{2}) \leqslant Sf'_r(\frac{\beta-\gamma}{2})$.

Step 3. We prove that $Vol_1([Sf \leqslant a]) = Vol([f \leqslant s])$ for any $s \in \mathbb{R}$. When $s < f(\gamma)$, the proof is same as in Case (1). When $s \geqslant f(\gamma)$, the proof is same as that when f is decreasing in (α,β). □

REMARK 3.2. 1) By Theorem 3.1, for any $x \in \mathbb{R}$, if $x = 0$, then $Sf(0) = \inf f$; if $x \neq 0$, then there exist three cases:

i) $Sf(x) = f(x') = f(x' - 2|x|)$ for some $x' \in \mathbb{R}$;

ii) $Sf(x) = f(x_0 - 2|x|)$ for some $x_0 \in \mathbb{R}$, where x_0 satisfies that $f(y) = +\infty$ for $y > x_0$ and $x_0 - 2|x| \in \{y \in \mathbb{R} : f(y) > \lim_{z \to x_0^-} f(z)\}$;

iii) $Sf(x) = f(x_0 + 2|x|)$ for some $x_0 \in \mathbb{R}$, where x_0 satisfies that $f(y) = +\infty$ for $y < x_0$ and $x_0 + 2|x| \in \{y \in \mathbb{R} : f(y) > \lim_{z \to x_0^+} f(z)\}$;

2) In Theorem 3.1, there exist three cases for $\mathrm{dom}Sf$: i) $\mathrm{dom}Sf = (-\delta,\delta)$; ii) $\mathrm{dom}Sf = [-\delta,\delta]$; iii) $\mathrm{dom}Sf = \mathbb{R}$. $\mathrm{dom}Sf = (-\delta,\delta)$ is corresponding to $\mathrm{dom}f = (\alpha,\beta)$, $\mathrm{dom}f = (\alpha,\beta]$ and $\mathrm{dom}f = [\alpha,\beta)$, where $\delta = \frac{\beta-\alpha}{2}$. $\mathrm{dom}Sf = [-\delta,\delta]$ is corresponding $\mathrm{dom}f = [\alpha,\beta]$, where $\delta = \frac{\beta-\alpha}{2}$. $\mathrm{dom}Sf = \mathbb{R}$ is corresponding to $\mathrm{dom}f = (-\infty,\beta)$, $\mathrm{dom}f = (-\infty,\beta]$, $\mathrm{dom}f = (\alpha,+\infty)$, $\mathrm{dom}f = [\alpha,+\infty)$ and $\mathrm{dom}f = \mathbb{R}$. By the defintion of Steiner symmetrization of non-empty convex set (Definition 2.1) and Definition 3.1, for coercive convex function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ and its Steiner symmetrization $S_u f$ about hyperplane $u^\perp$, the effective domains of f and $S_u f$ satisfy

$$\mathrm{dom}(S_u f) = S_{u^\perp}(\mathrm{dom}f). \tag{3.58}$$

We know that $\mathrm{dom}f$ is convex if f is convex and the Steiner symmetrization of a non-empty convex set is still a convex set, thus by equality (3.58), we have $\mathrm{dom}(S_u f)$ is a convex set.

3) For a convex function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, the *epigraph* of f is defined as

$$\mathrm{epi}f := \{(x,y) \in \mathbb{R}^{n+1} : x \in \mathrm{dom}f,\ y \geqslant f(x)\}. \tag{3.59}$$

By the definition of epigraph and Theorem 3.1, for one-dimensional coercive convex function $f : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$, we have $\mathrm{cl}(\mathrm{epi}Sf) = S_{e^\perp}(\mathrm{cl}(\mathrm{epi}f))$, where e is a unit vector along the x-axis and $\mathrm{cl}A$ denotes the closure of a subset $A \subset \mathbb{R}^n$. For $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is a coercive and convex function and $u \in S^{n-1}$. For any $x' \in u^\perp$ and $t \in \mathbb{R}$, if $\tilde{f}(t) = f(x' + tu)$ is considered as a one-dimensional function about t, then $S\tilde{f}(t) = S_u f(x' + tu)$. By Theorem 3.1, $\mathrm{cl}(\mathrm{epi}(S\tilde{f})) = S_{e^\perp}(\mathrm{cl}(\mathrm{epi}\tilde{f}))$. Since $x' \in u^\perp$ is arbitrary, thus we have

$$\mathrm{cl}(\mathrm{epi}(S_u f)) = S_{\tilde{u}^\perp}(\mathrm{cl}(\mathrm{epi}f)), \tag{3.60}$$

where $\tilde{u}^\perp \subset \mathbb{R}^{n+1}$ denotes the hyperplane through the origin and orthogonal to the unit vector $\tilde{u} = (u,0) \in \mathbb{R}^{n+1}$.

Next, by Definition 3.1 and Theorem 3.1, we first prove the following five theorems which are corresponding to the properties 1–5 in Table 1.

THEOREM 3.2. *If* $f:\mathbb{R}^n\to\mathbb{R}\cup\{+\infty\}$ *is a coercive convex function and* $u\in S^{n-1}$*, then* $S_u f$ *is a coercive convex function and symmetric about* $u^{\perp}$*.*

We first establish a lemma that will be used in the proof of Theorem 3.2.

LEMMA 3.3. *For* $f:\mathbb{R}^n\to\mathbb{R}\cup\{+\infty\}$*, not identically* $+\infty$*, let* $u\in S^{n-1}$ *and* $H=u^{\perp}$*, if*

i) f *is symmetric with respect to hyperplane* H*, i.e., for any* $x'\in H$ *and* $t\in\mathbb{R}$*,* $f(x'+tu)=f(x'-tu)$*;*

ii) *for any* $x'\in H$ *and* $t_1,\ t_2\in\mathbb{R}$*, if* $|t_1|\leqslant|t_2|$*, then* $f(x'+t_1u)\leqslant f(x'+t_2u)$*;*

iii) f *is convex on half-space* H^+*, where*

$$H^+=\{x'+tu:x'\in u^{\perp},t\geqslant 0\},\tag{3.61}$$

then f *is a convex function defined on* $\mathbb{R}^n$*.*

Proof. For any $x,y\in\mathbb{R}^n$, let $x=x'+tu$ and $y=y'+su$, where $x',\ y'\in H$.
If $t\geqslant 0$ and $s\geqslant 0$, for $\lambda\in(0,1)$, then by iii),

$$f(\lambda x+(1-\lambda)y)\leqslant\lambda f(x)+(1-\lambda)f(y).\tag{3.62}$$

If $t\geqslant 0$ and $s<0$, by i), ii) and iii), we have

$$\begin{aligned}\lambda f(x)+(1-\lambda)f(y) &= \lambda f(x'+tu)+(1-\lambda)f(y'+su)\\ &= \lambda f(x'+tu)+(1-\lambda)f(y'-su)\\ &\geqslant f(\lambda(x'+tu)+(1-\lambda)(y'-su))\\ &= f(\lambda x'+(1-\lambda)y'+(\lambda t-(1-\lambda)s)u)\\ &\geqslant f(\lambda x'+(1-\lambda)y'+(\lambda t+(1-\lambda)s)u)\\ &\geqslant f(\lambda x+(1-\lambda)y),\end{aligned}\tag{3.63}$$

where the second inequality is by $\lambda t-(1-\lambda)s=\lambda|t|+(1-\lambda)|s|\geqslant|\lambda t+(1-\lambda)s|$.
In the same method, we can get inequality (3.63) for $t<0$ and $s\geqslant 0$.
If $t<0$ and $s<0$, by i) and iii), we have

$$\begin{aligned}\lambda f(x)+(1-\lambda)f(y) &= \lambda f(x'+tu)+(1-\lambda)f(y'+su)\\ &= \lambda f(x'-tu)+(1-\lambda)f(y'-su)\\ &\geqslant f(\lambda(x'-tu)+(1-\lambda)(y'-su))\\ &= f(\lambda x'+(1-\lambda)y'-(\lambda t+(1-\lambda)s)u)\\ &= f(\lambda x'+(1-\lambda)y'+(\lambda t+(1-\lambda)s)u)\\ &= f(\lambda x+(1-\lambda)y).\end{aligned}\tag{3.64}$$

□

Next, we prove Theorem 3.2.

Proof. *Step* 1. We prove that $S_u f$ is proper.

For any $x \in \mathbb{R}^n$, let $x = x' + tu$, where $x' \in u^{\perp}$. Since f is a coercive convex function defined on $\mathbb{R}^n$, thus one dimensional function $f(x'+tu)$ about $t \in \mathbb{R}$ either is a coercive convex function or is identically $+\infty$. If $f(x'+tu)$ is a coercive convex function, then there exists $s \in \mathbb{R}$ such that $s = \inf\{f(x'+tu) : t \in \mathbb{R}\}$. Thus, we have

$$Sf(x) = \sup_{\lambda\in[0,1]} \inf_{t_1+t_2=t} [\lambda f(x'+2t_1u) + (1-\lambda)f(x'-2t_2u)] \geqslant s, \tag{3.65}$$

which implies that $Sf(x) > -\infty$. If $f(x'+tu)$ is identically $+\infty$, then $S_u f(x) = +\infty > -\infty$. By Definition 2.2, f is not identically $+\infty$, there exists $x \in \mathbb{R}^n$ such that $f(x) < +\infty$. Let $x = x_0 + tu$, where $x_0 \in u^{\perp}$, then by Definition 3.1 we have

$$\begin{aligned} S_u f(x_0) &= \sup_{\lambda\in[0,1]} \inf_{t_1+t_2=0} [\lambda f(x_0+2t_1u) + (1-\lambda)f(x_0-2t_2u)] \\ &= \inf_{t_1\in\mathbb{R}} f(x_0+2t_1u) \leqslant f(x) < +\infty, \end{aligned} \tag{3.66}$$

which implies that $S_u f$ is not identically $+\infty$.

Step 2. We prove that $S_u f$ is coercive.

Suppose that there exist $M_0 > 0$ and a sequence $\{x_n\}_{n=1}^{\infty} \subset \mathbb{R}^n$ satisfying that $|x_n| > n$ and $S_u f(x_n) < M_0$. For any positive integer $n \geqslant 1$, let $x_n = x'_n + t_n u$ and $x'_n \in u^{\perp}$. There exist two cases of $t_n \neq 0$ and $t_n = 0$.

(1) If $t_n \neq 0$, then by Theorem 3.1, there exist three cases:

i) $S_u f(x_n) = f(x'_n + t'_n u) = f(x'_n + (t'_n - 2t_n)u)$ for some $t'_n \in \mathbb{R}$;

ii) $S_u f(x_n) = f(x'_n + (t_0 - 2t_n)u)$ for some $t_0 \in \mathbb{R}$;

iii) $S_u f(x_n) = f(x'_n + (t_0 + 2t_n)u)$ for some $t_0 \in \mathbb{R}$.

For case i), since

$$|t'_n| + |t'_n - 2t_n| \geqslant 2|t_n|, \tag{3.67}$$

thus either $|t'_n| \geqslant |t_n|$ or $|t'_n - 2t_n| \geqslant |t_n|$. If $|t'_n| \geqslant |t_n|$, let $y_n = x'_n + t'_n u$, then $S_u f(x_n) = f(y_n)$ and

$$|y_n| = |x'_n| + |t'_n| \geqslant |x'_n| + |t_n| = |x_n|.$$

If $|t'_n - 2t_n| \geqslant |t_n|$, let $y_n = x'_n + (t'_n - 2t_n)u$, then $S_u f(x_n) = f(y_n)$

$$|y_n| = |x'_n| + |t'_n - 2t_n| \geqslant |x'_n| + |t_n| = |x_n|.$$

Since $|x_n| > n$, we have $|y_n| > n$ and $f(y_n) = S_u f(x_n) < M_0$.

For case ii), since

$$|t_0| + |t_0 - 2t_n| \geqslant 2|t_n|, \tag{3.68}$$

thus either $|t_0| \geqslant |t_n|$ or $|t_0 - 2t_n| \geqslant |t_n|$. If $|t_0 - 2t_n| \geqslant |t_n|$, let $y_n = x'_n + (t_0 - 2t_n)u$, then $S_u f(x_n) = f(y_n)$ and $|y_n| \geqslant |x_n|$. If $|t_0| \geqslant |t_n|$, let $y_n = x'_n + t_0 u$ if $x'_n + t_0 u \in \mathrm{dom} f$, otherwise let $y_n = x'_n + t'_0 u$, where t'_0 satisfies

a) $x'_n + t'_0 u \in \mathrm{dom} f$;

b) $|x'_n + t'_0 u| > n$;

c) $f(x'_n + t'_0 u) < f(x'_n + (t_0 - 2t_n)u)$,

where case c) can be satisfied for $\lim_{t\to t_0,\, t<t_0} f(x'_n + tu) \leqslant f(x'_n + (t_0 - 2t_n)u)$ by Theorem 3.1. Thus, we have $|y_n| > n$ and $f(y_n) < M_0$.

For case iii), we can construct $\{y_n\}$ with the same method as in case (ii).

(2) If $t_n = 0$, by Definition 3.1, we have $Sf(x_n) = \inf_{t\in\mathbb{R}} f(x'_n + tu)$. Since $S_u f(x_n) < M_0$, there exists $y_n = x'_n + t'u$ such that $f(y_n) < M_0$. Since $|y_n| = |x'_n| + |t'| \geqslant |x'_n| = |x_n|$, we have $|y_n| > n$ and $f(y_n) < M_0$.

By (1) and (2), we get that a sequence $\{y_n\}_{n=1}^{\infty}$ satisfying $|y_n| > n$ and $f(y_n) < M_0$ for given $M_0 > 0$, which is contradictory with f is coercive. Hence, $S_u f$ is coercive.

Step 3. We prove $S_u f$ is symmetric about $u^{\perp}$.

For any $x' \in u^{\perp}$ and $t \in \mathbb{R}$, if we consider $S_u f(x' + tu)$ as a one-dimensional function about t, then by Theorem 3.1 and Definition 3.1, we have $S_u f(x' + tu) = S_u f(x' - tu)$. Thus $S_u f$ is symmetric about $u^{\perp}$.

Step 4. We prove $S_u f$ is convex.

By Definition 3.1 and Theorem 3.1, for any $x' \in u^{\perp}$, one-dimensional function $S_u f(x' + tu)$ either is an even and coercive convex function about $t \in \mathbb{R}$ or is identically $+\infty$. Thus, $S_u f$ satisfies conditions i) and ii) in Lemma 3.3, thus to prove that $S_u f$ is convex, it suffices to prove that $S_u f$ satisfies condition iii) of Lemma 3.3. For any x, $y \in \{x' + tu : x' \in u^{\perp}, t \geqslant 0\}$ and $\lambda \in (0,1)$, if $x \notin \mathrm{dom}(S_u f)$ or $y \notin \mathrm{dom}(S_u f)$, then $S_u f(x) = +\infty$ or $S_u f(y) = +\infty$, thus

$$S_u f(\lambda x + (1-\lambda)y) \leqslant \lambda S_u f(x) + (1-\lambda) S_u f(y). \tag{3.69}$$

Since $\mathrm{dom} f$ is convex and $\mathrm{dom}(S_u f) = S_u(\mathrm{dom} f)$ (see equality (3.58)), thus $\mathrm{dom}(S_u f)$ is convex.

If $x \in \mathrm{dom}(S_u f)$ and $y \in \mathrm{dom}(S_u f)$, then $\lambda x + (1-\lambda)y \in \mathrm{dom}(S_u f)$. Let $x = x' + tu$ and $y = y' + su$, where x', $y' \in u^{\perp}$ and $t \geqslant 0$ and $s \geqslant 0$, then

$$\lambda x + (1-\lambda)y = [\lambda x' + (1-\lambda)y'] + [\lambda t + (1-\lambda)s]u.$$

Case 3.1. The case of $t = 0$ and $s = 0$. For the case we have x, $y \in u^{\perp}$, thus $\lambda x + (1-\lambda)y \in u^{\perp}$. By Definition 3.1, there exist t' and $s' \in \mathbb{R}$ satisfying

$$S_u f(x) = \inf_{t\in\mathbb{R}} f(x + tu) = \lim_{t\to t',\, f(x+tu)<+\infty} f(x + tu), \tag{3.70}$$

$$S_u f(y) = \inf_{s\in\mathbb{R}} f(y + su) = \lim_{s\to s',\, f(y+su)<+\infty} f(y + s'u) \tag{3.71}$$

and

$$S_u f(\lambda x + (1-\lambda)y) = \inf_{w\in\mathbb{R}} f(\lambda x + (1-\lambda)y + wu). \tag{3.72}$$

By f is a convex function and the above three equalities, we have

$$\begin{aligned}
&\lambda S_u f(x)+(1-\lambda)S_u f(y)\\
&=\lambda \lim_{t\to t',\, f(x+tu)<+\infty} f(x+tu)+(1-\lambda)\lim_{s\to s',\, f(y+su)<+\infty} f(y+su)\\
&=\lim_{t\to t',\, f(x+tu)<+\infty}\ \lim_{s\to s',\, f(y+su)<+\infty}[\lambda f(x+tu)+(1-\lambda)f(y+su)]\\
&\geqslant \lim_{t\to t',\, f(x+tu)<+\infty}\ \lim_{s\to s',\, f(y+su)<+\infty} f(\lambda x+(1-\lambda)y+(\lambda t+(1-\lambda)s)u)\\
&\geqslant \inf\{f(\lambda x+(1-\lambda)y+wu): w\in\mathbb{R}\}\\
&=S_u f(\lambda x+(1-\lambda)y).
\end{aligned}\tag{3.73}$$

Case 3.2. The case of $t>0$ and $s>0$. For $x=x'+tu\in \mathrm{dom}(S_u f)$, by Theorem 3.1, there exist three cases:

a_1) There exists some $t'\in\mathbb{R}$ such that

$$S_u f(x)=f(x'+t'u)=f(x'+(t'-2t)u);\tag{3.74}$$

a_2) There exists some $t_0\in\mathbb{R}$ such that

$$S_u f(x)=f(x'+(t_0-2t)u)\geqslant \lim_{t_0'\to t_0, t_0'<t_0} f(x'+t_0'u);\tag{3.75}$$

a_3) There exists some $t_0\in\mathbb{R}$ such that

$$S_u f(x)=f(x'+(t_0+2t)u)\geqslant \lim_{t_0'\to t_0, t_0'>t_0} f(x'+t_0'u).\tag{3.76}$$

For $y=y'+su\in \mathrm{dom}(S_u f)$, by Theorem 3.1, there exist three cases:

b_1) There exists some $s'\in\mathbb{R}$ such that

$$S_u f(y)=f(y'+s'u)=f(y'+(s'-2s)u);\tag{3.77}$$

b_2) There exists some $s_0\in\mathbb{R}$ such that

$$S_u f(y)=f(y'+(s_0-2s)u)\geqslant \lim_{s_0'\to s_0, s_0'<s_0} f(y'+s_0'u);\tag{3.78}$$

b_3) There exists some $s_0\in\mathbb{R}$ such that

$$S_u f(y)=f(y'+(s_0+2s)u)\geqslant \lim_{s_0'\to s_0, s_0'>s_0} f(y'+s_0'u).\tag{3.79}$$

We may assume that

$$\begin{aligned}
f(x'+t_0u)&=\lim_{t_0'\to t_0, t_0'<t_0} f(x'+t_0'u)\ \text{ for case } a_2),\\
f(x'+t_0u)&=\lim_{t_0'\to t_0, t_0'>t_0} f(x'+t_0'u)\ \text{ for case } a_3),\\
f(y'+s_0u)&=\lim_{s_0'\to s_0, s_0'<s_0} f(y'+s_0'u)\ \text{ for case } b_2),\\
f(y'+s_0u)&=\lim_{s_0'\to s_0, s_0'>s_0} f(y'+s_0'u)\ \text{ for case } b_3).
\end{aligned}\tag{3.80}$$

Let $(\tilde{t}_1,\tilde{t}_2)$ be a pair of real numbers satisfying

$$(\tilde{t}_1,\tilde{t}_2)=\begin{cases}(t'-2t,t') & \text{for case } a_1)\\ (t_0-2t,t_0) & \text{for case } a_2)\\ (t_0,t_0+2t) & \text{for case } a_3).\end{cases} \tag{3.81}$$

Let $(\tilde{s}_1,\tilde{s}_2)$ be a pair of real numbers satisfying

$$(\tilde{s}_1,\tilde{s}_2)=\begin{cases}(s'-2s,s') & \text{for case } b_1)\\ (s_0-2s,s_0) & \text{for case } b_2)\\ (s_0,s_0+2s) & \text{for case } b_3).\end{cases} \tag{3.82}$$

Since f is convex and by (3.74-3.79), for $i=1,2$, we have

$$\begin{aligned}\lambda S_u f(x)+(1-\lambda)S_u f(y) &\geqslant \lambda f(x'+\tilde{t}_i u)+(1-\lambda)f(y'+\tilde{s}_i u)\\ &\geqslant f(\lambda x'+(1-\lambda)y'+(\lambda\tilde{t}_i+(1-\lambda)\tilde{s}_i)u).\end{aligned} \tag{3.83}$$

By (3.81) and (3.82), we have

$$\begin{aligned}&[\lambda\tilde{t}_2+(1-\lambda)\tilde{s}_2]-[\lambda\tilde{t}_1+(1-\lambda)\tilde{s}_1]\\ &=\lambda(\tilde{t}_2-\tilde{t}_1)+(1-\lambda)(\tilde{s}_2-\tilde{s}_1)=2[\lambda t+(1-\lambda)s].\end{aligned} \tag{3.84}$$

Let

$$M=\max_{i=1,2} f(\lambda x'+(1-\lambda)y'+(\lambda\tilde{t}_i+(1-\lambda)\tilde{s}_i)u). \tag{3.85}$$

By $\lambda x+(1-\lambda)y=\lambda x'+(1-\lambda)y'+(\lambda t+(1-\lambda)s)u$ and Definition 3.1, we have

$$\begin{aligned}S_u f(\lambda x+(1-\lambda)y) &= \sup_{\lambda\in[0,1]}\inf_{\omega\in\mathbb{R}}\big[\lambda f(x'+(1-\lambda)y'+\omega u)\\ &\quad +(1-\lambda)f(x'+(1-\lambda)y'+(\omega-2(\lambda t+(1-\lambda)s)u)\big]\\ &\leqslant \sup_{\lambda\in[0,1]}\inf_{\omega\in\mathbb{R}}\big[\lambda f(x'+(1-\lambda)y'+(\lambda\tilde{t}_2+(1-\lambda)\tilde{s}_2)u)\\ &\quad +(1-\lambda)f(x'+(1-\lambda)y'+(\lambda\tilde{t}_1+(1-\lambda)\tilde{s}_1)u)\big]\\ &\leqslant M\leqslant \lambda S_u f(x)+(1-\lambda)S_u f(y),\end{aligned} \tag{3.86}$$

where the first inequality is by choosing $\omega=\lambda\tilde{t}_2+(1-\lambda)\tilde{s}_2$ and (3.84), the second inequality is by (3.85) and the last inequality is by (3.83).

Case 3.3. The case of $t=0$ and $s>0$ (or $t>0$ and $s=0$). In this case, there exists t_0 such that

$$S_u f(x)=\lim_{t\to t_0,\, f(x+tu)<+\infty} f(x+tu). \tag{3.87}$$

We may assume that

$$f(x+t_0 u)=\lim_{t\to t_0,\, f(x+tu)<+\infty} f(x+tu). \tag{3.88}$$

In the proof of Case 3.2, let $\tilde{t}_1=\tilde{t}_2=t_0$, we can get the required inequality by the process of proof of Case 3.2. □

THEOREM 3.3. *Let* $f:\mathbb{R}^n\to\mathbb{R}\cup\{+\infty\}$ *be a coercive convex function and* $u\in S^{n-1}$*, then*

$$\int_{\mathbb{R}^n} e^{-(S_uf)(x)}dx=\int_{\mathbb{R}^n} e^{-f(x)}dx. \tag{3.89}$$

Proof. By (3.60), for any $t\in\mathbb{R}$, we have

$$\mathrm{cl}[S_uf<t]=S_{u^\perp}(\mathrm{cl}[f<t]). \tag{3.90}$$

Since Steiner symmetrization of convex sets preserves volume, we have

$$Vol([S_uf<t])=Vol([f<t]). \tag{3.91}$$

By (3.91) and Fubini's theorem, we have

$$\begin{aligned}\int_{\mathbb{R}^n} e^{-(S_uf)(x)}dx&=\int_{\mathbb{R}^n}\left[\int_{(S_uf)(x)}^{+\infty}e^{-t}dt\right]dx\\&=\int_{\mathbb{R}}Vol([S_uf<t])e^{-t}dt\\&=\int_{\mathbb{R}}Vol([f<t])e^{-t}dt=\int_{\mathbb{R}^n}e^{-f(x)}dx.\end{aligned} \tag{3.92}$$

□

We say that a function $f:\mathbb{R}^n\mapsto\mathbb{R}\cup\{+\infty\}$ is *unconditional* if

$$f(x_1,\cdots,x_n)=f(|x_1|,\cdots,|x_n|)\text{ for every }(x_1,\cdots,x_n)\in\mathbb{R}^n. \tag{3.93}$$

THEOREM 3.4. *Any coercive convex function* $f:\mathbb{R}^n\to\mathbb{R}\cup\{+\infty\}$ *can be transformed into an unconditional function* $\overline{f}$ *using* n *Steiner symmetrizations.*

We first prove the following lemma. In the lemma, $\langle u_1,u_2\rangle$ denotes the inner product of unit vectors u_1 and u_2.

LEMMA 3.4. *Let* $u_1,u_2\in S^{n-1}$ *and* $\langle u_1,u_2\rangle=0$*. If* $f:\mathbb{R}^n\to\mathbb{R}\cup\{+\infty\}$ *is a coercive convex function and* f *is symmetric about* $u_1^\perp$*, then* $S_{u_2}f$ *is symmetric about both* $u_1^\perp$ *and* $u_2^\perp$*.*

Proof. By Theorem 3.2, $S_{u_2}f$ is symmetric about $u_2^\perp$. Next, we prove that $S_{u_2}f$ is symmetric about $u_1^\perp$. Since $\langle u_1,u_2\rangle=0$, then $u_1\in u_2^\perp$ and $u_2\in u_1^\perp$. For any $x'\in u_1^\perp$, let $x'=x''+t_{x'}u_2$, where $x''=x'|u_2^\perp$. Then $x''=x'-t_{x'}u_2\in u_1^\perp$, thus $x''+tu_2\in u_1^\perp$. Because that $x''\in u_2^\perp$ and $u_1\in u_2^\perp$, thus $x''+tu_1\in u_2^\perp$. Thus, for any $x'\in u_1^\perp$ and $t\in\mathbb{R}$, we have

$$\begin{aligned}&(S_{u_2}f)(x'+tu_1)=(S_{u_2}f)(x''+tu_1+t_{x'}u_2)\\&=\sup_{\lambda\in[0,1]}\inf_{t_1+t_2=t_{x'}}[\lambda f(x''+tu_1+2t_1u_2)+(1-\lambda)f(x''+tu_1-2t_2u_2)]\\&=\sup_{\lambda\in[0,1]}\inf_{t_1+t_2=t_{x'}}[\lambda f(x''-tu_1+2t_1u_2)+(1-\lambda)f(x''-tu_1-2t_2u_2)]\\&=(S_{u_2}f)(x''-tu_1+t_{x'}u_2)=(S_{u_2}f)(x'-tu_1),\end{aligned} \tag{3.94}$$

where the second equality is by f is symmetric about $u_1^{\perp}$ and $x''+tu_2 \in u_1^{\perp}$. This completes the proof. $\square$

Now we prove Theorem 3.4.

Proof. Let $\{u_1,\dots,u_n\}$ be an orthonormal basis of $\mathbb{R}^n$. By Theorem 3.2 and Lemma 3.4, $S_{u_n}\cdots S_{u_1}f$ is symmetric about $u_i^{\perp}$, $i=1,\cdots,n$, which implies that f can be transformed into an unconditional function

$$\bar{f} = S_{u_n}\cdots S_{u_1}f \tag{3.95}$$

using n Steiner symmetrizations. $\square$

THEOREM 3.5. *Let $f_1:\mathbb{R}^n \to \mathbb{R}\cup\{+\infty\}$ and $f_2:\mathbb{R}^n \to \mathbb{R}\cup\{+\infty\}$ be coercive convex functions and $u\in S^{n-1}$. If $f_1 \leqslant f_2$ (which implies that $f_1(x)\leqslant f_2(x)$ for any $x\in\mathbb{R}^n$), then $S_uf_1 \leqslant S_uf_2$.*

Proof. By Definition 3.1, for $x\in\mathbb{R}^n$, let $x=x'+tu$, where $x'\in u^{\perp}$, we have

$$\begin{aligned} S_uf_1(x) &= \sup_{\lambda\in[0,1]}\inf_{t_1+t_2=t}[\lambda f_1(x'+2t_1u)+(1-\lambda)f_1(x'-2t_2u)] \\ &\leqslant \sup_{\lambda\in[0,1]}\inf_{t_1+t_2=t}[\lambda f_2(x'+2t_1u)+(1-\lambda)f_2(x'-2t_2u)] \\ &= S_uf_2(x), \end{aligned} \tag{3.96}$$

where the inequality is by $f_1(x)\leqslant f_2(x)$ for any $x\in\mathbb{R}^n$. $\square$

We say a function f is even about point $z\in\mathbb{R}^n$ if $f(z+x)=f(z-x)$ for any $x\in\mathbb{R}^n$. Let $z|H$ denote the projection of z onto hyperplane H.

THEOREM 3.6. *Let $f:\mathbb{R}^n\to\mathbb{R}\cup\{+\infty\}$ be a coercive convex function and $u\in S^{n-1}$, if f is even about z, then S_uf is even about $z|u^{\perp}$.*

Proof. For any $x\in\mathbb{R}^n$, let $x=x'+tu$, where $x'=x|u^{\perp}$. Let $z=z'-t_0u$, where $z'=z|u^{\perp}$. By Definition 3.1, we have

$$\begin{aligned} &(S_uf)(z'+x) \\ &= (S_uf)(z'+x'+tu) \\ &= (S_uf)(z'+x'-tu) \\ &= \sup_{\lambda\in[0,1]}\inf_{t_1+t_2=-t}[\lambda f(z'+x'+2t_1u)+(1-\lambda)f(z'+x'-2t_2u)] \\ &= \sup_{\lambda\in[0,1]}\inf_{t_2\in\mathbb{R}}[\lambda f(z+t_0u+x'-2t_2u-2tu)+(1-\lambda)f(z+t_0u+x'-2t_2u)] \\ &= \sup_{\lambda\in[0,1]}\inf_{t_2\in\mathbb{R}}[\lambda f(z+x'-2t_2u-2tu)+(1-\lambda)f(z+x'-2t_2u)] \\ &= \sup_{\lambda'\in[0,1]}\inf_{t_2\in\mathbb{R}}[\lambda' f(z+x'-2t_2u)+(1-\lambda')f(z+x'-2t_2u-2tu)], \end{aligned} \tag{3.97}$$

where the second equality is by $S_u f$ is symmetric about $u^\perp$ and the fifth equality is by replacing $t_0 - 2t_2$ by $-2t_2$.

On the other hand, since f is even about z, we have

$$\begin{aligned}
&(S_u f)(z' - x) \\
=& (S_u f)(z' - x' - tu) \\
=& \sup_{\lambda\in[0,1]} \inf_{t_1+t_2=-t} [\lambda f(z' - x' + 2t_1 u) + (1-\lambda) f(z' - x' - 2t_2 u)] \\
=& \sup_{\lambda\in[0,1]} \inf_{t_1\in\mathbb{R}} [\lambda f(z + t_0 u - x' + 2t_1 u) + (1-\lambda) f(z + t_0 u - x' + 2t_1 u + 2tu)] \\
=& \sup_{\lambda\in[0,1]} \inf_{t_1\in\mathbb{R}} [\lambda f(z - x' + 2t_1 u) + (1-\lambda) f(z - x' + 2t_1 u + 2tu)] \\
=& \sup_{\lambda\in[0,1]} \inf_{t_1\in\mathbb{R}} [\lambda f(z + x' - 2t_1 u) + (1-\lambda) f(z + x' - 2t_1 u - 2tu)],
\end{aligned} \tag{3.98}$$

where the last equality is by f is even about z. By (3.97) and (3.98), we have $(S_u f)(z' + x) = (S_u f)(z' - x)$ for any $x \in \mathbb{R}^n$. □

4. The relation between new definition and former definitions

4.1. The relation between Definition 3.1 and Definition 1.2

First, look at some basic concepts. A function $f : \mathbb{R}^n \to [0,\infty)$ is called *log-concave* if $f = e^{-\phi}$, where ϕ is a convex function defined on $\mathbb{R}^n$. Given two functions $f, g : \mathbb{R}^n \to [0,\infty)$, their *Asplund product* is defined by

$$(f \star g)(x) = \sup_{x_1+x_2=x} f(x_1) g(x_2). \tag{4.1}$$

For $\lambda \in \mathbb{R}$, we define the λ*-homothety* of a function $f : \mathbb{R}^n \to [0,\infty)$, denoted by $\lambda \cdot f$, as

$$(\lambda \cdot f)(x) = f^\lambda\left(\frac{x}{\lambda}\right). \tag{4.2}$$

By the above concepts, Defintion 1.2 can be transformed into the following form.

DEFINITION 4.1. For a log-concave function $F = e^{-f}$, where $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is a convex function, and a hyperplane $H = u^\perp$ in $\mathbb{R}^n$, for any $x \in \mathbb{R}^n$, if $x = x' + tu$ and $x' \in u^\perp$, we define its Steiner symmetrization about $u^\perp$ by

$$\left(\widetilde{S}_u F\right)(x' + tu) = \exp\left\{ - \inf_{t_1+t_2=t} \left[\frac{1}{2} f(x' + 2t_1 u) + \frac{1}{2} f(x' - 2t_2 u)\right]\right\}. \tag{4.3}$$

If we define Steiner symmetrization for coercive convex function f using the above definition, we have

$$\left(\widetilde{S}_u f\right)(x' + tu) = \inf_{t_1+t_2=t} \left[\frac{1}{2} f(x' + 2t_1 u) + \frac{1}{2} f(x' - 2t_2 u)\right]. \tag{4.4}$$

Thus, $S_u f$ given in Definition 3.1 is in general larger than $\widetilde{S}_u f$. Look at the following example.

EXAMPLE 1. For one-dimensional coercive convex function

$$f(x)=\begin{cases} x^3 & \text{if } x\geqslant 0,\\ x^2 & \text{if } x\leqslant 0.\end{cases} \tag{4.5}$$

Compare Sf and $\widetilde{S}f$, where

$$Sf(x)=\sup_{\lambda\in[0,1]}\inf_{x_1+x_2=x}[\lambda f(2x_1)+(1-\lambda)f(-2x_2)]$$

and

$$\widetilde{S}f(x)=\inf_{x_1+x_2=x}\left[\frac{1}{2}f(2x_1)+\frac{1}{2}f(-2x_2)\right].$$

By calculation, we can get that

$$\widetilde{S}f(x)=\begin{cases} \dfrac{(-12x-1)\sqrt{1+12x}+18x+1}{27}+2x^2 & \text{if } x\geqslant 0,\\ \dfrac{(12x-1)\sqrt{1-12x}-18x+1}{27}+2x^2 & \text{if } x\leqslant 0.\end{cases} \tag{4.6}$$

and

$$Sf(x)=g^{-1}(|x|), \tag{4.7}$$

where g^{-1} is the inverse function of

$$g(x)=\frac{1}{2}(\sqrt[3]{x}+\sqrt{x}),\ \ x\in[0,\infty). \tag{4.8}$$

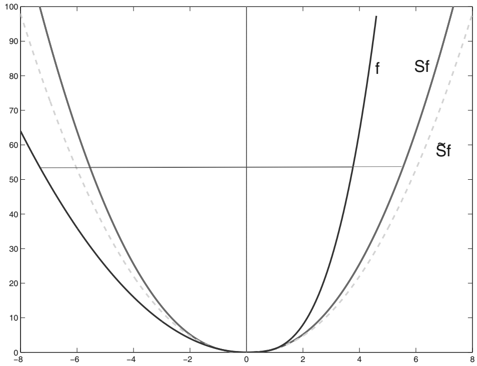

Figure 1: f, Sf and $\widetilde{S}f$.

By Matlab, we can draw their figures (see Figure 1). From the figure, we can find that the level sets of Sf and f have the same size and $Sf>\widetilde{S}f$.

REMARK 4.1. (i) For one-dimensional coercive convex function $f:\mathbb{R}\to\mathbb{R}\cup\{+\infty\}$, if f is symmetric about an axes $x=x_0$, i.e., $f(x_0-x)=f(x_0+x)$ for any $x\in\mathbb{R}$, then $Sf=\widetilde{S}f$.

(ii) For n-dimensional coercive convex function $f:\mathbb{R}^n\to\mathbb{R}\cup\{+\infty\}$ and $u\in S^{n-1}$, if for any $x'\in u^{\perp}$, one-dimensional function $f(x'+tu)$ about $t\in\mathbb{R}$ is symmetric about an axes $t=t_0$, then $S_uf=\widetilde{S}_uf$.

4.2. The relation between Definition 3.1 and Definition 1.1

In this section, we show that the two definitions is same for log-concave function (Theorem 4.1) and properties 6 and 7 in Table 1 are established (Theorem 4.2 and Theorem 4.3). First, look at some basic concepts.

DEFINITION 4.2. For function $f:E\to\mathbb{R}\cup\{-\infty,+\infty\}$, where $E\subset\mathbb{R}^n$ is a Lebesgue measurable set, if for any $t\in\mathbb{R}$,

$$\{x\in E: f(x)>t\} \tag{4.9}$$

is Lebesgue measurable, then f is said to be *Lebesgue measurable*.

We say that a non-negative measurable function f *vanishes at infinity* if

$$m([f>t])<+\infty \tag{4.10}$$

for all $t>0$, where $m([f>t])$ denotes Lebesgue measure of level set $\{x\in\mathbb{R}^n: f(x)>t\}$.

For a convex function $f:\mathbb{R}^n\to\mathbb{R}\cup\{+\infty\}$, the *subgraph* of f is defined as

$$\mathrm{sub}f:=\{(x,y)\in\mathbb{R}^{n+1}: x\in\mathrm{dom}f,\ y\leqslant f(x)\}. \tag{4.11}$$

For a convex function $f:\mathbb{R}^n\to\mathbb{R}\cup\{+\infty\}$, the *effective domain* of f is defined as

$$\mathrm{dom}f:=\{x\in\mathbb{R}^n: f(x)\neq+\infty\}. \tag{4.12}$$

LEMMA 4.1. *Let $f:\mathbb{R}^n\to\mathbb{R}\cup\{+\infty\}$ be a coercive convex function, if* $\mathrm{dom}f=\mathbb{R}^n$*, then log-concave function $F=e^{-f}$ is a nonnegative measurable function on $\mathbb{R}^n$ vanishes at infinity.*

Proof. It is clear that F is nonnegative and $F>0$. For any $t\in\mathbb{R}$, if $t\leqslant 0$, then

$$\{x\in\mathbb{R}^n: F(x)>t\}=\mathbb{R}^n \tag{4.13}$$

is measurable. If $t>0$, since that convex function is continuous in the interior of its effective domain, f is continuous on $\mathbb{R}^n$, thus

$$\{x\in\mathbb{R}^n: F(x)>t\}=\{x\in\mathbb{R}^n: f(x)<-\ln t\} \tag{4.14}$$

is an open set. Because that any open set is measurable, thus F is measurable. Since f is coercive, set $\{x \in \mathbb{R}^n : f(x) < -\ln t\}$ is bounded for any $t > 0$, by (4.14) we have

$$m(\{x \in \mathbb{R}^n : F(x) > t\}) < +\infty \tag{4.15}$$

for any $t > 0$, thus F is vanishes at infinity. □

LEMMA 4.2. *Let $F = e^{-f}$ be a log-concave function, where $f : \mathbb{R}^n \to \mathbb{R}$ is a coercive convex function, then*

$$[\bar{S}_u F > t] = S_{u^\perp}([F > t]) \tag{4.16}$$

where $\bar{S}_u F$ is given in Definition 1.1 and $S_u([F > t])$ is given in Definition 2.1.

Proof. By Definition 1.1, if $\bar{S}_u F(x) > t$, then $x \in S_u([F > t])$. On the other hand, if $x \in S_u([F > t])$, since $S_u([F > t])$ is an open set and F is continuous, then there exists $t' > t$ such that $x \in S_u([F > t'])$, by (1.1), we have $\bar{S}_u F(x) > t$. □

THEOREM 4.1. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a coercive convex function and $u \in S^{n-1}$, then*

$$e^{(-S_u f)} = \bar{S}_u(e^{-f}), \tag{4.17}$$

where $S_u f$ and $\bar{S}_u(e^{-f})$ are given in (1.3) and (1.1), respectively.

Proof. For $t > 0$, we have

$$[e^{(-S_u f)} > t] = [S_u f < -\ln t] = S_u([f < -\ln t]) = S_u([e^{-f} > t]), \tag{4.18}$$

where the second equality uses equality (3.60).

By Lemma 4.2, we have

$$[\bar{S}_u(e^{-f}) > t] = S_u([e^{-f} > t]). \tag{4.19}$$

By (4.18) and (4.19), we have

$$[e^{(-S_u f)} > t] = [\bar{S}_u(e^{-f}) > t]. \tag{4.20}$$

Using the "layer-cake representation" and (4.20), we have

$$e^{(-S_u f)} = \int_0^\infty \mathscr{X}_{[e^{(-S_u f)} > t]}(x)dt = \int_0^\infty \mathscr{X}_{[\bar{S}_u(e^{-f}) > t]}(x)dt = \bar{S}_u(e^{-f}). \tag{4.21}$$

□

Next, we will give two theorems, which are corresponding to Proposition 3 and Theorem 2 in [9], respectively. They show the continuity and convergence of Steiner symmetrization in L^p space, which are corresponding to the properties 6-7 in Table 1.

We first give the concept of rearrangements of sets and functions. Our presentation follows that of [9]. Let $\mathscr{M}$ denote the sigma-algebra consisting of all Lebesgue measurable subsets of $\mathbb{R}^n$ and m denote the corresponding Lebesgue measure.

DEFINITION 4.3. A *rearrangement* T is a map $T : \mathscr{M} \to \mathscr{M}$ that is both monotone ($A \subset B$ implies $T(A) \subset T(B)$) and measure preserving ($m(T(A)) = m(A)$ for all $A \in \mathscr{M}$). For a nonnegative measurable function f, if it vanishes at infinity, then we define its *rearrangement* Tf by using the "layer cake principle"

$$Tf(x) = \int_0^\infty \mathscr{X}_{T([f>t])}(x)dt. \tag{4.22}$$

By the definition, we can get functional rearrangement using rearrangement of sets.

For any $A \in \mathscr{M}$, there exists a unique open ball centered at the origin A^* with the same measure as A. A^* is called the *Schwarz symmetrization* of A. For a nonnegative measurable function f, if it vanishes at infinity then its rearrangement with respect to the Schwarz rearrangement, denoted by $f^*(x)$, is called the *symmetrization decreasing rearrangement*.

PROPOSITION 1. For a coercive convex function $f : \mathbb{R}^n \to \mathbb{R}$, we have

$$(e^{-f})^*(x) = e^{-g(|x|)}, \tag{4.23}$$

where $g(t)$ is an increasing convex function defined on $[0,+\infty)$.

Proof. For a coercive convex function $f : \mathbb{R}^n \to \mathbb{R}$, by Lemma 4.1, log-concave function $F = e^{-f}$ is a nonnegative measurable function on $\mathbb{R}^n$ vanishes at infinity. By (4.22), we have

$$\begin{aligned} F^*(x) = \int_0^\infty \mathscr{X}_{[F>t]^*}(x)dt &= \int_0^\infty \mathscr{X}_{[f<-\ln t]^*}(x)dt \\ &= \sup\{t : x \in [f < -\ln t]^*\}. \end{aligned} \tag{4.24}$$

If $|x| = 0$, then

$$\begin{aligned} F^*(0) = \sup\{t : [f < -\ln t]^* \neq \emptyset\} &= \sup\{t : \inf f < -\ln t\} \\ &= \sup\{t : t < e^{-\inf f}\} \\ &= e^{-\inf f}. \end{aligned} \tag{4.25}$$

If $|x| \neq 0$, let κ_n denote the volume of Euclidean unit ball, by (4.24) we have

$$F^*(x) = \sup\left\{ t : |x| < \left(\frac{Vol([f < -\ln t])}{\kappa_n} \right)^{1/n} \right\}. \tag{4.26}$$

Since $Vol([f < -\ln t])$ is decreasing about $t \in (0,+\infty)$, thus by (4.26) $F^*(x)$ equals t satisfying

$$|x| = \left(\frac{Vol([f < -\ln t])}{\kappa_n} \right)^{1/n}. \tag{4.27}$$

Let $s = -\ln t$, then $t = e^{-s}$, then $F^*(x)$ equals e^{-s}, where s satisfies

$$|x| = \left(\frac{Vol([f < s])}{\kappa_n}\right)^{1/n}. \tag{4.28}$$

Define one-dimensional function $h(s)$ about $s \in [\inf f, +\infty)$ as

$$h(s) = \left(\frac{Vol([f < s])}{\kappa_n}\right)^{1/n}. \tag{4.29}$$

Since $f : \mathbb{R}^n \to \mathbb{R}$ is convex function, thus epi$f$ is closed convex set. By Brunn's Theorem, $h(s)$ is a concave function. It is clear that $h(s)$ is increasing on $[\inf f, +\infty)$, thus its inverse function g is an increasing convex function defined on $[0, +\infty)$. Thus, for any $x \in \mathbb{R}^n$, by (4.25) and (4.27), we have $F^*(x) = e^{-s} = e^{-g(|x|)}$. □

If $u \in S^{n-1}$ and $A \in \mathscr{M}$ then, by Fubini's theorem, the set

$$A_{x'} = \{t \in \mathbb{R} : x' + tu \in A\} \tag{4.30}$$

is measurable in $\mathbb{R}$ for almost every $x' \in u^{\perp}$. If we let $A^*_{x'}$ equal the empty set whenever $A_{x'}$ is non-measurable then we denote by

$$S_u(A) = \bigcup_{x' \in u^{\perp}} \{x' + uA^*_{x'}\} \tag{4.31}$$

the *Steiner symmetrization* of A with respect to u. The rearrangement of f with respect to the Steiner symmetrization, denote by $S_u f$, is called the *Steiner symmetrization*.

For a non-empty convex set $K \subset \mathbb{R}^n$, its Steiner symmetrization in Definition 2.1 is not same with that defined by (4.31) in the boundary, but $S_u f$ is same for the two definitions of the Steiner symmetrization of convex sets.

Let $L^p(\mathbb{R}^n)$ $(1 \leqslant p < \infty)$ denote the L^p space; i.e., the set of all measurable functions defined on $\mathbb{R}^n$ satisfying

$$\|f\|_p = \left(\int_{\mathbb{R}^n} |f(x)|^p dx\right)^{1/p} < \infty. \tag{4.32}$$

The L^p distance of two function f, $g \in L^p(\mathbb{R})$ is defined as

$$\|f - g\|_p = \left(\int_{\mathbb{R}^n} |f(x) - g(x)|^p dx\right)^{1/p}. \tag{4.33}$$

THEOREM 4.2. ([9, Proposition 3]) *If $1 \leqslant p < \infty$ and T is any rearrangement, then*

$$\|T(f) - T(g)\|_p \leqslant \|f - g\|_p \tag{4.34}$$

for any $(f, g) \in L^p(\mathbb{R}^n) \times L^p(\mathbb{R}^n)$.

Let $S_{u_1,\cdots,u_i}$ denote $S_{u_i}S_{u_{i-1}}\cdots S_{u_1}f$.

THEOREM 4.3. ([9, Theorem 2]) *Let $\{u_n\}_{n=1}^{\infty}\subset S^{n-1}$ be a dense subset of S^{n-1} with respect to the Euclidean metric. If $f\in L^p(\mathbb{R}^n)$ and*

$$f_{n+1}=\begin{cases} S_{u_1}(f), & n=0,\\ S_{u_1,\ldots,u_{n+1}}(f_n), & n\geqslant 1\end{cases} \tag{4.35}$$

then f_n converges to f^* in the L^p distance.

Acknowledgement. I am indebted to the referees for their careful reading and very valuable comments for improvements.

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(Received March 31, 2017)

Lin Youjiang
School of Mathematics and Statistics
Chongqing Technology and Business University
Chongqing 400067, PR China
e-mail: `lxyoujiang@126.com`