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# ANALYTICAL, ASYMPTOTIC AND INTEGRAL REPRESENTATIONS FOR A DOUBLE SUM

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*Abstract.* We give an analytical representation for the double sum

$$\sum_{1\leqslant i<j\leqslant n} \frac{1}{[(n+i)(n+j)]^k}$$

in terms of the polygamma functions, where $k$ is any given positive integer. Based on this result, we present an asymptotic formula as $n\to\infty$ and an integral representation for this sum.

## 1. Introduction

Izán Péraz, in a private communication to the first author, formulated the following limit:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{1\leqslant i<j\leqslant n} \frac{1}{\sqrt{(n+i)(n+j)}} = 6-4\sqrt{2}. \tag{1.1}$$

We give here a generalization of this limit by considering the double sum

$$S_r(n) = \sum_{1\leqslant i<j\leqslant n} \frac{1}{\left[(n+i)(n+j)\right]^r}, \tag{1.2}$$

where in this section the index $r>0$ is an arbitrary real quantity, but will subsequently be restricted to be a positive integer.

It is easy to see that

$$n^{2r-2}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{1}{\left[(n+i)(n+j)\right]^r} = \left(n^{r-1}\sum_{i=1}^{n}\frac{1}{(n+i)^r}\right)^2 = \left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{\left(1+\frac{i}{n}\right)^r}\right)^2.$$

By the definition of a definite integral, we therefore have

$$\begin{aligned}\lim_{n\to\infty} n^{2r-2}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{1}{\left[(n+i)(n+j)\right]^r} &= \left(\int_0^1 \frac{1}{(1+x)^r}\mathrm{d}x\right)^2 \\ &= \begin{cases} \left(\dfrac{2^{1-r}-1}{1-r}\right)^2 & r\neq 1 \\ (\ln 2)^2 & r=1. \end{cases}\end{aligned} \tag{1.3}$$

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In a similar manner, we have

$$n^{2r-1}\sum_{i=1}^{n}\frac{1}{(n+i)^{2r}}=\frac{1}{n}\sum_{i=1}^{n}\frac{1}{\left(1+\frac{i}{n}\right)^{2r}}\longrightarrow\begin{cases}\dfrac{2^{1-2r}-1}{1-2r} & r\neq 1\\ \ln 2 & r=1,\end{cases}$$

as $n\to\infty$, which implies that

$$\lim_{n\to\infty}n^{2r-2}\sum_{i=1}^{n}\frac{1}{(n+i)^{2r}}=0. \tag{1.4}$$

Let

$$a_{ij}=\frac{1}{\left[(n+i)(n+j)\right]^{r}}\qquad (i,j=1,2,\ldots,n),$$

where $r>0$. Then it follows that

$$S_r(n)=\sum_{1\leqslant i<j\leqslant n}a_{ij}=\frac{1}{2}\left\{\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}-\sum_{j=1}^{n}a_{jj}\right\}. \tag{1.5}$$

Noting that (1.4) holds, we have

$$\lim_{n\to\infty}n^{2r-2}\sum_{j=1}^{n}a_{jj}=0.$$

Then from (1.3) and (1.5) we finally obtain

$$\begin{aligned}\lim_{n\to\infty}n^{2r-2}S_r(n)&=\frac{1}{2}\lim_{n\to\infty}n^{2r-2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{\left[(n+i)(n+j)\right]^{r}}\\&=\begin{cases}\dfrac{1}{2}\left(\dfrac{2^{1-r}-1}{1-r}\right)^{2} & r\neq 1,\\ \dfrac{1}{2}(\ln 2)^2 & r=1,\end{cases}\end{aligned} \tag{1.6}$$

which generalizes the formula (1.1). The choice $r=1/2$ in (1.6) yields (1.1).

Now let $r=k$, where $k$ is any given positive integer. In this paper, we establish an analytical representation for the sum $S_k(n)$ defined in (1.2) in terms of the polygamma functions. The polygamma functions $\psi^{(n)}(x)$ for $n=1,2,\ldots$ are defined by

$$\psi^{(n)}(x)=\frac{\mathrm{d}^{n+1}}{\mathrm{d}x^{n+1}}\ln\Gamma(x)=\frac{\mathrm{d}^{n}}{\mathrm{d}x^{n}}\psi(x),$$

where $\Gamma(x)$ is the gamma function, and $\psi(x)$ (also known as the digamma function) is defined by

$$\psi(x)=\frac{\mathrm{d}}{\mathrm{d}x}\ln\Gamma(x)\quad\text{and}\quad\psi^{(0)}(x)=\psi(x).$$

Based on this exact representation for $S_k(n)$, we present an asymptotic formula as $n\to\infty$ in Section 2 and an integral representation for this sum in Section 3.

## 2. An asymptotic formula as $n \to \infty$

We first show how the double sum $S_k(n)$ can be evaluated in terms of polygamma functions. We have the following theorem.

THEOREM 2.1. *For positive integer $k$, the sum $S_k(n)$ has the following analytical representation:*

$$S_k(n) = \frac{1}{2}\left\{ \left( \frac{\psi^{(k-1)}(2n+1) - \psi^{(k-1)}(n+1)}{(k-1)!} \right)^2 + \frac{\psi^{(2k-1)}(2n+1) - \psi^{(2k-1)}(n+1)}{(2k-1)!} \right\}. \tag{2.1}$$

*Proof.* The polygamma function has the expression (see [2, Eq. 1.2(54)] and [3, Eq. 1.3(54)])

$$\psi^{(n)}(x+m) = \psi^{(n)}(x) + (-1)^n n! \sum_{j=1}^{m} \frac{1}{(x+j-1)^{n+1}}, \quad m,n = 0,1,2,\dots. \tag{2.2}$$

The choice $(x,m,n) = (n+1,n,k-1)$ in (2.2) yields

$$\sum_{j=1}^{n} \frac{1}{(n+j)^k} = (-1)^{k-1} \frac{\psi^{(k-1)}(2n+1) - \psi^{(k-1)}(n+1)}{(k-1)!}.$$

We then obtain

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{1}{\left[(n+i)(n+j)\right]^k} = \left( \sum_{j=1}^{n} \frac{1}{(n+j)^k} \right)^2 = \left( \frac{\psi^{(k-1)}(2n+1) - \psi^{(k-1)}(n+1)}{(k-1)!} \right)^2. \tag{2.3}$$

The choice $(x,m,n) = (n+1,n,2k-1)$ in (2.2) yields

$$\sum_{j=1}^{n} \frac{1}{(n+j)^{2k}} = -\frac{\psi^{(2k-1)}(2n+1) - \psi^{(2k-1)}(n+1)}{(2k-1)!}. \tag{2.4}$$

We then obtain from (2.3) and (2.4)

$$\sum_{1 \leqslant i < j \leqslant n} \frac{1}{\left[(n+i)(n+j)\right]^k} = \frac{1}{2}\left\{ \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{1}{\left[(n+i)(n+j)\right]^k} - \sum_{j=1}^{n} \frac{1}{(n+j)^{2k}} \right\}$$
$$= \frac{1}{2}\left\{ \left( \frac{\psi^{(k-1)}(2n+1) - \psi^{(k-1)}(n+1)}{(k-1)!} \right)^2 + \frac{\psi^{(2k-1)}(2n+1) - \psi^{(2k-1)}(n+1)}{(2k-1)!} \right\}.$$

This completes the proof.

We now give the asymptotic expansion of $S_k(n)$.

THEOREM 2.2. *For $k = 1, 2, \ldots$, the sum $S_k(n)$ has the asymptotic expansion*

$$S_k(n) = \frac{1}{2n^{2k-2}}\left\{C_0(k) + \frac{C_1(k)}{n} + \frac{C_2(k)}{n^2} + O(n^{-3})\right\} \tag{2.5}$$

*as $n \to \infty$, where the coefficients $C_s(k)$ ( $s \leqslant 2$ ) when $k \geqslant 2$ are given by*

$$C_0(k) = \left(\frac{1-2^{1-k}}{k-1}\right)^2, \quad C_1(k) = -\frac{(1-2^{-k})(1-2^{1-k})}{k-1} - \frac{1-2^{1-2k}}{2k-1},$$

$$C_2(k) = \left(\frac{1-2^{-k}}{2}\right)^2 + \frac{1-2^{-2k}}{2} + \frac{k(1-2^{-1-k})(1-2^{1-k})}{6(k-1)}. \tag{2.6}$$

*When $k = 1$, we have*

$$C_0(1) = (\ln 2)^2, \qquad C_1(1) = -\frac{1}{2}(1+\ln 2), \qquad C_2(1) = \frac{1}{16}(7+2\ln 2). \tag{2.7}$$

*Proof.* To demonstrate (2.5), we require the asymptotic formula [1, Eq. (6.4.11)]

$$(-1)^{k+1}\psi^{(k)}(x) \sim \frac{(k-1)!}{x^k} + \frac{k!}{2x^{k+1}} + \sum_{s=1}^{\infty} B_{2s}\frac{(2s+k-1)!}{(2s)!x^{2s+k}} \tag{2.8}$$

for $x \to \infty$, where $B_n$ are the Bernoulli numbers defined by

$$\frac{t}{e^t-1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!} \qquad (|t| < 2\pi).$$

The expansion (2.8) shows that

$$\frac{\psi^{(k-1)}(n+1)}{(k-1)!} = \frac{(-1)^k}{n^{k-1}}\left\{\frac{1}{k-1} - \frac{1}{2n} + \frac{k}{12n^2} + O(n^{-3})\right\} \qquad (n \to \infty)$$

upon using the fact that $B_2 = \frac{1}{6}$. Thus we find

$$\begin{aligned}\Psi_1(n,k) &:= \frac{\psi^{(k-1)}(2n+1) - \psi^{(k-1)}(n+1)}{(k-1)!} \\ &= \frac{(-1)^{k-1}}{n^{k-1}}\left\{A_0(k) + \frac{A_1(k)}{n} + \frac{A_2(k)}{n^2} + O(n^{-3})\right\}\end{aligned}$$

and

$$\begin{aligned}\Psi_2(n,k) &:= \frac{\psi^{(2k-1)}(2n+1) - \psi^{(2k-1)}(n+1)}{(2k-1)!} \\ &= \frac{-1}{n^{2k-1}}\left\{A_0(2k) + \frac{A_1(2k)}{n} + O(n^{-2})\right\}\end{aligned}$$

as $n \to \infty$, where

$$A_0(k) = \frac{1-2^{1-k}}{k-1}, \qquad A_1(k) = -\frac{1}{2}(1-2^{-k}), \qquad A_2(k) = \frac{1}{12}(1-2^{-1-k}).$$

It then follows that

$$\Psi_1^2(n,k) + \Psi_2(n,k) = \frac{1}{n^{2k-2}}\left\{C_0(k) + \frac{C_1(k)}{n} + \frac{C_2(k)}{n^2} + O(n^{-3})\right\} \qquad (n \to \infty),$$

where

$$\begin{aligned}
C_0(k) &= A_0(k)^2 = \left(\frac{1-2^{1-k}}{k-1}\right)^2, \\
C_1(k) &= 2A_0(k)A_1(k) - A_0(2k) = -\frac{(1-2^{-k})(1-2^{1-k})}{k-1} - \frac{1-2^{1-2k}}{2k-1}, \\
C_2(k) &= A_1(k)^2 + 2A_0(k)A_2(k) - A_1(2k) \\
&= \left(\frac{1-2^{-k}}{2}\right)^2 + \frac{1-2^{-2k}}{2} + \frac{k(1-2^{-1-k})(1-2^{1-k})}{6(k-1)}.
\end{aligned}$$

Substitution of this expansion into (2.1) then yields (2.5).

In the case $k=1$, we proceed in the same manner making use of the expansion [1, Eq. (6.4.11)]

$$\psi(x) \sim \ln x - \frac{1}{2x} - \sum_{s=1}^{\infty} \frac{B_{2s}}{2sx^{2s}} \qquad (x \to \infty),$$

to find

$$\Psi_1(n,1) = \ln 2 - \frac{1}{4n} + \frac{1}{16n^2} + O(n^{-3}), \qquad \Psi_2(n,1) = -\frac{1}{2n} + \frac{3}{8n^2} + O(n^{-3}).$$

Hence we obtain

$$S_1(n) = \frac{(\ln 2)^2}{2} - \frac{1+\ln 2}{4n} + \frac{7+2\ln 2}{32n^2} + O(n^{-3}) \qquad (n \to \infty).$$

This completes the proof.

REMARK 2.1. *Noting that*

$$\lim_{k \to 1} \frac{1-2^{1-k}}{k-1} = \ln 2,$$

*we see that the limiting values of the coefficients $C_s(k)$ in (2.6) as $k \to 1$ agree with those given in (2.7).*

## 3. An integral representation of $S_k(n)$

In this section we obtain an integral representation for the sum $S_k(n)$. We have

THEOREM 3.1. *For $k = 2,3,\ldots$ we have the integral representation for $S_k(n)$ given by*

$$S_k(n) = \frac{1}{2}\Big\{I^2(n,k) + I(n,2k)\Big\}, \tag{3.1}$$

*where*

$$I(n,k) = \frac{1}{\Gamma(k)}\int_0^\infty \left\{\frac{2^{-k}}{e^{t/2}-1} - \frac{1}{e^t-1}\right\} t^{k-1}\, e^{-nt}\,\mathrm{d}t.$$

*Proof.* Using the representation [1, Eq. (6.4.1)] for positive integer $n$

$$\psi^{(n)}(x) = (-1)^{n+1}\int_0^\infty \frac{t^n e^{-xt}}{1-e^{-t}}\,\mathrm{d}t \qquad (x>0),$$

we find

$$\psi^{(n)}(x+1) = (-1)^{n+1}\int_0^\infty \frac{t^n e^{-xt}}{e^t-1}\,\mathrm{d}t \tag{3.2}$$

and

$$\psi^{(n)}(2x+1) = \frac{(-1)^{n+1}}{2^{n+1}}\int_0^\infty \frac{t^n e^{-xt}}{e^{t/2}-1}\,\mathrm{d}t. \tag{3.3}$$

We then obtain

$$\begin{aligned}\Psi_1^2(n,k) + \Psi_2(n,k) &= \left(\frac{1}{\Gamma(k)}\int_0^\infty \left\{\frac{2^{-k}}{e^{t/2}-1} - \frac{1}{e^t-1}\right\} t^{k-1} e^{-nt}\,\mathrm{d}t\right)^2 \\ &\quad + \frac{1}{\Gamma(2k)}\int_0^\infty \left\{\frac{2^{-2k}}{e^{t/2}-1} - \frac{1}{e^t-1}\right\} t^{2k-1} e^{-nt}\,\mathrm{d}t.\end{aligned} \tag{3.4}$$

Finally, substituting the expression (3.4) into (2.1) yields (2.5). This completes the proof.

The case $k=1$ is covered by the following theorem.

THEOREM 3.2. . *When $k=1$, we have the representation*

$$S_1(n) = \frac{1}{2}\left[\left\{\ln 2 - \frac{1}{2}\int_0^\infty \frac{e^{-nt}}{e^{t/2}+1}\,\mathrm{d}t\right\}^2 + \int_0^\infty \left\{\frac{1}{4(e^{t/2}-1)} - \frac{1}{e^t-1}\right\} te^{-nt}\,\mathrm{d}t\right]. \tag{3.5}$$

*Proof.* It follows from (3.2) and (3.3) that

$$\psi'(2n+1)-\psi'(n+1)=\int_0^\infty \left[\frac{1}{4(e^{t/2}-1)}-\frac{1}{e^t-1}\right\} te^{-nt}\,\mathrm{d}t. \tag{3.6}$$

Using the recurrence formula $\psi(x+1)=\psi(x)+1/x$, the duplication formula [1, Eq. (6.3.8)]:

$$\psi(2x)=\frac{1}{2}\psi(x)+\frac{1}{2}\psi\left(x+\frac{1}{2}\right)+\ln 2$$

and the representation [1, Eq. (6.3.21)]:

$$\psi(x)=\int_0^\infty\left(\frac{e^{-t}}{t}-\frac{e^{-xt}}{1-e^{-t}}\right)\mathrm{d}t,$$

we find

$$\begin{aligned}\psi(2n+1)-\psi(n+1)&=\ln 2+\frac{1}{2}\left\{\psi\left(n+\frac{1}{2}\right)-\psi(n+1)\right\}\\&=\ln 2-\frac{1}{2}\int_0^\infty\frac{e^{-nt}}{e^{t/2}+1}\,\mathrm{d}t.\end{aligned} \tag{3.7}$$

We then obtain from (2.1), (3.6) and (3.7)

$$\begin{aligned}S_1(n)&=\frac{1}{2}\left\{\Big(\psi(2n+1)-\psi(n+1)\Big)^2+\psi'(2n+1)-\psi'(n+1)\right\}\\&=\frac{1}{2}\left[\left\{\ln 2-\frac{1}{2}\int_0^\infty\frac{e^{-nt}}{e^{t/2}+1}\,\mathrm{d}t\right\}^2+\int_0^\infty\left\{\frac{1}{4(e^{t/2}-1)}-\frac{1}{e^t-1}\right\}te^{-nt}\,\mathrm{d}t\right].\end{aligned}$$

This completes the proof.

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