Journal of Classical Analysis
Volume 12, Number 2 (2018), 55–81 doi:10.7153/jca-2018-12-06

EVALUATION OF APÉRY–LIKE SERIES THROUGH MULTISECTION METHOD

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Abstract. By combining the multisection series method with the power series expansion of arcsin-function, we investigate Apéry-like infinite series involving the central binomial coefficients in denominators. By constructing and resolving systems of linear equations, numerous remarkable infinite series formulae (generated by using an appropriate computer algebra system) for π and special values of the logarithm function are established, including some recent results due to Almkvist *et al.* (2003) and Zheng (2008).

1. Introduction and motivation

In 1979, Apéry [2] (cf. [20] also) proved the irrationality of $\zeta(2)$ and $\zeta(3)$ by making use of the following formulae

$$\zeta(2)=3\sum_{n=1}^{\infty}\frac{1}{n^2\binom{2n}{n}} \quad\text{and}\quad \zeta(3)=\frac{5}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^3\binom{2n}{n}}.$$

Since then, much research interests have turned to investigate the Apéry-like series for values of Riemann zeta function (cf. [5, 7, 8, 9, 14, 25]), Ramanujan-like series for π (cf. [4, 10, 16, 17] and other infinite series involving central binomial coefficients (cf. [6, 11, 15, 21, 23, 22]. Recently, special attention has been paid to find fast convergent series for evaluating π [1, 3, 24]. For example, both Almkvist *et al.* [1] and Zheng [24] found several infinite series expressions of the following type for π

$$\sum_{n=0}^{\infty}\frac{P_m(n)}{\binom{2mn}{mn}}\alpha^n,$$

where $\alpha\in\mathbb{R}$ and $P_m(n)$ is a polynomial of degree m in n.

For a nonnegative integer n and an indeterminate x, define the falling factorial by

$$\langle x\rangle_0 := 1 \quad\text{and}\quad \langle x\rangle_n := \prod_{k=0}^{n-1}(x-k), \quad n\in\mathbb{N}.$$

In this paper, we will investigate the following expressions of Apéry-like infinite series

Mathematics subject classification (2010): Primary 11Y60, Secondary 14G10.
Keywords and phrases: Multisection series, Apéry-like series, central binomial coefficient.

Paper JCA-12-06

for π and special values of the logarithm function

$$\Omega_m^\gamma(x) := \sum_{n=0}^{\infty} \frac{\Lambda_m(n)}{\binom{2mn+2\gamma}{mn+\gamma}} x^n, \tag{1}$$

where $\Lambda_m(n)$ is a polynomial of degree m in n, expressed through the real connection coefficients $\{\lambda_k\}_{k=0}^m$ in terms of falling factorials:

$$\Lambda_m(n) := \sum_{k=0}^{m} \lambda_k \langle 2mn + 2\gamma \rangle_k. \tag{2}$$

Numerous remarkable multisection series for $\Omega_m^\gamma(x)$ are expressed in terms of π and special values of the logarithm function, constitute several classes of identities for Apéry-like series. They not only recover, in particular for $\gamma = 0$, the results obtained previously by Almkvist *et al.* [1] and Zheng [24], but also discover numerous infinite series identities involving the central binomial coefficients $\binom{2mn+2\gamma}{mn+\gamma}$ covering all the residue classes of γ modulo m. It is remarkable that for these formulae corresponding to $\gamma \not\equiv 0 \pmod{m}$, there has not been hitherto even a single example in mathematical literature.

The rest of the paper will be organized as follows. In the next section, we shall present, as preliminaries, higher order derivatives of the arcsin-function, the multisection series method and special values tabulated for the function $\mathrm{hyp}(y)$ (see (3) for its definition). Then in Section 3, we shall transform the $\Omega_m^\gamma(x)$-series, for odd m, into a finite sum associated with a linear system that will be resolved to give six examples of infinite series identities. A similar approach will be employed in Section 4 to examining the alternating series that will lead to further fifteen examples of alternating series for π and for special values of the logarithm function. Finally in Section 5, we record the *Mathematica* commands that have been utilized by the authors to produce infinite series identities.

Throughout the paper, we shall utilize the following usual Kronecker symbol $\delta_{i,j}$ with $\delta_{i,j} = 1$ for $i = j$ and $\delta_{i,j} = 0$ for $i \neq j$.

2. Preliminary background

As preliminaries, we discuss below higher order derivatives of the function $h(x)$ (see equation (4) for definition) related to the arcsin-function, multisection series method and special values of the function $\mathrm{hyp}(y)$ defined by

$$\mathrm{hyp}(y) := \frac{\arcsin(y)}{y\sqrt{1-y^2}}. \tag{3}$$

2.1. Higher order derivatives of $h(x)$

For the reciprocals of central binomial coefficients, there exists the following interesting generating function [18, eq. (15)]

$$h(x) := \mathscr{D}_x \frac{\arcsin(x)}{\sqrt{1-x^2}} = \frac{1}{1-x^2} + \frac{x\arcsin(x)}{\sqrt{(1-x^2)^3}} = \sum_{n=0}^{\infty} \frac{(2x)^{2n}}{\binom{2n}{n}}, \tag{4}$$

where $|x| < 1$. Several variants of this expansion and applications to infinite series involving reciprocals of central binomial coefficients can be found in [6, 11, 13, 19, 25].

LEMMA 1. *For the k-th derivative of the function $h(x)$, there exist polynomials $\{P_k, Q_k\}$ such that*

$$\mathscr{D}_x^k h(ex) = \frac{e^k P_k(ex)}{(1-e^2x^2)^{k+1}} + \frac{e^k Q_k(ex)}{(1-e^2x^2)^{k+1}} \cdot \frac{\arcsin(ex)}{\sqrt{1-e^2x^2}}, \tag{5}$$

where $\{P_k, Q_k\}$ satisfy the recurrence relations

$$P_{k+1}(y) = (1-y^2)P_k'(y) + (2k+2)yP_k(y) + Q_k(y), \tag{6}$$

$$Q_{k+1}(y) = (1-y^2)Q_k'(y) + (2k+3)yQ_k(y); \tag{7}$$

with the initial conditions

$$P_0(y) = 1, \quad Q_0(y) = y. \tag{8}$$

Proof. This can be done by the induction principle. When $k = 0$, the statement of the lemma is true because in this case, the equality (5) is equivalent to the initial conditions given by (8) in view of (4). Suppose that the statement of the lemma is valid for k. We have to verify it for $k+1$ by differentiating the right member of (5) as follows:

$$\begin{aligned}
\mathscr{D}_x^{k+1} h(ex) &= \mathscr{D}\left\{\frac{e^k P_k(ex)}{(1-e^2x^2)^{k+1}}\right\} + \mathscr{D}\left\{\frac{e^k Q_k(ex)}{(1-e^2x^2)^{k+1}} \cdot \frac{\arcsin(ex)}{\sqrt{1-e^2x^2}}\right\} \\
&= e^{k+1} \frac{(1-e^2x^2)P_k'(ex) + (2k+2)exP_k(ex) + Q_k(ex)}{(1-e^2x^2)^{k+2}} \\
&\quad + e^{k+1} \frac{(1-e^2x^2)Q_k'(ex) + (2k+3)exQ_k(ex)}{(1-e^2x^2)^{k+2}} \cdot \frac{\arcsin(ex)}{\sqrt{1-e^2x^2}}.
\end{aligned}$$

However, by (6) and (7), one can rewrite the last expression for $\mathscr{D}_x^{k+1} h(ex)$ as

$$\mathscr{D}_x^{k+1} h(ex) = \frac{e^{k+1} P_{k+1}(ex)}{(1-e^2x^2)^{k+2}} + \frac{e^{k+1} Q_{k+1}(ex)}{(1-e^2x^2)^{k+2}} \cdot \frac{\arcsin(ex)}{\sqrt{1-e^2x^2}},$$

which is exactly formula (5) when writing $k+1$ instead of k. This concludes the proof by induction. □

2.2. Multisection series

In classical combinatorics, there is the following useful expression of multisection series for formal power series (cf. Comtet [12, p. 84] for example). Let $f(x)$ be a formal power series with complex coefficients defined by $f(x) := \sum_{n=0}^{\infty} A_n x^n$. Then for a natural number m and an integer γ subject to $0 \leqslant \gamma < m$, the following formula holds:

$$\sum_{n=0}^{\infty} A_{mn+\gamma} x^{mn+\gamma} = \frac{1}{m} \sum_{k=1}^{m} \omega_m^{-k\gamma} f(x\omega_m^k), \tag{9}$$

where $\omega_m := \exp(2\pi i/m)$ is the m-th root of unity. It is not hard to check that for an even function $g(x) := \sum_{n=0}^{\infty} B_n x^{2n}$, there holds similarly the formula

$$\sum_{n=0}^{\infty} B_{mn+\gamma} x^{2mn+2\gamma} = \frac{1}{m} \sum_{k=1}^{m} \omega_{2m}^{-2k\gamma} g(x\omega_{2m}^k). \tag{10}$$

In fact, writing the right-hand side of (10) as a double series and then interchanging the summation order, we have

$$\begin{aligned} \frac{1}{m} \sum_{k=1}^{m} \omega_{2m}^{-2k\gamma} g(x\omega_{2m}^k) &= \frac{1}{m} \sum_{k=1}^{m} \sum_{n=0}^{\infty} B_n x^{2n} \omega_{2m}^{2k(n-\gamma)} \\ &= \frac{1}{m} \sum_{n=0}^{\infty} B_n x^{2n} \sum_{k=1}^{m} \omega_{2m}^{2k(n-\gamma)}. \end{aligned}$$

Then the identity (10) follows from the almost trivial sum

$$\sum_{k=1}^{m} \omega_{2m}^{2k(n-\gamma)} = \begin{cases} m, & n \equiv \gamma \pmod{m}; \\ 0, & n \not\equiv \gamma \pmod{m}. \end{cases}$$

In an analogous manner, one can verify that (10) admits the following counterpart

$$\sum_{n=0}^{\infty} B_{mn+\gamma} x^{2mn+2\gamma} = \frac{1}{m} \sum_{k=1}^{m} \omega_m^{-2k\gamma} g(x\omega_m^k) \quad \text{for an odd} \quad m \in \mathbb{N}. \tag{11}$$

It should be pointed out that under the replacements $g \to h$ and $B_n \to 4^n/\binom{2n}{n}$, both (10) and (11) remain valid because the h-series defined by (4) represents an even function.

2.3. Special function values

Recall that $\Omega_m^{\gamma}(x)$ introduced in (1) results in a multisection series related to h-series given by (4). In order to evaluate $\Omega_m^{\gamma}(x)$ for specific integers m, γ and a real number x, it will be necessary for us to know special values of the function $\text{hyp}(y)$ defined in (3). They are tabulated below for our later needs.

Special values for $hyp(x \cdot e^{i\theta})$

$x \backslash \theta$	π	$\frac{\pi}{2}$	$\frac{\pi}{4}$	$\frac{\pi}{8}$
$\frac{1}{2}$	$\frac{2\pi}{3\sqrt{3}}$	$\frac{4}{\sqrt{5}} \ln \frac{1+\sqrt{5}}{2}$		
$\frac{1}{\sqrt{2}}$	$\frac{\pi}{2}$	$\frac{\ln(2+\sqrt{3})}{\sqrt{3}}$		
$\frac{\sqrt{3}}{2}$	$\frac{4\pi}{3\sqrt{3}}$	$\frac{2}{\sqrt{21}} \ln \frac{5+\sqrt{21}}{2}$	$\frac{4-2i}{15\sqrt{3}}\{\pi + 3i\ln(2+\sqrt{3})\}$	
$\frac{1}{\sqrt[4]{2}}$		$\sqrt{\frac{2}{1+\sqrt{2}}} \ln \frac{1+\sqrt{1+\sqrt{2}}}{\sqrt[4]{2}}$		$\frac{\pi+2i\ln(1+\sqrt{2})}{2\sqrt{2}}$
$\frac{1}{\sqrt[4]{8}}$		$\sqrt{\frac{8}{1+2\sqrt{2}}} \ln \frac{1+\sqrt{1+2\sqrt{2}}}{\sqrt[4]{8}}$	$\frac{\sqrt{2}-i}{6\sqrt{2}}\{\pi + 4i\ln(1+\sqrt{2})\}$	
$\frac{1}{\sqrt[4]{32}}$		$\sqrt{\frac{32}{1+4\sqrt{2}}} \ln \frac{1+\sqrt{1+4\sqrt{2}}}{\sqrt[4]{32}}$		$\frac{3-i}{10}(\pi + 2i\ln 2)$

These particular values can be realized by routine calculations, or simply by executing *Mathematica* command ‘`FunctionExpand`’.

3. Summation formulae for m odd

Let γ and m be nonnegative integers with m being odd and $0 \leqslant \gamma < m$. This section will be devoted to examining the Ω-series (1) by determining the $\Lambda_m(n)$-polynomials (2) so that the series have closed forms in terms of π and special values of the logarithm function. This will be carried out by constructing and then resolving special systems of linear equations.

Combining (5) with (11), we can reformulate the following series

$$\begin{aligned}
\Omega_m^\gamma\big((2x)^{2m}\big) &= \sum_{k=0}^{m} x^{k-2\gamma} \frac{\lambda_k}{4^\gamma} \mathscr{D}_x^k \sum_{n=0}^{\infty} \frac{(2x)^{2mn+2\gamma}}{\binom{2mn+2\gamma}{mn+\gamma}} \\
&= \sum_{k=0}^{m} \frac{\lambda_k x^{k-2\gamma}}{4^\gamma m} \mathscr{D}_x^k \sum_{\ell=1}^{m} \omega_m^{-2\ell\gamma} h(x\omega_m^\ell) \\
&= \sum_{k=0}^{m} \frac{\lambda_k}{4^\gamma m} \sum_{\ell=1}^{m} \frac{(x\omega_m^\ell)^{k-2\gamma} P_k(x\omega_m^\ell)}{(1-x^2\omega_m^{2\ell})^{k+1}} \\
&\quad + \sum_{k=0}^{m} \frac{\lambda_k}{4^\gamma m} \sum_{\ell=1}^{m} \frac{(x\omega_m^\ell)^{k-2\gamma} Q_k(x\omega_m^\ell)}{(1-x^2\omega_m^{2\ell})^{k+1}} \cdot \frac{\arcsin(x\omega_m^\ell)}{\sqrt{1-x^2\omega_m^{2\ell}}}.
\end{aligned}$$

Interchanging the summation order, we get the following equation:

$$\Omega_m^\gamma\big((2x)^{2m}\big) = \sum_{k=0}^{m} \frac{\lambda_k}{4^\gamma m} \sum_{\ell=1}^{m} \frac{(x\omega_m^\ell)^{k-2\gamma} P_k(x\omega_m^\ell)}{(1-x^2\omega_m^{2\ell})^{k+1}}$$

$$+\sum_{\ell=1}^{m}\text{hyp}(x\omega_m^\ell)\sum_{k=0}^{m}\frac{\lambda_k}{4\gamma m}\cdot\frac{(x\omega_m^\ell)^{1+k-2\gamma}Q_k(x\omega_m^\ell)}{(1-x^2\omega_m^{2\ell})^{k+1}}.$$

From the last expression, we may construct the system of linear equations

$$\begin{cases} V_0=\sum_{k=0}^{m}\frac{\lambda_k}{4\gamma m}\sum_{\ell=1}^{m}\frac{(x\omega_m^\ell)^{k-2\gamma}}{(1-x^2\omega_m^{2\ell})^{k+1}}P_k(x\omega_m^\ell),\\ V_\ell=\sum_{k=0}^{m}\frac{\lambda_k}{4\gamma m}\cdot\frac{(x\omega_m^\ell)^{1+k-2\gamma}}{(1-x^2\omega_m^{2\ell})^{k+1}}Q_k(x\omega_m^\ell),\ \ell=1,2,\cdots,m.\end{cases}\tag{12}$$

By resolving this system of equations with $m+1$ variables $\{\lambda_k\}_{k=0}^m$, we derive the following general identity.

THEOREM 2. *For any given $m+1$ constants $\{V_k\}_{k=0}^m$, let $\{\lambda_k\}_{k=0}^m$ be the solution of the linear system* (12)*. Then for an arbitrary odd $m\in\mathbb{N}$ and a real x subject to $|x|<1$, the Ω-series* (1) *determined by the corresponding Λ-polynomial* (2) *is evaluated by*

$$\Omega_m^\gamma\big((2x)^{2m}\big)=V_0+\sum_{\ell=1}^{m}V_\ell\cdot\text{hyp}(x\omega_m^\ell).$$

It is not hard to check, by means of D'Alembert's test, that the above Ω-series is convergent for all the real x subject to $|x|<1$. Observe further that

$$\text{hyp}(x\omega_m^m)=\text{hyp}(x)=\frac{\arcsin(x)}{x\sqrt{1-x^2}}$$

is always real and can be expressed in terms of π or special values of the logarithm function for some properly chosen x. Therefore for the fixed constants $V_\ell=\delta_{\ell,m}$ with $0\leqslant\ell\leqslant m$, the corresponding solution of the last linear system will lead to the evaluation $\Omega_m^\gamma\big((2x)^{2m}\big)=V_m\cdot\text{hyp}(x)$.

We shall provide six examples for the multisection series $\Omega_m^\gamma(x)$ with $m=3$ and 5, where each example will be derived from Theorem 2 by specifying parameters as highlighted in its header.

3.1. Formulae corresponding to $m=3$

According to $\text{hyp}(\frac{e^{i\pi}}{2})=\frac{2\pi}{3\sqrt{3}}$, we can work out the first example.

EXAMPLE 1. (Theorem 2: $m=3$, $x=1/2$ and $V_i=\delta_{i,3}$ for $i=0,\cdots,3$)

(a) $\displaystyle\frac{200\pi}{81\sqrt{3}}=\sum_{n=0}^{\infty}\frac{\Lambda_3(n)}{\binom{6n}{3n}}$, where $\Lambda_3(n)=-8+103n-273n^2+378n^3$;

(b) $\displaystyle\frac{40\pi}{81\sqrt{3}}=\sum_{n=0}^{\infty}\frac{\Lambda_3(n)}{\binom{6n+2}{3n+1}}$, where $\Lambda_3(n)=2+29n+168n^2-189n^3$;

(c) $\displaystyle\frac{80\pi}{81\sqrt{3}}=\sum_{n=0}^{\infty}\frac{\Lambda_3(n)}{\binom{6n+4}{3n+2}}$, where $\Lambda_3(n)=-56-75n+882n^2+1701n^3$.

Analogously according to $\mathrm{hyp}(\frac{e^{i\pi}}{\sqrt{2}}) = \frac{\pi}{2}$, we get the second example.

EXAMPLE 2. (Theorem 2: $m = 3$, $x = 1/\sqrt{2}$ and $V_i = \delta_{i,3}$ for $i = 0, \cdots, 3$)

(a) $$\frac{75\pi}{2} = \sum_{n=0}^{\infty} \frac{\Lambda_3(n)}{\binom{6n}{3n}} 8^n, \text{ where } \Lambda_3(n) = -162 + 850n - 2961n^2 + 1638n^3;$$

(b) $$\frac{15\pi}{4} = \sum_{n=0}^{\infty} \frac{\Lambda_3(n)}{\binom{6n+2}{3n+1}} 8^n, \text{ where } \Lambda_3(n) = 168 + 674n + 1512n^2 - 1449n^3;$$

(c) $$\frac{5\pi}{8} = \sum_{n=0}^{\infty} \frac{\Lambda_3(n)}{\binom{6n+4}{3n+2}} 8^n, \text{ where } \Lambda_3(n) = -244 - 564n + 21n^2 + 756n^3.$$

Finally according to $\mathrm{hyp}(\frac{\sqrt{3}}{2} e^{i\pi}) = \frac{4\pi}{3\sqrt{3}}$, we have the third example.

EXAMPLE 3. (Theorem 2: $m = 3$, $x = \sqrt{3}/2$ and $V_i = \delta_{i,3}$ for $i = 0, \cdots, 3$)

(a) $$1200\sqrt{3}\pi = \sum_{n=0}^{\infty} \frac{\Lambda_3(n)}{\binom{6n}{3n}} 27^n,$$
where $\Lambda_3(n) = -7668 + 5827n - 110145n^2 + 27306n^3$;

(b) $$240\sqrt{3}\pi = \sum_{n=0}^{\infty} \frac{\Lambda_3(n)}{\binom{6n+2}{3n+1}} 27^n,$$
where $\Lambda_3(n) = 33966 + 117259n + 166104n^2 - 54279n^3$;

(c) $$\frac{160\pi}{3\sqrt{3}} = \sum_{n=0}^{\infty} \frac{\Lambda_3(n)}{\binom{6n+4}{3n+2}} 27^n,$$
where $\Lambda_3(n) = -19872 - 46589n - 24778n^2 + 12099n^3$.

3.2. Formulae corresponding to $m = 5$

According to $\mathrm{hyp}(\frac{e^{i\pi}}{2}) = \frac{2\pi}{3\sqrt{3}}$, we can work out the first example.

EXAMPLE 4. (Theorem 2: $m = 5$, $x = 1/2$ and $V_i = \delta_{i,5}$ for $i = 0, \cdots, 5$)

(a) $$\frac{7840\pi}{81\sqrt{3}} = \sum_{n=0}^{\infty} \frac{\Lambda_5(n)}{\binom{10n}{5n}},$$
where $$\Lambda_5(n) = -324 + 9348n - 79475n^2 + 331250n^3 - 570625n^4 + 426250n^5;$$

(b) $$\frac{7840\pi}{81\sqrt{3}} = \sum_{n=0}^{\infty} \frac{\Lambda_5(n)}{\binom{10n+2}{5n+1}},$$
where $$\Lambda_5(n) = -288 + 8046n + 76925n^2 - 398875n^3 + 818125n^4 - 213125n^5;$$

$$\text{(c)}\quad \frac{2240\pi}{81\sqrt{3}}=\sum_{n=0}^{\infty}\frac{\Lambda_5(n)}{\binom{10n+4}{5n+2}},$$
$$\text{where } \Lambda_5(n)=-1224-882n+68275n^2 -121625n^3-611875n^4+1491875n^5;$$

$$\text{(d)}\quad \frac{224\pi}{81\sqrt{3}}=\sum_{n=0}^{\infty}\frac{\Lambda_5(n)}{\binom{10n+6}{5n+3}},$$
$$\text{where } \Lambda_5(n)=936+5250n+34895n^2 +64150n^3-196625n^4-426250n^5;$$

$$\text{(e)}\quad \frac{896\pi}{81\sqrt{3}}=\sum_{n=0}^{\infty}\frac{\Lambda_5(n)}{\binom{10n+8}{5n+4}},$$
$$\text{where } \Lambda_5(n)=-12096-41910n+255205n^2 +1996525n^3+4197875n^4+2770625n^5.$$

Analogously according to $\text{hyp}(\frac{e^{i\pi}}{\sqrt{2}})=\frac{\pi}{2}$, we get the second example.

EXAMPLE 5. (Theorem 2: $m=5$, $x=1/\sqrt{2}$ and $V_i=\delta_{i,5}$ for $i=0,\cdots,5$)

$$\text{(a)}\quad \frac{59535\pi}{2}=\sum_{n=0}^{\infty}\frac{\Lambda_5(n)}{\binom{10n}{5n}}32^n,$$
$$\text{where } \Lambda_5(n)=-104856+1545449n-9968350n^2 +22046625n^3-26065625n^4+10191250n^5;$$

$$\text{(b)}\quad \frac{6615\pi}{4}=\sum_{n=0}^{\infty}\frac{\Lambda_5(n)}{\binom{10n+2}{5n+1}}32^n,$$
$$\text{where } \Lambda_5(n)=61296+590656n+2340075n^2 -4977500n^3+16093125n^4-9513125n^5;$$

$$\text{(c)}\quad \frac{945\pi}{8}=\sum_{n=0}^{\infty}\frac{\Lambda_5(n)}{\binom{10n+4}{5n+2}}32^n,$$
$$\text{where } \Lambda_5(n)=-126648-674973n-1200250n^2 -2000375n^3-3235625n^4+4805000n^5;$$

$$\text{(d)}\quad \frac{567\pi}{16}=\sum_{n=0}^{\infty}\frac{\Lambda_5(n)}{\binom{10n+6}{5n+3}}32^n,$$
$$\text{where } \Lambda_5(n)=330816+1469380n+2816755n^2 +2704200n^3-1532125n^4-4785625n^5;$$

$$\text{(e)}\quad \frac{189\pi}{32}=\sum_{n=0}^{\infty}\frac{\Lambda_5(n)}{\binom{10n+8}{5n+4}}32^n,$$

$$\text{where } \Lambda_5(n) = -187824 - 703285n - 910170n^2 + 125375n^3 + 1592625n^4 + 1201250n^5.$$

Finally according to $\text{hyp}(\frac{\sqrt{3}}{2}e^{i\pi}) = \frac{4\pi}{3\sqrt{3}}$, we have the third example.

EXAMPLE 6. (Theorem 2: $m = 5$, $x = \sqrt{3}/2$ and $V_i = \delta_{i,5}$ for $i = 0, \cdots, 5$)

(a) $$\sqrt{3}\pi = \sum_{n=0}^{\infty} \frac{\Lambda_5(n)}{\binom{10n}{5n}} \frac{3^{5n}}{102876480},$$

$$\text{where } \Lambda_5(n) = -549773892 + 1921522776n - 40788238625n^2 + 44230474250n^3 - 95132351875n^4 + 22597258750n^5;$$

(b) $$\sqrt{3}\pi = \sum_{n=0}^{\infty} \frac{\Lambda_5(n)}{\binom{10n+2}{5n+1}} \frac{3^{5n}}{3810240},$$

$$\text{where } \Lambda_5(n) = 1058881248 + 7967417106n + 23519492675n^2 + 15230532875n^3 + 48362359375n^4 - 15059144375n^5;$$

(c) $$\sqrt{3}\pi = \sum_{n=0}^{\infty} \frac{\Lambda_5(n)}{\binom{10n+4}{5n+2}} \frac{3^{5n}}{362880},$$

$$\text{where } \Lambda_5(n) = -2524970232 - 13752921654n - 28705576275n^2 - 31929532375n^3 - 22200688125n^4 + 10040243125n^5;$$

(d) $$\sqrt{3}\pi = \sum_{n=0}^{\infty} \frac{\Lambda_5(n)}{\binom{10n+6}{5n+3}} \frac{3^{5n}}{36288},$$

$$\text{where } \Lambda_5(n) = 2237207688 + 9899119050n + 17463183815n^2 + 15225220000n^3 + 4053902875n^4 - 3346585000n^5;$$

(e) $$\sqrt{3}\pi = \sum_{n=0}^{\infty} \frac{\Lambda_5(n)}{\binom{10n+8}{5n+4}} \frac{3^{5n}}{16128},$$

$$\text{where } \Lambda_5(n) = -3361774752 - 12542542170n - 18201581285n^2 - 11418939725n^3 - 470993875n^4 + 2231219375n^5.$$

4. Alternating series identities

In this section, we are going to investigate the following alternating series

$$\Omega_m^{\gamma}\big(-(2x)^{2m}\big) := \sum_{n=0}^{\infty} (-1)^n \frac{\Lambda_m(n)}{\binom{2mn+2\gamma}{mn+\gamma}} (2x)^{2mn}.$$

Analogously, this series converges for all the real x subject to $|x| < 1$.

Replacing g by h and x by x/ω_{4m} in (10) yields an alternative relation

$$\sum_{n=0}^{\infty}(-1)^n\frac{(2x)^{2mn+2\gamma}}{\binom{2mn+2\gamma}{mn+\gamma}}=\frac{1}{m}\sum_{k=1}^{m}\omega_{4m}^{2\gamma(1-2k)}h(x\omega_{4m}^{2k-1}).$$

Combining this with (5), we have the following expression

$$\begin{aligned}\Omega_m^{\gamma}\big(-(2x)^{2m}\big)&=\sum_{k=0}^{m}x^{k-2\gamma}\frac{\lambda_k}{4^{\gamma}}\mathscr{D}_x^k\sum_{n=0}^{\infty}(-1)^n\frac{(2x)^{2mn+2\gamma}}{\binom{2mn+2\gamma}{mn+\gamma}}\\&=\sum_{k=0}^{m}\frac{\lambda_k x^{k-2\gamma}}{4^{\gamma}m}\mathscr{D}_x^k\sum_{\ell=1}^{m}\omega_{4m}^{2\gamma(1-2\ell)}h(x\omega_{4m}^{2\ell-1})\\&=\sum_{k=0}^{m}\frac{\lambda_k}{4^{\gamma}m}\sum_{\ell=1}^{m}\frac{(x\omega_{4m}^{2\ell-1})^{k-2\gamma}P_k(x\omega_{4m}^{2\ell-1})}{(1-x^2\omega_{4m}^{4\ell-2})^{k+1}}\\&\quad+\sum_{k=0}^{m}\frac{\lambda_k}{4^{\gamma}m}\sum_{\ell=1}^{m}\frac{(x\omega_{4m}^{2\ell-1})^{k-2\gamma}Q_k(x\omega_{4m}^{2\ell-1})}{(1-x^2\omega_{4m}^{4\ell-2})^{k+1}}\cdot\frac{\arcsin(x\omega_{4m}^{2\ell-1})}{\sqrt{1-x^2\omega_{4m}^{4\ell-2}}}.\end{aligned}$$

Interchanging the summation order, we get further the following equation:

$$\begin{aligned}\Omega_m^{\gamma}\big(-(2x)^{2m}\big)&=\sum_{k=0}^{m}\frac{\lambda_k}{4^{\gamma}m}\sum_{\ell=1}^{m}\frac{(x\omega_{4m}^{2\ell-1})^{k-2\gamma}P_k(x\omega_{4m}^{2\ell-1})}{(1-x^2\omega_{4m}^{4\ell-2})^{k+1}}\\&\quad+\sum_{\ell=1}^{m}\operatorname{hyp}(x\omega_{4m}^{2\ell-1})\sum_{k=0}^{m}\frac{\lambda_k}{4^{\gamma}m}\cdot\frac{(x\omega_{4m}^{2\ell-1})^{1+k-2\gamma}Q_k(x\omega_{4m}^{2\ell-1})}{(1-x^2\omega_{4m}^{4\ell-2})^{k+1}}.\end{aligned}$$

From the last expression, we may construct the following system of equations

$$\begin{cases}W_0=\displaystyle\sum_{k=0}^{m}\frac{\lambda_k}{4^{\gamma}m}\sum_{\ell=1}^{m}\frac{(x\omega_{4m}^{2\ell-1})^{k-2\gamma}}{(1-x^2\omega_{4m}^{4\ell-2})^{k+1}}P_k(x\omega_{4m}^{2\ell-1}),\\ W_\ell=\displaystyle\sum_{k=0}^{m}\frac{\lambda_k}{4^{\gamma}m}\cdot\frac{(x\omega_{4m}^{2\ell-1})^{1+k-2\gamma}}{(1-x^2\omega_{4m}^{4\ell-2})^{k+1}}Q_k(x\omega_{4m}^{2\ell-1}),\ \ell=1,2,\cdots,m.\end{cases}\tag{13}$$

Resolving this system of equations will result in the following general identity.

THEOREM 3. *For any given $m+1$ constants $\{W_k\}_{k=0}^m$, let $\{\lambda_k\}_{k=0}^m$ be the solution of the linear system* (13). *Then for an arbitrary $m\in\mathbb{N}$ and a real subject to $|x|<1$, the Ω-series* (1) *determined by the corresponding Λ-polynomial* (2) *is evaluated by*

$$\Omega_m^{\gamma}\big(-(2x)^{2m}\big)=W_0+\sum_{\ell=1}^{m}W_\ell\cdot\operatorname{hyp}(x\omega_{4m}^{2\ell-1}).$$

It should be pointed out that the last theorem cannot be deduced from Theorem 2 by simply replacing x by x/ω_{4m} because Theorem 3 is valid for all the $m\in\mathbb{N}$, instead of odd $m\in\mathbb{N}$.

When $\mathrm{hyp}(x\omega_{4m}^{2k-1})$ has a "good" expression in terms of π and special values of the logarithm function given in Subsection 2.3, we may specify the corresponding $W_k = 1$ and the others $W_\ell = 0$ for $\ell \neq k$. Resolving the corresponding linear system will give rise to infinite series formulae concerning π and special values of the logarithm function. We record fifteen examples as exemplification.

4.1. Formulae corresponding to $m = 2$

Firstly, by separating π from the logarithmic value in

$$\mathrm{hyp}\Big(\frac{\sqrt{3}}{2}\mathrm{e}^{\frac{\pi i}{4}}\Big) = \frac{4-2i}{15\sqrt{3}}\{\pi + 3i\ln(2+\sqrt{3})\} \tag{14}$$

we can work out the following two examples.

EXAMPLE 7. (Theorem 3: $m = 2,\ x = \sqrt{3}/2$ and $W_k = \delta_{k,1}$ for $k = 0,1,2$)

(a) $36\sqrt{3}\pi = \sum_{n=0}^{\infty} \frac{\Lambda_2(n)}{\binom{4n}{2n}}(-9)^n$, where $\Lambda_2(n) = -1341 - 1250n - 4000n^2$;

(b) $2\sqrt{3}\pi = \sum_{n=0}^{\infty} \frac{\Lambda_2(n)}{\binom{4n+2}{2n+1}}(-9)^n$, where $\Lambda_2(n) = 423 + 580n + 700n^2$.

EXAMPLE 8. (Theorem 3: $m = 2,\ x = \sqrt{3}/2$ and $W_k = \delta_{k,1}$ for $k = 0,1,2$)

(a) $27\sqrt{3}\ln(2+\sqrt{3}) = \sum_{n=0}^{\infty} \frac{\Lambda_2(n)}{\binom{4n}{2n}}(-9)^n$, where $\Lambda_2(n) = 153 - 25n + 1000n^2$;

(b) $3\sqrt{3}\ln(2+\sqrt{3}) = \sum_{n=0}^{\infty} \frac{\Lambda_2(n)}{\binom{4n+2}{2n+1}}(-9)^n$, where $\Lambda_2(n) = -243 - 430n - 325n^2$.

Then by separating π from the logarithmic value in

$$\mathrm{hyp}\Big(\frac{\mathrm{e}^{\frac{\pi i}{4}}}{\sqrt[4]{8}}\Big) = \frac{\sqrt{2}-i}{6\sqrt{2}}\{\pi + 4i\ln(1+\sqrt{2})\} \tag{15}$$

we have similarly the next two examples.

EXAMPLE 9. (Theorem 3: $m = 2,\ x = 1/\sqrt[4]{8}$ and $W_k = \delta_{k,1}$ for $k = 0,1,2$)

(a) $\frac{9\pi}{2} = \sum_{n=0}^{\infty} \frac{\Lambda_2(n)}{\binom{4n}{2n}}(-2)^n$, where $\Lambda_2(n) = -65 + 146n - 576n^2$;

(b) $\frac{3\pi}{4} = \sum_{n=0}^{\infty} \frac{\Lambda_2(n)}{\binom{4n+2}{2n+1}}(-2)^n$, where $\Lambda_2(n) = 41 + 68n + 180n^2$.

EXAMPLE 10. (Theorem 3: $m=2, x=1/\sqrt[4]{8}$ and $W_k=\delta_{k,1}$ for $k=0,1,2$)

$$\text{(a)}\quad \frac{9\ln(1+\sqrt{2})}{\sqrt{2}}=\sum_{n=0}^{\infty}\frac{\Lambda_2(n)}{\binom{4n}{2n}}(-2)^n, \quad \text{where} \quad \Lambda_2(n)=-2-79n+72n^2;$$

$$\text{(b)}\quad \frac{3\ln(1+\sqrt{2})}{2\sqrt{2}}=\sum_{n=0}^{\infty}\frac{\Lambda_2(n)}{\binom{4n+2}{2n+1}}(-2)^n, \quad \text{where} \quad \Lambda_2(n)=-13-40n-63n^2.$$

4.2. Formulae corresponding to $m=3$

According to $\operatorname{hyp}(\frac{e^{i\pi/2}}{2})=\frac{4}{\sqrt{5}}\ln\frac{1+\sqrt{5}}{2}$, we get the following first example.

EXAMPLE 11. (Theorem 3: $m=3, x=1/2$ and $W_k=\delta_{k,2}$ for $k=0,\cdots,3$)

$$\text{(a)}\quad 48\sqrt{5}\ln\frac{1+\sqrt{5}}{2}=\sum_{n=0}^{\infty}(-1)^n\frac{\Lambda_3(n)}{\binom{6n}{3n}},$$

where $\Lambda_3(n)=-252-5809n+24291n^2-26910n^3$;

$$\text{(b)}\quad \frac{48}{\sqrt{5}}\ln\frac{1+\sqrt{5}}{2}=\sum_{n=0}^{\infty}(-1)^n\frac{\Lambda_3(n)}{\binom{6n+2}{3n+1}},$$

where $\Lambda_3(n)=222-205n-1512n^2+9945n^3$;

$$\text{(c)}\quad \frac{32}{\sqrt{5}}\ln\frac{1+\sqrt{5}}{2}=\sum_{n=0}^{\infty}(-1)^n\frac{\Lambda_3(n)}{\binom{6n+4}{3n+2}},$$

where $\Lambda_3(n)=-208-1329n-4866n^2-5265n^3$.

Then according to $\operatorname{hyp}(\frac{e^{i\pi/2}}{\sqrt{2}})=\frac{\ln(2+\sqrt{3})}{\sqrt{3}}$, we obtain the next example.

EXAMPLE 12. (Theorem 3: $m=3, x=1/\sqrt{2}$ and $W_k=\delta_{k,2}$ for $k=0,\cdots,3$)

$$\text{(a)}\quad \frac{25}{3\sqrt{3}}\ln(2+\sqrt{3})=\sum_{n=0}^{\infty}\frac{\Lambda_3(n)}{\binom{6n}{3n}}(-8)^n,$$

where $\Lambda_3(n)=-52-452n+735n^2-1026n^3$;

$$\text{(b)}\quad \frac{5}{6\sqrt{3}}\ln(2+\sqrt{3})=\sum_{n=0}^{\infty}\frac{\Lambda_3(n)}{\binom{6n+2}{3n+1}}(-8)^n,$$

where $\Lambda_3(n)=42+88n+48n^2+297n^3$;

$$\text{(c)}\quad \frac{5}{12\sqrt{3}}\ln(2+\sqrt{3})=\sum_{n=0}^{\infty}\frac{\Lambda_3(n)}{\binom{6n+4}{3n+2}}(-8)^n,$$

where $\Lambda_3(n)=-37-99n-153n^2-81n^3$.

Finally according to $\operatorname{hyp}(\frac{\sqrt{3}}{2}e^{i\pi/2})=\frac{2}{\sqrt{21}}\ln\frac{5+\sqrt{21}}{2}$, we have the third example.

EXAMPLE 13. (Theorem 3: $m=3$, $x=\sqrt{3}/2$ and $W_k=\delta_{k,2}$ for $k=0,\cdots,3$)

(a) $$\frac{\sqrt{7}}{\sqrt{3}}\ln\frac{\sqrt{3}+\sqrt{7}}{2}=\frac{49}{10800}\sum_{n=0}^{\infty}\frac{\Lambda_3(n)}{\binom{6n}{3n}}(-27)^n,$$
where $\Lambda_3(n)=-3456-21815n+4377n^2-30186n^3$;

(b) $$\frac{\sqrt{7}}{\sqrt{3}}\ln\frac{\sqrt{3}+\sqrt{7}}{2}=\frac{49}{2160}\sum_{n=0}^{\infty}\frac{\Lambda_3(n)}{\binom{6n+2}{3n+1}}(-27)^n,$$
where $\Lambda_3(n)=7398+19159n+16632n^2+20241n^3$;

(c) $$\frac{\sqrt{7}}{\sqrt{3}}\ln\frac{\sqrt{3}+\sqrt{7}}{2}=\frac{49}{160}\sum_{n=0}^{\infty}\frac{\Lambda_3(n)}{\binom{6n+4}{3n+2}}(-27)^n,$$
where $\Lambda_3(n)=-744-1399n-1334n^2-39n^3$.

4.3. Formulae corresponding to $m=4$

Firstly, by separating π from the logarithmic value in

$$\mathrm{hyp}\Big(\frac{e^{i\pi/8}}{\sqrt[4]{2}}\Big)=\frac{\pi+2i\ln(1+\sqrt{2})}{2\sqrt{2}}$$

we can work out the following two examples, where the formula (a) in Example 14 is due to Zheng [24, Example 4.1].

EXAMPLE 14. (Theorem 3: $m=4$, $x=1/\sqrt[4]{2}$ and $W_k=\delta_{k,1}$ for $k=0,\cdots,4$)

(a) $$\frac{11025\pi}{8\sqrt{2}}=\sum_{n=0}^{\infty}\frac{\Lambda_4(n)}{\binom{8n}{4n}}(-64)^n,$$
where $\Lambda_4(n)=-5856+17803n-223184n^2+163232n^3-133120n^4$;

(b) $$\frac{225\pi}{64\sqrt{2}}=\sum_{n=0}^{\infty}\frac{\Lambda_4(n)}{\binom{8n+2}{4n+1}}(-64)^n,$$
where $\Lambda_4(n)=480+2611n+5630n^2-544n^3+5440n^4$;

(c) $$\frac{315\pi}{64\sqrt{2}}=\sum_{n=0}^{\infty}\frac{\Lambda_4(n)}{\binom{8n+4}{4n+2}}(-64)^n,$$
where $\Lambda_4(n)=-10656-40305n-51512n^2-33888n^3-37120n^4$;

(d) $$\frac{315\pi}{256\sqrt{2}}=\sum_{n=0}^{\infty}\frac{\Lambda_4(n)}{\binom{8n+6}{4n+3}}(-64)^n,$$
where $\Lambda_4(n)=12000+36211n+43900n^2+34784n^3+18560n^4$.

EXAMPLE 15. (Theorem 3: $m=4$, $x=1/\sqrt[4]{2}$ and $W_k=\delta_{k,1}$ for $k=0,\cdots,4$)

(a) $$\frac{\ln(3+2\sqrt{2})}{2\sqrt{2}}=\sum_{n=0}^{\infty}\frac{\Lambda_4(n)}{\binom{8n}{4n}}\frac{(-64)^n}{11025},$$

where $\Lambda_4(n) = 14019 - 30932n + 525856n^2 - 293248n^3 + 327680n^4$;

(b) $$\frac{\ln(3+2\sqrt{2})}{4\sqrt{2}} = \sum_{n=0}^{\infty} \frac{\Lambda_4(n)}{\binom{8n+2}{4n+1}} \frac{(-64)^n}{1575},$$

where $\Lambda_4(n) = -29595 - 159992n - 341920n^2 + 51968n^3 - 327680n^4$;

(c) $$\frac{\ln(3+2\sqrt{2})}{4\sqrt{2}} = \sum_{n=0}^{\infty} \frac{\Lambda_4(n)}{\binom{8n+4}{4n+2}} \frac{(-64)^n}{315},$$

where $\Lambda_4(n) = 95715 + 363312n + 468928n^2 + 316416n^3 + 343040n^4$;

(d) $$\frac{\ln(3+2\sqrt{2})}{8\sqrt{2}} = \sum_{n=0}^{\infty} \frac{\Lambda_4(n)}{\binom{8n+6}{4n+3}} \frac{(-64)^n}{15},$$

where $\Lambda_4(n) = -10415 - 31392n - 37696n^2 - 29184n^3 - 15360n^4$.

Then by separating π from the logarithmic value in

$$\text{hyp}\Big(\frac{e^{i\pi/8}}{\sqrt[4]{32}}\Big) = \frac{3-i}{10}\big(\pi + 2i\ln 2\big)$$

we get the next two examples, where the formula (a) in the first example has previously appeared in [1, Example 3.2] and [24, Example 3.1].

EXAMPLE 16. (Theorem 3: $m = 4$, $x = 1/\sqrt[4]{32}$ and $W_k = \delta_{k,1}$ for $k = 0, \cdots, 4$)

(a) $$\frac{\pi}{2} = \sum_{n=0}^{\infty} \frac{\Lambda_4(n)}{\binom{8n}{4n}} \frac{(-1/4)^n}{11025},$$

where $\Lambda_4(n) = -44643 + 1937974n - 17485067n^2 + 55101236n^3 - 57596800n^4$;

(b) $$\pi = \sum_{n=0}^{\infty} \frac{\Lambda_4(n)}{\binom{8n+2}{4n+1}} \frac{(-1/4)^n}{1575},$$

where $\Lambda_4(n) = -3405 + 239056n + 2384255n^2 - 4060174n^3 - 5403800n^4$;

(c) $$\pi = \sum_{n=0}^{\infty} \frac{\Lambda_4(n)}{\binom{8n+4}{4n+2}} \frac{(-1/4)^n}{1260},$$

where $\Lambda_4(n) = -26541 + 404190n + 6175603n^2 - 877488n^3 - 37375600n^4$;

(d) $$\pi = \sum_{n=0}^{\infty} \frac{\Lambda_4(n)}{\binom{8n+6}{4n+3}} \frac{(-1/4)^n}{105},$$

where $\Lambda_4(n) = 25610 + 136695n + 1637332n^2 + 5851587n^3 + 5571900n^4$.

EXAMPLE 17. (Theorem 3: $m=4$, $x=1/\sqrt[4]{32}$ and $W_k=\delta_{k,1}$ for $k=0,\cdots,4$)

(a) $$\ln 2=\sum_{n=0}^{\infty}\frac{\Lambda_4(n)}{\binom{8n}{4n}}\frac{(-1/4)^n}{176400},$$
where $\Lambda_4(n)=-88491-14550862n+122888096n^2-432150368n^3+266598400n^4$;

(b) $$\ln 2=\sum_{n=0}^{\infty}\frac{\Lambda_4(n)}{\binom{8n+2}{4n+1}}\frac{(-1/4)^n}{12600},$$
where $\Lambda_4(n)=-51465-783382n-3795860n^2-812072n^3-29946400n^4$;

(c) $$\ln 2=\sum_{n=0}^{\infty}\frac{\Lambda_4(n)}{\binom{8n+4}{4n+2}}\frac{(-1/4)^n}{2520},$$
where $\Lambda_4(n)=-13887-338820n-2605204n^2-6862416n^3-5379200n^4$;

(d) $$\ln 2=\sum_{n=0}^{\infty}\frac{\Lambda_4(n)}{\binom{8n+6}{4n+3}}\frac{(-1/4)^n}{1260},$$
where $\Lambda_4(n)=-22380-570260n-4486931n^2-12273646n^3-10340200n^4$.

4.4. Formulae corresponding to $m=6$

By separating π from the logarithmic value in (14), we get the next two examples.

EXAMPLE 18. (Theorem 3: $m=6$, $x=\sqrt{3}/2$ and $W_k=\delta_{k,2}$ for $k=0,\cdots,6$)

(a) $$\frac{2\pi}{\sqrt{3}}=\sum_{n=0}^{\infty}\frac{\Lambda_6(n)}{\binom{12n}{6n}}\frac{(-729)^n}{2593344600},$$
where $\Lambda_6(n)=-5387310000+651979343999n-1261120210685n^2+14199336146970n^3+413588141460n^4+102129696936n^5+3808298851200n^6$;

(b) $$\frac{2\pi}{\sqrt{3}}=\sum_{n=0}^{\infty}\frac{\Lambda_6(n)}{\binom{12n+2}{6n+1}}\frac{(-729)^n}{117879300},$$
where $\Lambda_6(n)=91789293960+940806119161n+3901783340435n^2+5542344867780n^3+17690071993920n^4-7853003377056n^5+7288661134800n^6$;

(c) $$\frac{2\pi}{\sqrt{3}}=\sum_{n=0}^{\infty}\frac{\Lambda_6(n)}{\binom{12n+4}{6n+2}}\frac{(-729)^n}{2910600},$$

$$\begin{aligned}\text{where } \Lambda_6(n) = -102492581310 - 750044788543n - 2208603591930n^2 \\ - 3497295656225n^3 - 3039634783770n^4 \\ - 51546143052n^5 - 1512229073400n^6;\end{aligned}$$

(d) $$\frac{2\pi}{\sqrt{3}} = \sum_{n=0}^{\infty} \frac{\Lambda_6(n)}{\binom{12n+6}{6n+3}} \frac{(-729)^n}{207900},$$

$$\begin{aligned}\text{where } \Lambda_6(n) = 90162698130 + 534126652561n + 1300924281070n^2 \\ + 1656674437695n^3 + 1090050880050n^4 \\ + 444261799764n^5 + 445362283800n^6;\end{aligned}$$

(e) $$\frac{2\pi}{\sqrt{3}} = \sum_{n=0}^{\infty} \frac{\Lambda_6(n)}{\binom{12n+8}{6n+4}} \frac{(-729)^n}{166320},$$

$$\begin{aligned}\text{where } \Lambda_6(n) = -436029305040 - 2219297544437n - 4649560710464n^2 \\ - 5110164469629n^3 - 3290093603622n^4 \\ - 1779805345500n^5 - 918360585000n^6;\end{aligned}$$

(f) $$\frac{2\pi}{\sqrt{3}} = \sum_{n=0}^{\infty} \frac{\Lambda_6(n)}{\binom{12n+10}{6n+5}} \frac{(-729)^n}{15400},$$

$$\begin{aligned}\text{where } \Lambda_6(n) = 128261863800 + 574203431967n + 1062022202180n^2 \\ + 1068407754575n^3 + 701225415870n^4 \\ + 369737572788n^5 + 125525694600n^6.\end{aligned}$$

EXAMPLE 19. (Theorem 3: $m = 6$, $x = \sqrt{3}/2$ and $W_k = \delta_{k,2}$ for $k = 0, \cdots, 6$)

(a) $$\frac{2}{\sqrt{3}} \ln(2+\sqrt{3}) = \sum_{n=0}^{\infty} \frac{\Lambda_6(n)}{\binom{12n}{6n}} \frac{(-729)^n}{1945008450},$$

$$\begin{aligned}\text{where } \Lambda_6(n) = 10595832750 + 49798089833n + 817260572230n^2 \\ + 815892243615n^3 - 693432589680n^4 \\ + 953786473812n^5 - 78652749600n^6;\end{aligned}$$

(b) $$\frac{2}{\sqrt{3}} \ln(2+\sqrt{3}) = \sum_{n=0}^{\infty} \frac{\Lambda_6(n)}{\binom{12n+2}{6n+1}} \frac{(-729)^n}{176818950},$$

$$\begin{aligned}\text{where } \Lambda_6(n) = -15770158110 - 149005318351n - 565100692085n^2 \\ - 246648769605n^3 - 3657975117345n^4 \\ + 1847707970796n^5 - 1293585414300n^6;\end{aligned}$$

(c) $$\frac{2}{\sqrt{3}} \ln(2+\sqrt{3}) = \sum_{n=0}^{\infty} \frac{\Lambda_6(n)}{\binom{12n+4}{6n+2}} \frac{(-729)^n}{1091475},$$

$$\text{where } \Lambda_6(n) = 7306203240 + 54411302972n + 164720036220n^2 + 272113093525n^3 + 243664907205n^4 - 6224902092n^5 + 123240671100n^6;$$

(d) $$\frac{2}{\sqrt{3}}\ln(2+\sqrt{3}) = \sum_{n=0}^{\infty} \frac{\Lambda_6(n)}{\binom{12n+6}{6n+3}} \frac{(-729)^n}{311850},$$

$$\text{where } \Lambda_6(n) = -31885057080 - 190348868126n - 464930555495n^2 - 588548910120n^3 - 384784965675n^4 - 162263479824n^5 - 156644223300n^6;$$

(e) $$\frac{2}{\sqrt{3}}\ln(2+\sqrt{3}) = \sum_{n=0}^{\infty} \frac{\Lambda_6(n)}{\binom{12n+8}{6n+4}} \frac{(-729)^n}{31185},$$

$$\text{where } \Lambda_6(n) = 16812253080 + 84112833224n + 173205332153n^2 + 188379121608n^3 + 121806913419n^4 + 65560363200n^5 + 32555722500n^6;$$

(f) $$\frac{2}{\sqrt{3}}\ln(2+\sqrt{3}) = \sum_{n=0}^{\infty} \frac{\Lambda_6(n)}{\binom{12n+10}{6n+5}} \frac{(-729)^n}{11550},$$

$$\text{where } \Lambda_6(n) = -15326028900 - 68789106111n - 128385658315n^2 - 130058450725n^3 - 84793886085n^4 - 44917548804n^5 - 16004544300n^6.$$

Finally, by separating π from the logarithmic value in (15), we can deduce two further examples below.

EXAMPLE 20. (Theorem 3: $m = 6$, $x = 1/\sqrt[4]{8}$ and $W_k = \delta_{k,2}$ for $k = 0, \cdots, 6$)

(a) $$\frac{\pi}{2} = \sum_{n=0}^{\infty} \frac{\Lambda_6(n)}{\binom{12n}{6n}} \frac{(-8)^n}{4002075},$$

$$\text{where } \Lambda_6(n) = -32011150 - 73727270n + 1577312553n^2 - 27977606010n^3 + 57401953020n^4 - 19518288120n^5 - 18785945088n^6;$$

(b) $$\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{\Lambda_6(n)}{\binom{12n+2}{6n+1}} \frac{(-8)^n}{363825},$$

$$\text{where } \Lambda_6(n) = -3854690 - 23407438n + 19118307n^2 - 2339283240n^3 + 1285157880n^4 + 8008538688n^5 - 8649007632n^6;$$

(c) $$\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{\Lambda_6(n)}{\binom{12n+4}{6n+2}} \frac{(-8)^n}{13475},$$

$$\text{where } \Lambda_6(n) = -857640 - 6488388n - 7686494n^2 - 241188885n^3 - 377839170n^4 + 2190796308n^5 - 1960969176n^6;$$

(d) $$\frac{\pi}{8} = \sum_{n=0}^{\infty} \frac{\Lambda_6(n)}{\binom{12n+6}{6n+3}} \frac{(-8)^n}{17325},$$

$$\text{where } \Lambda_6(n) = 17300680 + 104361526n + 296658012n^2 + 191790855n^3 - 1243291950n^4 + 517195044n^5 + 6318382968n^6;$$

(e) $$\frac{\pi}{8} = \sum_{n=0}^{\infty} \frac{\Lambda_6(n)}{\binom{12n+8}{6n+4}} \frac{(-8)^n}{3465},$$

$$\text{where } \Lambda_6(n) = -16662380 - 83047832n - 140221389n^2 - 74442132n^3 - 665677008n^4 - 2634020208n^5 - 2536395120n^6;$$

(f) $$\frac{\pi}{16} = \sum_{n=0}^{\infty} \frac{\Lambda_6(n)}{\binom{12n+10}{6n+5}} \frac{(-8)^n}{1925},$$

$$\text{where } \Lambda_6(n) = 10890740 + 47641520n + 110751371n^2 + 374701950n^3 + 1118189700n^4 + 1594640520n^5 + 816433344n^6.$$

EXAMPLE 21. (Theorem 3: $m = 6$, $x = 1/\sqrt[4]{8}$ and $W_k = \delta_{k,2}$ for $k = 0, \cdots, 6$)

(a) $$\frac{\ln(1+\sqrt{2})}{\sqrt{2}} = \sum_{n=0}^{\infty} \frac{\Lambda_6(n)}{\binom{12n}{6n}} \frac{(-8)^n}{4002075},$$

$$\text{where } \Lambda_6(n) = 2918435 - 271350755n + 5660009634n^2 - 35892192555n^3 + 113980760820n^4 - 146537688060n^5 + 65954251296n^6;$$

(b) $$\frac{\ln(1+\sqrt{2})}{2\sqrt{2}} = \sum_{n=0}^{\infty} \frac{\Lambda_6(n)}{\binom{12n+2}{6n+1}} \frac{(-8)^n}{363825},$$

$$\text{where } \Lambda_6(n) = -2000285 - 21263899n - 95907309n^2 + 124700310n^3 + 564955695n^4 - 604884996n^5 - 560633076n^6;$$

(c) $$\frac{\ln(1+\sqrt{2})}{2\sqrt{2}} = \sum_{n=0}^{\infty} \frac{\Lambda_6(n)}{\binom{12n+4}{6n+2}} \frac{(-8)^n}{13475},$$

where $\Lambda_6(n) = -212205 - 2503944n - 12153437n^2 + 7990605n^3 + 99010260n^4 - 103699116n^5 - 67149648n^6$;

(d) $$\frac{\ln(1+\sqrt{2})}{4\sqrt{2}} = \sum_{n=0}^{\infty} \frac{\Lambda_6(n)}{\binom{12n+6}{6n+3}} \frac{(-8)^n}{17325},$$

where $\Lambda_6(n) = -1573085 - 12501287n - 46362354n^2 - 31045005n^3 + 245791395n^4 + 71259912n^5 - 836544996n^6$;

(e) $$\frac{\ln(1+\sqrt{2})}{2\sqrt{2}} = \sum_{n=0}^{\infty} \frac{\Lambda_6(n)}{\binom{12n+8}{6n+4}} \frac{(-8)^n}{3465},$$

where $\Lambda_6(n) = 6289325 + 28861733n + 38190702n^2 + 16709787n^3 + 285022206n^4 + 1000720980n^5 + 901866312n^6$;

(f) $$\frac{\ln(1+\sqrt{2})}{4\sqrt{2}} = \sum_{n=0}^{\infty} \frac{\Lambda_6(n)}{\binom{12n+10}{6n+5}} \frac{(-8)^n}{1925},$$

where $\Lambda_6(n) = -2536835 - 12875405n - 42055148n^2 - 128241225n^3 - 286069140n^4 - 344692260n^5 - 162222912n^6$.

5. Appendix: Mathematica commands

In order to evaluate the infinite series in Section 3, we have appropriately devised, according to Theorem 2, the following *Mathematica* commands, where ‘hyp’ stands for the “hyp”-function whose special values are given in Subsection 2.3, and ‘vpiv’ corresponds to the Λ-polynomials under the setting $V_k = \delta_{k,m}$ for $k = 0, 1, \cdots, m$.

```
        (*pp[k,y] and qq[k,y] polynomials*)
hh[x_]:=1/(1-x^2)+x*ArcSin[x]/(1-x^2)^(3/2)
pp[k_,x_]:=Factor[(1-y^2)^(k+1)*(D[ArcSin[y]/Sqrt[1-y^2],{y,k+1}]-
          ArcSin[y]*D[1/Sqrt[1-y^2],{y,k+1}])/.y->x]
qq[k_,x_]:=Factor[(1-y^2)^(k+3/2)*D[1/Sqrt[1-y^2],{y,k+1}]/.y->x]

        (*Special values for variable x*)
ww[m_]:=Exp[2Pi*I/m]
xx:={2^(-1),2^(-1/2),3^(1/2)/2,2^(-1/4),2^(-3/4),2^(-5/4)}
xy[k_]:=Part[xx,k]
hpy[y_]:=Normal[I/y*Log[Sqrt[1-y^2]-y*I]/Sqrt[1-y^2]]
hyp[m_,k_,x_]:=FunctionExpand[hpy[x*ww[m]^k]]
```

```
        (*Hyperbolic representations*)
pq[r_,k_,y_]:=pp[k,y]*y^(k-2r)/(1-y^2)^(k+1)
qp[r_,k_,y_]:=qq[k,y]*y^(1+k-2r)/(1-y^2)^(k+1)
uv[0,m_,r_,x_]:=Sum[mm[i]/(4^r*m)*Sum[pq[r,i,x*ww[m]^k],
                  {k,1,m}],{i,0,m}]
uv[k_,m_,r_,x_]:=Sum[mm[i]/(4^r*m)*If[k==m,qp[r,i,x*ww[m]^k],
If[k<m/2,-2Im[qp[r,i,x*ww[m]^k]],2Re[qp[r,i,x*ww[m]^k]]]],{i,0,m}]

        (*Equation system and solutions*)
system[m_,r_,x_]:=FullSimplify[Table[uv[k,m,r,x]==0,{k,0,m}]]
wpiw[m_,r_,x_]:=Solve[ReplacePart[system[m,r,x],
                     uv[m,m,r,x]==1,1+m],Table[mm[k],{k,0,m}]]
weew[ee_,m_,r_,x_]:=Solve[ReplacePart[system[m,r,x],
                     uv[ee,m,r,x]==1,1+ee],Table[mm[k],{k,0,m}]]

        (*Weight polynomials*)
vv[m_,r_,n_]:=Sum[k!*mm[k]*Binomial[2m*n+2r,k],{k,0,m}]
vpiv[m_,r_,n_,x_]:=
    Factor[vv[m,r,n]/.Flatten[FullSimplify[wpiw[m,r,x]]]]
veev[ee_,m_,r_,n_,x_]:=
    Factor[vv[m,r,n]/.Flatten[FullSimplify[weew[ee,m,r,x]]]]
```

For instance, we can confirm Example 1 by executing the command below

```
SequenceForm[hyp[3,3,xy[1]],": ",
  ColumnForm[Table[vpiv[3,r,n,xy[1]],{r,0,2}]]]
```

that gives the following output:

$$\frac{2\pi}{3\sqrt{3}}: \quad \begin{cases} \frac{27}{100}(378n^3-273n^2+103n-8), & r=0; \\ \frac{27}{20}(-189n^3+168n^2+29n+2), & r=1; \\ \frac{27}{40}(1701n^3+882n^2-75n-56, & r=2. \end{cases}$$

Instead, for the alternating series in Section 4, we have to first replace the two functions "hyp" and "uv" in the above package by the following three commands:

```
hyp[m_, k_, x_] :=FullSimplify[hpy[x*ww[4m]^(2k - 1)]]
uv[0,m_,r_,x_]:=
    Sum[mm[i]/(4^r*m)*Sum[pq[r,i,x*ww[4m]^(2k-1)],{k,1,m}],{i,0,m}]
uv[k_,m_,r_,x_]:=Sum[mm[i]/(4^r*m)*qp[r,i,x*ww[4m]^(2k-1)],{i,0,m}]
```

Then we run the commands 'hyp' and 'veev' to determine the hyp-values and the Λ-polynomials. Finally, separating their real part from the imaginary part, we can derive, in view of Theorem 3, the corresponding alternating series identities.

For example, by carrying out the command below

```
SequenceForm[hyp[2,1,xy[3]],": ",
  ColumnForm[Table[veev[1,2,r,n,xy[3]],{r,0,1}]]]
```

we get the following expressions

$$\operatorname{hyp}\Big(\frac{\sqrt{3}}{2}e^{\frac{\pi i}{4}}\Big) = \frac{\frac{4+8i}{5}\ln\big(\frac{1-i}{4}\sqrt{6}+\frac{\sqrt{4-3i}}{2}\big)}{\sqrt{3}} = \frac{4-2i}{15\sqrt{3}}\big\{\pi+3i\ln(2+\sqrt{3})\big\}, \tag{16}$$

$$\Lambda_4(n) = \begin{cases} \frac{2+i}{162}\big\{(160+1120i)n^2-(166-188i)n-(63-288i)\big\}, & r=0; \\ -\frac{2+i}{9}\big\{(20+190i)n^2+(68+196i)n+(27+126i)\big\}, & r=1. \end{cases} \tag{17}$$

For the sake of brevity, let $\Re\boxed{\text{xx}}$ and $\Im\boxed{\text{xx}}$ stand, respectively, for the the real part and the imaginary part of the expression displayed in equation (xx). By separating the real part and the imaginary part in (16)

$$\Re\boxed{16} = \frac{4\pi+6\ln(2+\sqrt{3})}{15\sqrt{3}}, \qquad \Im\boxed{16} = \frac{12\ln(2+\sqrt{3})-2\pi}{15\sqrt{3}};$$

we can combine them to express π and a special logarithmic value

$$2\Re\boxed{16} - \Im\boxed{16} = \frac{2\pi}{3\sqrt{3}}, \tag{18}$$

$$\Re\boxed{16} + 2\Im\boxed{16} = \frac{2\ln(2+\sqrt{3})}{\sqrt{3}}. \tag{19}$$

Analogously, for the Λ-polynomial in (17), we have

$$\Re\boxed{17} = \begin{cases} \frac{1}{81}(-400n^2-260n-207), & r=0; \\ \frac{2}{3}(25n^2+10n+12), & r=1; \end{cases}$$

$$\Im\boxed{17} = \begin{cases} \frac{1}{54}(800n^2+70n+171), & r=0; \\ \frac{1}{9}(-400n^2-460n-279), & r=1. \end{cases}$$

Their linear combinations yield the following expressions

$$2\Re\boxed{17} - \Im\boxed{17} = \begin{cases} \frac{-1}{162}(4000n^2+1250n+1341), & r=0; \\ \frac{1}{9}(700n^2+580n+423), & r=1; \end{cases} \tag{20}$$

$$\Re\boxed{17} + 2\Im\boxed{17} = \begin{cases} \frac{2}{81}(1000n^2-25n+153), & r=0; \\ -\frac{2}{9}(325n^2+430n+243), & r=1. \end{cases} \tag{21}$$

According to Theorem 3, we obtain the four identities given in Examples 7 and 8 by relating (18) and (19), respectively, to (20) and (21).

Concluding remarks. The multisection series method has been successfully used to derive numerous formulae for π and special values of the logarithm function with $m = 2,3,4,5,6$ and $0 \leqslant \gamma \leqslant m$, where all the formulae corresponding to $\gamma \neq 0$ seem new. Following the procedure described in this paper, it is possible to work out further formulae. However, we shall not produce them due to the space limitation.

Acknowledgement. The authors express sincerely their gratitude to the anonymous referee for the careful reading, critical comments and constructive suggestions, that lead to a substantial improvement of this manuscript.

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(Received December 17, 2017)

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