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NON–ARCHIMEDEAN HYPERSTABILITY OF A CAUCHY–JENSEN TYPE FUNCTIONAL EQUATION

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Abstract. In this paper, we establish some hyperstability results concerning the following Cauchy - Jensen functional equation

$$f\left(\frac{x+y}{2}\right)+f\left(\frac{x-y}{2}\right)=f(x)$$

in Non-Archimedean normed spaces.

1. Introduction

The starting point of studying the stability of functional equations seems to be the famous talk of Ulam [33] in 1940, in which he discussed a number of important unsolved problems. Among those was the question concerning the stability of group homomorphisms.

Let G_1 be a group and let G_2 be a metric group with a metric $d(.,.)$. Given $\varepsilon > 0$, does there exists a $\delta > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality $d\big(h(xy), h(x)h(y)\big) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d\big(h(x), H(x)\big) < \varepsilon$ for all $x \in G_1$.

The first partial answer, in the case of Cauchy's equation in Banach spaces, to Ulam's question was given by Hyers [23]. Later, the result of Hyers was significantly generalized by Aoki [6], Rassias [31] and Găvruţa [20]. Since then, the stability problems of several functional equations have been extensively investigated.
We say a functional equation is *hyperstable* if any function f satisfying the equation approximately (in some sense) must be actually solutions to it. It seems that the first hyperstability result was published in [12] and concerned the ring homomorphisms. However, The term *hyperstability* has been used for the first time in [27]. Quite often the hyperstability is confused with superstability, which admits also bounded functions. Numerous papers on this subject have been published and we refer to [1]-[5], [11], [14]-[19], [22], [27], [30], [32].

Throughout this paper, $\mathbb{N}$ stands for the set of all positive integers, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, N_{m_0} the set of integers $\geqslant m_0$, $\mathbb{R}_+ := [0,\infty)$ and we use the notation X_0 for the set

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$X \setminus \{0\}$.

Let us recall (see, for instance, [26]) some basic definitions and facts concerning non-Archimedean normed spaces.

DEFINITION 1. By a *non-Archimedean* field we mean a field $\mathbb{K}$ equipped with a function (*valuation*) $|\cdot| : \mathbb{K} \to [0,\infty)$ such that for all $r,s \in \mathbb{K}$, the following conditions hold:

1. $|r| = 0$ if and only if $r = 0$,
2. $|rs| = |r||s|$,
3. $|r+s| \leqslant \max\{|r|,|s|\}$.

The pair $(\mathbb{K},|.|)$ is called a *valued field*.

In any non-Archimedean field we have $|1| = |-1| = 1$ and $|n| \leqslant 1$ for $n \in \mathbb{N}_0$. In any field $\mathbb{K}$ the function $|\cdot| : \mathbb{K} \to \mathbb{R}_+$ given by

$$|x| := \begin{cases} 0, & x = 0, \\ 1, & x \neq 0, \end{cases}$$

is a valuation which is called *trivial*, but the most important examples of non-Archimedean fields are p-adic numbers which have gained the interest of physicists for their research in some problems coming from quantum physics, p-adic strings and superstrings.

DEFINITION 2. Let X be a vector space over a scalar field $\mathbb{K}$ with a non-Archimedean non-trivial valuation $|\cdot|$. A function $||\cdot||_* : X \to \mathbb{R}$ is a *non-Archimedean norm (valuation)* if it satisfies the following conditions:

1. $||x||_* = 0$ if and only if $x = 0$,
2. $||rx||_* = |r|\ ||x||_*$ $(r \in \mathbb{K}, x \in X)$,
3. The strong triangle inequality (ultrametric); namely

$$||x+y||_* \leqslant \max\{||x||_*, ||y||_*\} \quad x,y \in X.$$

Then $(X, ||\cdot||_*)$ is called *a non-Archimedean normed space* or *an ultrametric normed space*.

DEFINITION 3. Let $\{x_n\}$ be a sequence in a non-Archimedean normed space X.

1. A sequence $\{x_n\}_{n=1}^{\infty}$ in a non-Archimedean space is *a Cauchy sequence* iff the sequence $\{x_{n+1} - x_n\}_{n=1}^{\infty}$ converges to zero;
2. The sequence $\{x_n\}$ is said to be *convergent* if, there exists $x \in X$ such that, for any $\varepsilon > 0$, there is a positive integer N such that $||x_n - x||_* \leqslant \varepsilon$, for all $n \geqslant N$. Then the point $x \in X$ is called the *limit* of the sequence $\{x_n\}$, which is denoted by $lim_{n\to\infty} x_n = x$;

3. If every Cauchy sequence in X converges, then the non-Archimedean normed space X is called a *non-Archimedean Banach space* or an *ultrametric Banach space*.

Let X, Y be normed spaces. A function $f: X \to Y$ is Cauchy-Jensen provided it satisfies the functional equation

$$f\left(\frac{x+y}{2}\right)+f\left(\frac{x-y}{2}\right)=f(x) \quad \text{for all } x,y \in X, \tag{1}$$

and we can say that $f: X \to Y$ is Cauchy-Jensen on X_0 if it satisfies (1) for all $x,y \in X_0$. Recently, interesting results concerning the Cauchy-Jensen functional equation (1) have been obtained in [7], [21], [25], [28] and [29].

In 2013, A. Bahyrycz and al. [8] used the fixed point theorem from [18, Theorem 1] to prove the stability results for a generalization of p-Wright affine equation in Non-Archimedean spaces. Recently, corresponding results for more general functional equations (in classical spaces) have been proved in [9], [10], [14], [34] and [35].

In this paper, we make Non-Archimedean versions of results in [2]. Indeed, by using the fixed point method derived from [11], [14] and [13], we present some hyperstability results for the equation (1) in Non-Archimedean Banach spaces. Before proceeding to the main results, we state Theorem 1 which is useful for our purpose. To present it, we introduce the following three hypotheses:

(**H1**) X is a nonempty set, Y is an Non-Archimedean Banach space over a non-Archimedean field, $f_1,\ldots,f_k : X \longrightarrow X$ and $L_1,\ldots,L_k : X \longrightarrow \mathbb{R}_+$ are given.

(**H2**) $\mathcal{T}: Y^X \longrightarrow Y^X$ is an operator satisfying the inequality

$$\left\|\mathcal{T}\xi(x)-\mathcal{T}\mu(x)\right\|_* \leqslant \max_{1\leqslant i\leqslant k}\left\{L_i(x)\left\|\xi\left(f_i(x)\right)-\mu\left(f_i(x)\right)\right\|_*\right\},\ \xi,\mu \in Y^X,\ x \in X.$$

(**H3**) $\Lambda : \mathbb{R}_+^X \longrightarrow \mathbb{R}_+^X$ is a linear operator defined by

$$\Lambda\delta(x) := \max_{1\leqslant i\leqslant k}\left\{L_i(x)\delta\left(f_i(x)\right)\right\}, \qquad \delta \in \mathbb{R}_+^X, \qquad x \in X.$$

Thanks to a result due to J. Brzdęk and K. Ciepliñski [18, Remark 2], we state a slightly modified version of the fixed point theorem [17, Theorem 1] in Non-Archimedean spaces. We use it to assert the existence of a fixed point of operator $\mathcal{T}: Y^X \longrightarrow Y^X$.

THEOREM 1. *Let hypotheses (**H1**)-(**H3**) be valid and functions $\varepsilon : X \longrightarrow \mathbb{R}_+$ and $\varphi : X \longrightarrow Y$ fulfil the following two conditions*

$$\|\mathcal{T}\varphi(x)-\varphi(x)\|_* \leqslant \varepsilon(x), \qquad x \in X,$$
$$\lim_{n\to\infty}\Lambda^n\varepsilon(x)=0, \qquad x \in X.$$

Then there exists a fixed point $\psi \in Y^X$ *of* $\mathscr{T}$ *with*

$$\|\varphi(x) - \psi(x)\|_* \leqslant \sup_{n\in\mathbb{N}_0} \Lambda^n \varepsilon(x), \qquad x \in X.$$

Moreover

$$\psi(x) := \lim_{n\to\infty} \mathscr{T}^n \varphi(x), \qquad x \in X.$$

2. Main results

In this section, using Theorem 1 as a basic tool to prove the hyperstability results of Cauchy functional equation in Non-Archimedean Banach spaces.

THEOREM 2. *Let* $(X, \|\cdot\|)$ *and* $(Y, \|\cdot\|_*)$ *be normed space and Non-Archimedean Banach space respectively,* $c \geqslant 0$, $p, q \in \mathbb{R}$, $p+q < 0$ *and let* $f : X \to Y$ *satisfy*

$$\left\| f\left(\frac{x+y}{2}\right) + f\left(\frac{x-y}{2}\right) - f(x) \right\|_* \leqslant c\,\|x\|^p\,\|y\|^q, \tag{2}$$

for all $x, y \in X_0$. *Then* f *is Cauchy-Jensen on* X_0.

Proof. Take $m \in \mathbb{N}$ such that

$$\alpha_m := \left|\frac{1-m}{2}\right|^{p+q} < 1 \text{ and } m \geqslant m_0.$$

Since $p+q<0$, one of p,q must be negative. Assume that $q<0$ and replace y by mx in (2). Thus

$$\left\| f\left(\left(\frac{1+m}{2}\right)x\right) + f\left(\left(\frac{1-m}{2}\right)x\right) - f(x) \right\|_* \leqslant cm^q \|x\|^{p+q}, \quad x \in X_0. \tag{3}$$

Define operators $\mathscr{T}_m : Y^{X_0} \to Y^{X_0}$ and $\Lambda_m : \mathbb{R}_+^{X_0} \to \mathbb{R}_+^{X_0}$ by

$$\mathscr{T}_m \xi(x) := \xi\left(\left(\frac{1+m}{2}\right)x\right) + \xi\left(\left(\frac{1-m}{2}\right)x\right), \quad \xi \in Y^{X_0},\, x \in X_0, \tag{4}$$

$$\Lambda_m \delta(x) := \max\left\{ \delta\left(\left(\frac{1+m}{2}\right)x\right), \delta\left(\left(\frac{1-m}{2}\right)x\right)\right\}, \quad \delta \in \mathbb{R}_+^{X_0},\, x \in X_0 \tag{5}$$

and write

$$\varepsilon_m(x) := c\, m^q \|x\|^{p+q}, \quad x \in X_0. \tag{6}$$

It is easily seen that Λ_m has the form described in (**H3**) with $k=2$, $f_1(x) = \left(\frac{1+m}{2}\right)x$, $f_2(x) = \left(\frac{1-m}{2}\right)x$ and $L_1(x) = 1$, $L_2(x) = 1$. Further, (3) can be written in the following way

$$\|\mathscr{T}_m f(x) - f(x)\|_* \leqslant \varepsilon_m(x), \quad x \in X_0.$$

Moreover, for every $\xi,\mu \in Y^{X_0}$, $x \in X_0$

$$
\begin{aligned}
\left\| \mathscr{T}_m\xi(x) - \mathscr{T}_m\mu(x)\right\|_* &= \left\| \xi\left(\left(\frac{1+m}{2}\right)x\right) + \xi\left(\left(\frac{1-m}{2}\right)x\right) \right. \\
&\quad \left. -\mu\left(\left(\frac{1+m}{2}\right)x\right) - \mu\left(\left(\frac{1-m}{2}\right)x\right)\right\|_* \\
&\leqslant \max\left\{ \left\| \xi\left(\left(\frac{1+m}{2}\right)x\right) - \mu\left(\left(\frac{1+m}{2}\right)x\right)\right\|_*, \right. \\
&\quad \left. \left\| \xi\left(\left(\frac{1-m}{2}\right)x\right) - \mu\left(\left(\frac{1-m}{2}\right)x\right)\right\|_* \right\}.
\end{aligned}
$$

So, (**H2**) is valid.

By using mathematical induction, we will show that for each $x \in X_0$ we have

$$
\Lambda_m^n \varepsilon_m(x) = c\, m^q \|x\|^{p+q}\, \alpha_m^n \tag{7}
$$

where $\alpha_m = \left|\frac{1-m}{2}\right|^{p+q}$. From (6), we obtain that (7) holds for $n=0$. Next, we will assume that (7) holds for $n=k$, where $k \in \mathbb{N}$. Then we have

$$
\begin{aligned}
\Lambda_m^{k+1}\varepsilon_m(x) = \Lambda_m\left(\Lambda_m^k\varepsilon_m(x)\right) &= \max\left\{ \Lambda_m^k\varepsilon_m\left(\left(\frac{1+m}{2}\right)x\right), \Lambda_m^k\varepsilon_m\left(\left(\frac{1-m}{2}\right)x\right)\right\} \\
&= c\, m^q \|x\|^{p+q}\, \alpha_m^k \max\left\{ \left|\frac{1+m}{2}\right|^{p+q}, \left|\frac{1-m}{2}\right|^{p+q}\right\} \\
&= c\, m^q \|x\|^{p+q}\, \alpha_m^{k+1}, \quad x \in X_0.
\end{aligned}
$$

This shows that (7) holds for $n=k+1$. Now we can conclude that the inequality (7) holds for all $n \in \mathbb{N}_0$. From (7), we obtain

$$
\lim_{n\to\infty} \Lambda^n \varepsilon_m(x) = 0,
$$

for all $x \in X_0$. Hence, according to Theorem 1, there exists a solution $J_m : X_0 \to Y$ of the equation

$$
J_m(x) = J_m\left(\left(\frac{1+m}{2}\right)x\right) + J_m\left(\left(\frac{1-m}{2}\right)x\right), \quad x \in X_0 \tag{8}
$$

such that

$$
\|f(x) - J_m(x)\|_* \leqslant \sup_{n\in\mathbb{N}_0} \left\{ c\, m^q \|x\|^{p+q}\, \alpha_m^n \right\}, \quad x \in X_0. \tag{9}
$$

Moreover,

$$
J_m(x) := \lim_{n\to\infty} \mathscr{T}_m^n f(x)
$$

for all $x \in X_0$. Now we show that

$$\left\| \mathscr{T}_m^n f\left(\frac{x+y}{2}\right) + \mathscr{T}_m^n f\left(\frac{x-y}{2}\right) - \mathscr{T}_m^n f(x) \right\|_* \leqslant c\, \alpha_m^n \|x\|^p \, \|y\|^q, \tag{10}$$

for every $x, y \in X_0$. Since the case $n = 0$ is just (2), take $k \in \mathbb{N}$ and assume that (10) holds for $n = k$ and every $x, y \in X_0$. Then

$$\begin{aligned}
&\left\| \mathscr{T}_m^{k+1} f\left(\frac{x+y}{2}\right) + \mathscr{T}_m^{k+1} f\left(\frac{x-y}{2}\right) - \mathscr{T}_m^{k+1} f(x) \right\|_* \\
&= \left\| \mathscr{T}_m^k f\left(\left(\frac{1+m}{2}\right)\left(\frac{x+y}{2}\right)\right) + \mathscr{T}_m^k f\left(\left(\frac{1-m}{2}\right)\left(\frac{x+y}{2}\right)\right) \right. \\
&\qquad + \mathscr{T}_m^k f\left(\left(\frac{1+m}{2}\right)\left(\frac{x-y}{2}\right)\right) + \mathscr{T}_m^k f\left(\left(\frac{1-m}{2}\right)\left(\frac{x-y}{2}\right)\right) \\
&\qquad \left. - \mathscr{T}_m^k f\left(\left(\frac{1+m}{2}\right)x\right) - \mathscr{T}_m^k f\left(\left(\frac{1-m}{2}\right)x\right) \right\|_* \\
&\leqslant \max\left\{ \left\| \mathscr{T}_m^k f\left(\left(\frac{1+m}{2}\right)\left(\frac{x+y}{2}\right)\right) \right.\right. \\
&\qquad \left. + \mathscr{T}_m^k f\left(\left(\frac{1+m}{2}\right)\left(\frac{x-y}{2}\right)\right) - \mathscr{T}_m^k f\left(\left(\frac{1+m}{2}\right)x\right) \right\|_*, \\
&\qquad \left\| \mathscr{T}_m^k f\left(\left(\frac{1-m}{2}\right)\left(\frac{x+y}{2}\right)\right) + \mathscr{T}_m^k f\left(\left(\frac{1-m}{2}\right)\left(\frac{x-y}{2}\right)\right) \right. \\
&\qquad \left.\left. - \mathscr{T}_m^k f\left(\left(\frac{1-m}{2}\right)x\right) \right\|_* \right\} \\
&\leqslant \max\left\{ c\, \alpha_m^k \|x\|^p \, \|y\|^q \left|\frac{1+m}{2}\right|^{p+q}, c\, \alpha_m^k \|x\|^p \, \|y\|^q \left|\frac{1-m}{2}\right|^{p+q} \right\} \\
&= c\, \alpha_m^k \|x\|^p \, \|y\|^q \max\left\{ \left|\frac{1+m}{2}\right|^{p+q}, \left|\frac{1-m}{2}\right|^{p+q} \right\} \\
&\leqslant c\, \alpha_m^{k+1} \|x\|^p \, \|y\|^q
\end{aligned}$$

for all $x, y \in X_0$. Thus, by induction we have shown that (10) holds for every $n \in \mathbb{N}_0$. Letting $n \to \infty$ in (10), we obtain that

$$J_m\left(\frac{x+y}{2}\right) + J_m\left(\frac{x-y}{2}\right) = J_m(x),$$

for all $x, y \in X_0$. In this way we obtain a sequence $\{J_m\}_{m \geqslant m_0}$ of Cauchy-Jensen functions on X_0 such that

$$\|f(x) - J_m(x)\|_* \leqslant \sup_{n \in \mathbb{N}_0} \left\{ c\, m^q \|x\|^{p+q} \, \alpha_m^n \right\}, \quad x \in X_0,$$

this implies that

$$\|f(x)-J_m(x)\|_* \leqslant c\, m^q \|x\|^{p+q}, \quad x \in X_0,$$

It follows, with $m \to \infty$, that f is Cauchy-Jensen on X_0.

In a similar way we can prove the following theorem.

THEOREM 3. *Let $(X,\|\cdot\|)$ and $(Y,\|\cdot\|_*)$ be normed space and Non-Archimedean Banach space respectively, $c \geqslant 0$, $p,q \in \mathbb{R}$, $p+q>0$ and let $f : X \to Y$ satisfy*

$$\left\| f\left(\frac{x+y}{2}\right) + f\left(\frac{x-y}{2}\right) - f(x) \right\|_* \leqslant c\, \|x\|^p\, \|y\|^q, \tag{11}$$

for all $x,y \in X_0$. Then f is Cauchy-Jensen on X_0.

Proof. Take $m \in \mathbb{N}$ such that

$$\alpha_m := \left|\frac{m+1}{2m}\right|^{p+q} < 1 \text{ and } m \geqslant m_0.$$

Since $p+q>0$, one of p,q must be positive; let $q>0$ and replace y by $\frac{1}{m}x$ in (11). Thus

$$\left\| f\left(\left(\frac{m+1}{2m}\right)x\right) + f\left(\left(\frac{m-1}{2m}\right)x\right) - f(x) \right\|_* \leqslant c\, m^{-q}\|x\|^{p+q}, \quad x \in X_0. \tag{12}$$

Write

$$\mathcal{T}_m \xi(x) := \xi\left(\left(\frac{m+1}{2m}\right)x\right) + \xi\left(\left(\frac{m-1}{2m}\right)x\right) \quad \xi \in Y^{X_0},\, x \in X_0, \tag{13}$$

and

$$\varepsilon_m(x) := c\, m^{-q}\|x\|^{p+q}, \quad x \in X_0, \tag{14}$$

then (12) takes form

$$\|\mathcal{T}_m f(x) - f(x)\|_* \leqslant \varepsilon_m(x), \quad x \in X_0.$$

Define

$$\Lambda_m \delta(x) := \max\left\{ \delta\left(\left(\frac{m+1}{2m}\right)x\right),\, \delta\left(\left(\frac{m-1}{2m}\right)x\right) \right\}, \quad \delta \in \mathbb{R}_+^{X_0},\, x \in X_0. \tag{15}$$

Then it is easily seen that Λ_m has the form described in (**H3**) with $k=2$, $f_1(x) = \left(\frac{m+1}{2m}\right)x$, $f_2(x) = \left(\frac{m-1}{2m}\right)x$ and $L_1(x)=1$, $L_2(x)=1$.

Moreover, for every $\xi,\mu \in Y^{X_0}$, $x \in X_0$

$$\begin{aligned}\left\|\mathscr{T}_m\xi(x)-\mathscr{T}_m\mu(x)\right\|_* &= \Big\|\xi\left(\left(\frac{m+1}{2m}\right)x\right)+\xi\left(\left(\frac{m-1}{2m}\right)x\right)\\ &\quad -\mu\left(\left(\frac{m+1}{2m}\right)x\right)-\mu\left(\left(\frac{m-1}{2m}\right)x\right)\Big\|_*\\ &\leqslant \max\left\{\left\|\xi\left(\left(\frac{m+1}{2m}\right)x\right)-\mu\left(\left(\frac{m+1}{2m}\right)x\right)\right\|_*,\right.\\ &\qquad \left.\left\|\xi\left(\left(\frac{m-1}{2m}\right)x\right)-\mu\left(\left(\frac{m-1}{2m}\right)x\right)\right\|_*\right\}.\end{aligned}$$

So, (**H2**) is valid.

By using mathematical induction, we will show that for each $x\in X_0$ we have

$$\Lambda_m^n\varepsilon_m(x)= c\,m^{-q}\|x\|^{p+q}\,\alpha_m^n \tag{16}$$

where $\alpha_m=\left|\frac{m+1}{2m}\right|^{p+q}$. From (14), we obtain that (16) holds for $n=0$. Next, we will assume that (16) holds for $n=k$, where $k\in\mathbb{N}$. Then we have

$$\begin{aligned}\Lambda_m^{k+1}\varepsilon_m(x) &= \Lambda_m\left(\Lambda_m^k\varepsilon_m(x)\right)=\max\left\{\Lambda_m^k\varepsilon_m\left(\left(\frac{m+1}{2m}\right)x\right),\ \Lambda_m^k\varepsilon_m\left(\left(\frac{m-1}{2m}\right)x\right)\right\}\\ &= \max\left\{c\,m^{-q}\|x\|^{p+q}\,\alpha_m^k\left|\frac{m+1}{2m}\right|^{p+q},\ c\,m^{-q}\|x\|^{p+q}\,\alpha_m^k\left|\frac{m-1}{2m}\right|^{p+q}\right\}\\ &= c\,m^{-q}\|x\|^{p+q}\,\alpha_m^k\max\left\{\left|\frac{m+1}{2m}\right|^{p+q},\ \left|\frac{m-1}{2m}\right|^{p+q}\right\}\\ &= c\,m^{-q}\|x\|^{p+q}\,\alpha_m^{k+1},\quad x\in X_0.\end{aligned}$$

This shows that (16) holds for $n=k+1$. Now we can conclude that the inequality (16) holds for all $n\in\mathbb{N}_0$. From (16), we obtain

$$\lim_{n\to\infty}\Lambda^n\varepsilon_m(x)=0,$$

for all $x\in X_0$. Hence, according to Theorem 1, there exists a solution $J_m:X_0\to Y$ of the equation

$$J_m(x)=J_m\left(\left(\frac{m+1}{2m}\right)x\right)+J_m\left(\left(\frac{m-1}{2m}\right)x\right),\quad x\in X_0 \tag{17}$$

such that

$$\|f(x)-J_m(x)\|_*\leqslant \sup_{n\in\mathbb{N}_0}\left\{c\,m^{-q}\|x\|^{p+q}\,\alpha_m^n\right\},\quad x\in X_0. \tag{18}$$

Moreover,

$$J_m(x):=\lim_{n\to\infty}\mathscr{T}_m^n f(x)$$

for all $x \in X_0$. We show that

$$\left\| \mathscr{T}_m^n f\left(\frac{x+y}{2}\right) + \mathscr{T}_m^n f\left(\frac{x-y}{2}\right) - \mathscr{T}_m^n f(x) \right\|_* \leqslant c\, \alpha_m^n \|x\|^p\, \|y\|^q, \tag{19}$$

for every $x, y \in X_0$. Since the case $n = 0$ is just (11), take $k \in \mathbb{N}$ and assume that (19) holds for $n = k$ and every $x, y \in X_0$. Then

$$\begin{aligned}
&\left\| \mathscr{T}_m^{k+1} f\left(\frac{x+y}{2}\right) + \mathscr{T}_m^{k+1} f\left(\frac{x-y}{2}\right) - \mathscr{T}_m^{k+1} f(x) \right\|_* \\
&= \Bigg\| \mathscr{T}_m^k f\left(\left(\frac{m+1}{2m}\right)\left(\frac{x+y}{2}\right)\right) + \mathscr{T}_m^k f\left(\left(\frac{m-1}{2m}\right)\left(\frac{x+y}{2}\right)\right) \\
&\quad + \mathscr{T}_m^k f\left(\left(\frac{m+1}{2m}\right)\left(\frac{x-y}{2}\right)\right) + \mathscr{T}_m^k f\left(\left(\frac{m-1}{2m}\right)\left(\frac{x-y}{2}\right)\right) \\
&\quad - \mathscr{T}_m^k f\left(\left(\frac{m+1}{2m}\right)x\right) - \mathscr{T}_m^k f\left(\left(\frac{m-1}{2m}\right)x\right) \Bigg\|_* \\
&\leqslant \max\Bigg\{ \Bigg\| \mathscr{T}_m^k f\left(\left(\frac{m+1}{2m}\right)\left(\frac{x+y}{2}\right)\right) + \mathscr{T}_m^k f\left(\left(\frac{m+1}{2m}\right)\left(\frac{x-y}{2}\right)\right) \\
&\qquad - \mathscr{T}_m^k f\left(\left(\frac{m+1}{2m}\right)x\right) \Bigg\|_*, \\
&\quad \Bigg\| \mathscr{T}_m^k f\left(\left(\frac{m-1}{2m}\right)\left(\frac{x+y}{2}\right)\right) + \mathscr{T}_m^k f\left(\left(\frac{m-1}{2m}\right)\left(\frac{x-y}{2}\right)\right) \\
&\qquad - \mathscr{T}_m^k f\left(\left(\frac{m-1}{2m}\right)x\right) \Bigg\|_* \Bigg\} \\
&\leqslant \max\left\{ c\, \alpha_m^k \|x\|^p\, \|y\|^q \left|\frac{m+1}{2m}\right|^{p+q}, c\, \alpha_m^k \|x\|^p\, \|y\|^q \left|\frac{m-1}{2m}\right|^{p+q} \right\} \\
&= c\, \alpha_m^k \|x\|^p\, \|y\|^q \max\left\{ \left|\frac{m+1}{2m}\right|^{p+q}, \left|\frac{m-1}{2m}\right|^{p+q} \right\} \\
&\leqslant c\, \alpha_m^{k+1} \|x\|^p\, \|y\|^q
\end{aligned}$$

for all $x, y \in X_0$. Thus, by induction we have shown that (19) holds for every $n \in \mathbb{N}_0$. Letting $n \to \infty$ in (19), we obtain that

$$J_m\left(\frac{x+y}{2}\right) + J_m\left(\frac{x-y}{2}\right) = J_m(x),$$

for all $x, y \in X_0$. In this way we obtain a sequence $\{J_m\}_{m \geqslant m_0}$ of Cauchy-Jensen functions on X_0 such that

$$\|f(x) - J_m(x)\|_* \leqslant \sup_{n \in \mathbb{N}_0} \left\{ c\, m^{-q} \|x\|^{p+q}\, \alpha_m^n \right\}, \quad x \in X_0,$$

this implies that

$$\|f(x) - J_m(x)\|_* \leqslant c\, m^{-q} \|x\|^{p+q}, \quad x \in X_0.$$

It follows, with $m \to \infty$, that f is Cauchy-Jensen on X_0.

In the following theorem, we prove the hyperstability of the Cauchy-Jensen equation (1) on the set containing 0.

THEOREM 4. *Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|_*)$ be normed space and Non-Archimedean Banach space respectively, $c \geqslant 0$, $p, q > 0$, and let $f : X \to Y$ satisfy*

$$\left\| f\left(\frac{x+y}{2}\right) + f\left(\frac{x-y}{2}\right) - f(x) \right\|_* \leqslant c\, \|x\|^p\, \|y\|^q, \tag{20}$$

for all $x, y \in X$. Then f is Cauchy-Jensen on X.

Proof. Putting $y = 0$ in (20), we get that

$$f\left(\frac{x}{2}\right) = \frac{1}{2} f(x), \quad x \in X.$$

The function f satisfies (20) and

$$f(x) = 2f\left(\frac{x}{2}\right), \quad x \in X. \tag{21}$$

Replacing x by $2x$ in (21) we get

$$f(x) = \frac{1}{2} f(2x), \quad x \in X. \tag{22}$$

Using (20) and (22) we can prove by induction that for every $n \in \mathbb{N}_0$

$$\left\| f\left(\frac{x+y}{2}\right) + f\left(\frac{x-y}{2}\right) - f(x) \right\|_* \leqslant c \left(\frac{1}{2^{p+q}}\right)^n \|x\|^p\, \|y\|^q \tag{23}$$

for all $x, y \in X$.

Indeed, for $n = 0$ (23) is simply (20). So, take $k \in \mathbb{N}$ and assume that (23) holds for $n = k$. Then using (22) to (23) we have

$$\left\| \frac{1}{2} f(x+y) + \frac{1}{2} f(x-y) - \frac{1}{2} f(2x) \right\|_* \leqslant c \left(\frac{1}{2^{p+q}}\right)^k \|x\|^p\, \|y\|^q, \quad x, y \in X,$$

and

$$\frac{1}{|2|} \left\| f(x+y) + f(x-y) - f(2x) \right\|_* \leqslant c \left(\frac{1}{2^{p+q}}\right)^k \|x\|^p\, \|y\|^q, \quad x, y \in X.$$

Replacing x by $\frac{x}{2}$ and y by $\frac{y}{2}$ in the last inequality, we obtain

$$\left\| f\left(\frac{x+y}{2}\right) + f\left(\frac{x-y}{2}\right) - f(x) \right\|_* \leqslant c \left(\frac{1}{2^{p+q}}\right)^{k+1} \|x\|^p\, \|y\|^q$$

for all $x, y \in X$, so (23) holds for every $n \in \mathbb{N}_0$. With $n \to \infty$ in the inequality (23), we obtain that f is Cauchy-Jensen on X.

The above theorems imply in particular the following corollary, which shows their simple application.

COROLLARY 1. *Let* $(X,\|\cdot\|)$ *and* $(Y,\|\cdot\|_*)$ *be normed space and Non-Archimedean Banach space respectively,* $G: X^2 \to Y$ *and* $G(x,y) \neq 0$ *for some* $x,y \in X$ *and*

$$\left\|G(x,y)\right\|_* \leqslant c\,\|x\|^p\,\|y\|^q, \quad x,y \in X \tag{24}$$

where $c \geqslant 0$, $p,q \in \mathbb{R}$. *Assume that the numbers* p,q *satisfy one of the following conditions:*

1. $p+q<0$, *and (2) holds for all* $x,y \in X_0$,
2. $p+q>0$, *and (11) holds for all* $x,y \in X_0$,
3. $p,q>0$ *and (20) holds for all* $x,y \in X$.

Then the functional equation

$$g\left(\frac{x+y}{2}\right)+g\left(\frac{x-y}{2}\right)=g(x)+G(x,y), \;\; x,y \in X \tag{25}$$

has no solution in the class of functions $g: X \to Y$.

Proof. Suppose that $g: X \to Y$ is a solution to (25). Then (1) holds, and consequently, according to the above theorems, g is Cauchy-Jensen on X_0, which means that $G(x,y)=0$ for some $x,y \in X$. This is a contradiction.

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