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ON SHARPENING OF A THEOREM OF T. J. RIVLIN

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Abstract. Let $p(z)=a_0+a_1z+a_2z^2+a_3z^3+\cdots+a_nz^n$ be a polynomial of degree n. According to a well-known theorem of Rivlin [11], if $p(z)$ is a polynomial of degree n having no zeros inside the unit circle, then for $0<r\leqslant 1$,

$$\max_{|z|=r}|p(z)|\geqslant\left(\frac{r+1}{2}\right)^n\max_{|z|=1}|p(z)|.$$

In this paper, we generalize and sharpen the above result of Rivlin. Our result also sharpens a recently proved result of Govil and Nwaeze [3]. Also, we present some examples to show that in some cases the improvement obtained by our theorem can be considerably significant.

1. Introduction

Let $p(z)=\sum_{j=0}^{n}a_jz^j$ be a polynomial of degree n, $M(p,r):=\max_{|z|=r}|p(z)|$, $r>0$, and $\|p\|:=\max_{|z|=1}|p(z)|$. Then it is well known that

$$M(p',1)\leqslant n\|p\|, \tag{1}$$

and

$$M(p,R)\leqslant R^n\|p\|,\quad R\geqslant 1. \tag{2}$$

The above inequalities are known as Bernstein inequalities, and have been the starting point of a considerable literature in approximation theory. Several papers and research monographs have been written on this subject (see, for example Milovanović, Mitrinović and Rassias [6], Rahman [8], and Rahman and Schmeisser [9, 10]).

For polynomials of degree n not vanishing in the interior of the unit circle, the above inequalities have been replaced by:

$$M(p',1)\leqslant\frac{n}{2}\|p\|,$$

and

$$M(p,R)\leqslant\left(\frac{R^n+1}{2}\right)\|p\|,\quad R\geqslant 1.$$

Both the above inequalities are sharp and equality holds for polynomials having all their zeros on the unit circle.

If one applies Inequality (2) to the polynomial $P(z):=z^np(1/z)$ and use maximum modulus principle, one easily gets

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THEOREM 1. *Let* $p(z)=\sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree* n. *Then for* $0<r\leqslant 1$,

$$M(p,r)\geqslant r^n\|p\|. \tag{3}$$

Equality holds for $p(z)=\alpha z^n$, α *being a complex number.*

The above result is due to Varga [12] who attributes it to E. H. Zarantonello.

It was shown by Govil, Qazi and Rahman [5] that the inequalities (1), (2) and (3) are all equivalent in the sense that any of these inequalities can be derived from the other.

The analogue of Inequality (3) for polynomials not vanishing in the interior of a unit circle was proved in 1960 by Rivlin [11], who in fact proved

THEOREM 2. *Let* $p(z)=\sum_{j=0}^{n} a_j z^j\neq 0$ *in* $|z|<1$. *Then for* $0<r\leqslant 1$,

$$M(p,r)\geqslant\left(\frac{r+1}{2}\right)^n\|p\|.$$

The inequality is best possible and equality holds for $p(z)=\left(\frac{\alpha+\beta z}{2}\right)^n$, *where* $|\alpha|=|\beta|=1$.

Govil [1] generalized Theorem 2 by proving

THEOREM 3. *Let* $p(z)=\sum_{j=0}^{n} a_j z^j\neq 0$ *in* $|z|<1$. *Then for* $0<r\leqslant R\leqslant 1$,

$$M(p,r)\geqslant\left(\frac{1+r}{1+R}\right)^n M(p,R). \tag{4}$$

The result is best possible and equality holds for the polynomial $p(z)=\left(\frac{1+z}{1+R}\right)^n$.

There are many extensions of Inequality (4) (see, for example Govil, Qazi and Rahman [5], Govil and Qazi [4], and Qazi [7]).

Recently, Govil and Nwaeze [3] proved the following refinement of Theorem 3.

THEOREM 4. *Let* $p(z)=\sum_{j=0}^{n} a_j z^j\neq 0$ *in* $|z|<K$, $K\geqslant 1$. *Then for* $0<r\leqslant R\leqslant 1$,

$$M(p,r)\geqslant\frac{(1+r)^n}{(1+r)^n+(R+K)^n-(r+K)^n}\left[M(p,R)+n\min_{|z|=K}|p(z)|\ln\left(\frac{R+K}{r+K}\right)\right]. \tag{5}$$

2. Main results

In this paper, we prove the following result which sharpens Theorem 4 due to Govil and Nwaeze [3]. Also, our result sharpens Theorem 3 due to Govil [1] and Theorem 2 due to Rivlin [11].

THEOREM 5. *Let* $p(z) = \sum_{j=0}^{n} a_j z^j$. *If* $p(z) \neq 0$ *in* $|z| < K$, $K \geqslant 1$, *then for* $0 < r < R \leqslant 1$,

$$M(p,r) \geqslant \frac{(1+r)^n}{(1+r)^n + nh(n)} \left[M(p,R) + n \min_{|z|=K} |p(z)| \ln\left(\frac{R+K}{r+K}\right) \right],$$

where

$$h(n) = \sum_{m=0}^{n-1} (-1)^m (K-1)^m \left(\frac{(1+R)^{n-m} - (1+r)^{n-m}}{n-m} \right) + (-1)^n (K-1)^n \ln\left(\frac{R+K}{r+K}\right).$$

REMARK 1. It has been shown in Lemma 2 that

$$nh(n) = n \int_r^R \frac{(1+t)^n}{K+t} dt \leqslant n \int_r^R (K+t)^{n-1} dt = (R+K)^n - (r+K)^n. \tag{6}$$

Also, it is clear that $h(n) \geqslant 0$. Using this and inequality (6) it follows immediately that our Theorem 5 sharpens Theorem 4 due to Govil and Nwaeze [3]. Moreover, for some polynomials improvement can be considerably significant, and this has been shown in Section 5, where we have, by using MATLAB, constructed polynomials to prove this assertion.

Note that, by Lemma 2, for $K = 1$ we have $nh(n) = (1+R)^n - (1+r)^n$ and therefore Theorem 5 reduces to the following result which is Corollary 2.1 of Govil and Nwaeze [3].

COROLLARY 1. *Let* $p(z) = \sum_{j=0}^{n} a_j z^j$. *If* $p(z) \neq 0$ *in* $|z| < 1$, *then for* $0 < r < R \leqslant 1$,

$$M(p,r) \geqslant \left(\frac{1+r}{1+R}\right)^n \left[M(p,R) + n \min_{|z|=1} |p(z)| \ln\left(\frac{R+1}{r+1}\right) \right].$$

It is easy to see that the above Corollary 1 improves the bound in Theorem 3 due to Govil [1].

For $R = 1$, the Theorem 5, gives

COROLLARY 2. *Let* $p(z) = \sum_{j=0}^{n} a_j z^j$. *If* $p(z) \neq 0$ *in* $|z| < K$, $K \geqslant 1$, *then for* $0 < r < 1$,

$$M(p,r) \geqslant \frac{(1+r)^n}{(1+r)^n + nH(n)} \left[M(p,1) + n \min_{|z|=K} |p(z)| \ln\left(\frac{1+K}{r+K}\right) \right], \tag{7}$$

where

$$H(n)=\sum_{m=0}^{n-1}(-1)^m(K-1)^m\left(\frac{2^{n-m}-(1+r)^{n-m}}{n-m}\right)+(-1)^n(K-1)^n\ln\left(\frac{1+K}{r+K}\right). \quad (8)$$

Since, by Lemma 2 we have $nH(n)\leqslant(1+K)^n-(r+K)^n$ the above inequality (7), in particular gives

COROLLARY 3. *Let* $p(z)=\sum_{j=0}^{n}a_jz^j$. *If* $p(z)\neq 0$ *in* $|z|<K$, $K\geqslant 1$, *then for* $0<r<1$,

$$M(p,r)\geqslant\frac{(1+r)^n}{(1+r)^n+(1+K)^n-(r+K)^n}\left[M(p,1)+n\min_{|z|=K}|p(z)|\ln\left(\frac{1+K}{r+K}\right)\right]. \quad (9)$$

The inequality (9), for $K=1$ clearly reduces to

$$M(p,r)\geqslant\left(\frac{1+r}{2}\right)^n\left[M(p,1)+n\min_{|z|=1}|p(z)|\ln\left(\frac{2}{r+1}\right)\right], \quad (10)$$

which is Corollary 2.3 of Govil and Nwaeze [3]. It is clear that inequality (10) sharpens Theorem 2 due to Rivlin [11], and excepting the case when $\min_{|z|=1}|p(z)|=0$, the inequality (10) always gives a bound that is sharper than the bound obtainable from Theorem 2.

3. Lemmas

For the proof of Theorem 5, we will need the following lemmas. Our first lemma is a special case of a result due to Govil [2, Theorem 1].

LEMMA 1. *Let* $p(z)$ *be a polynomial of degree* n *having no zeros in* $|z|<K$, $K\geqslant 1$. *Then*

$$\max_{|z|=1}|p'(z)|\leqslant\frac{n}{1+K}\left[\max_{|z|=1}|p(z)|-\min_{|z|=K}|p(z)|\right].$$

LEMMA 2. *Let* $h(n)=\int_r^R\frac{(1+t)^n}{K+t}dt$ *for* $n\geqslant 0$. *Then*

$$h(n)=\sum_{m=0}^{n-1}(-1)^m(K-1)^m\left(\frac{(1+R)^{n-m}-(1+r)^{n-m}}{n-m}\right)+(-1)^n(K-1)^n\ln\left(\frac{R+K}{r+K}\right). \quad (11)$$

Proof. We have for $n \geqslant 0$,

$$\begin{aligned} h(n) &= \int_r^R \frac{(1+t)^n}{K+t} dt \\ &= \int_{1+r}^{1+R} \frac{t^n}{(t+K-1)} dt \\ &= \int_{1+r}^{1+R} \frac{t^n}{(t+a)} dt, \quad \text{where } a = K-1. \end{aligned} \tag{12}$$

It is easy to see that

$$\begin{aligned} h(n) + ah(n-1) &= \int_{1+r}^{1+R} t^{n-1} dt \\ &= \frac{(1+R)^n - (1+r)^n}{n} = g(n), \quad \text{(say)}. \end{aligned}$$

Therefore

$$h(n) = g(n) - ah(n-1). \tag{13}$$

Solving the recurrence relation (13), we get

$$h(n) = \sum_{m=0}^{n-1} (-1)^m a^m g(n-m) + (-1)^n a^n h(0), \tag{14}$$

where $h(0) = \int_r^R \frac{1}{K+t} dt = \ln(\frac{R+K}{r+K})$.

Now, substituting the value of $g(n)$ and $h(0)$ in (14), and noting that by (12) we have $a = K-1$, we get (11), which completes the proof of the Lemma 2. □

4. Proof of the Theorem

Proof. Let $0 < r < R \leqslant 1$, and $\theta \in [0, 2\pi)$. Then we have:

$$\left| p(Re^{i\theta}) - p(re^{i\theta}) \right| = \left| \int_r^R e^{i\theta} p'(te^{i\theta}) dt \right|,$$

which implies

$$\left| p(Re^{i\theta}) \right| \leqslant \left| p(re^{i\theta}) \right| + \left| \int_r^R e^{i\theta} p'(te^{i\theta}) dt \right|. \tag{15}$$

Since $p(z) \neq 0$ in $|z| < K$, $K \geqslant 1$, hence $p(tz) \neq 0$ in $|z| < K/t$. Further, if $0 < t \leqslant 1$, then $1/t \geqslant 1$ and $K/t \geqslant 1$, therefore on applies Lemma 1 to the polynomial $p(tz)$, we get

$$|p'(tz)| \leqslant \frac{n}{K+t} \left[M(p,t) - \min_{|z|=K} |p(z)| \right], \quad |z| = 1. \tag{16}$$

Combining (15) and (16), we get

$$\left|p(Re^{i\theta})\right| \leqslant \left|p(re^{i\theta})\right| + \int_r^R \frac{n}{K+t} M(p,t)dt - n \min_{|z|=K} |p(z)| \int_r^R \frac{1}{K+t} dt,$$

which implies

$$M(p,R) \leqslant M(p,r) + \int_r^R \frac{n}{K+t} M(p,t)dt - n \min_{|z|=K} |p(z)| \int_r^R \frac{1}{K+t} dt.$$

Now using Inequality (4), we obtain

$$\begin{aligned} M(p,R) &\leqslant M(p,r) + \int_r^R \frac{n}{K+t}\left(\frac{1+t}{1+r}\right)^n M(p,r)dt - n \min_{|z|=K} |p(z)| \int_r^R \frac{1}{K+t} dt \\ &= M(p,r) + \frac{nM(p,r)}{(1+r)^n} \int_r^R \frac{(1+t)^n}{K+t} dt - n \min_{|z|=K} |p(z)| \int_r^R \frac{1}{K+t} dt, \end{aligned}$$

which when combined with Lemma 2, gives

$$M(p,R) \leqslant M(p,r) + \frac{nM(p,r)}{(1+r)^n} h(n) - n \min_{|z|=K} |p(z)| \ln\left(\frac{R+K}{r+K}\right).$$

Therefore

$$M(p,r)\left(\frac{(1+r)^n + nh(n)}{(1+r)^n}\right) \geqslant M(p,R) + n \min_{|z|=K} |p(z)| \ln\left(\frac{R+K}{r+K}\right),$$

which is equivalent to

$$M(p,r) \geqslant \frac{(1+r)^n}{(1+r)^n + nh(n)}\left[M(p,R) + n \min_{|z|=K} |p(z)| \ln\left(\frac{R+K}{r+K}\right)\right],$$

and the proof of the theorem is now complete. □

5. Examples

In this section we present examples of polynomials to show that in some cases the improvement can be considerably significant, and we do this by using MATLAB.

(a). Let $p(z) = z^3 + 64$, a polynomial of degree $n = 3$. Then one can easily get that the zeros of this polynomial are : -4, $2+3.4641i$, and $2-3.4641i$, implying $p(z) \neq 0$ in $|z| < 3.9$. If we use Theorem 4 of Govil and Nwaeze [3] with $R = 1$, $r = 0.1$ and $K = 3.9$, for which $m = 4.681$ and $M(p,1) = 65$, we get

$$M(p,r) \geqslant (0.0242)M(p,1) + 0.0689 = 1.6419,$$

and on using our Theorem 5, we easily get

$$M(p,r) \geqslant (0.3576)M(p,1) + 1.0191 = 24.2631,$$

an improvement of more than 22.6 over the bound obtained by Theorem 4.

(b). If in the above example we take $r = 0.2$ instead of $r = 0.1$, then Theorem 4 of Govil and Nwaeze [3], gives

$$M(p,r) \geqslant (0.0342)M(p,1) + 0.0856 = 2.3086,$$

while from our Theorem 5 we easily get

$$M(p,r) \geqslant (0.4313)M(p,1) + 1.0796 = 29.1141,$$

an improvement of more than 26.8 over the bound obtained by Theorem 4.

REMARK 2. Since $\{(R+K)^n - (r+K)^n\} - nh(n) = n\int_r^R \frac{(K+t)^n-(1+t)^n}{K+t}dt$ is a non-negative and an increasing function of K it easily follows from this that for large value of K, the improvement in the bound can be made considerably large by choosing K large. In this regard we present the following example.

Let $p(z) = z^3 + 1000$, a polynomial of degree $n = 3$. Then one can easily get that the zeros of this polynomial are : -10, $5+8.6602i$, and $5-8.6602i$, implying $p(z) \neq 0$ in $|z| < 10$. If we use Theorem 4 of Govil and Nwaeze [3] with $R = 1$, $r = 0.1$ and $K = 10$, for which $m = 0$ and $M(p,1) = 1001$, we get

$$M(p,r) \geqslant (0.0044)M(p,1) = 4.4044,$$

while by using our Theorem 5, we easily get

$$M(p,r) \geqslant (0.5656)M(p,1) = 566.1656,$$

an improvement of more than 561.7 over the bound obtained by Theorem 4.

REMARK 3. We observe that for $K > 1$, an alternative expression of $h(n)$ can be given in terms of Lerch function $z \mapsto \Phi(z,s,a)$, defined by

$$\Phi(z,s,a) = \frac{1}{\Gamma(s)}\int_0^\infty \frac{t^{s-1}e^{-at}}{1-ze^{-t}}dt \left(= \sum_{\nu=0}^{\infty} \frac{z^\nu}{(\nu+a)^s}\right). \tag{17}$$

If

$$I(q) = \int_{-1}^{q} \frac{(1+t)^n}{K+t}dt, \tag{18}$$

then as is easy to see

$$h(n) = \int_r^R \frac{(1+t)^n}{K+t}dt = I(R) - I(r). \tag{19}$$

If in (18) we make the change of variable $1+t=(q+1)e^{-\xi}$, we get

$$I(q)=\int_0^{\infty}\frac{(q+1)^n e^{-n\xi}}{K-1+(q+1)e^{-\xi}}(q+1)e^{-\xi}d\xi$$
$$=\frac{(q+1)^{n+1}}{K-1}\int_0^{\infty}\frac{e^{-(n+1)\xi}}{1+\frac{q+1}{K-1}e^{-\xi}}d\xi$$
$$=\frac{(q+1)^{n+1}}{K-1}\Phi\left(-\frac{q+1}{K-1},1,n+1\right), \tag{20}$$

if we take $s=1$, $a=n+1$, and $z=-(q+1)/(K-1)$ in (17). Therefore

$$I(R)=\frac{(R+1)^{n+1}}{K-1}\Phi\left(-\frac{R+1}{K-1},1,n+1\right), \tag{21}$$

and

$$I(r)=\frac{(r+1)^{n+1}}{K-1}\Phi\left(-\frac{r+1}{K-1},1,n+1\right). \tag{22}$$

Now combining (19), (21) and (22), we get

$$h(n)=\frac{1}{K-1}\left\{(R+1)^{n+1}\Phi\left(-\frac{R+1}{K-1},1,n+1\right)-(r+1)^{n+1}\Phi\left(-\frac{r+1}{K-1},1,n+1\right)\right\}.$$

The function $\Phi(z,s,a)$ is implimented in Wolfram's MATHEMATICA as LerchPhi [z, s, a], and it is suitable for both symbolic and numerical manipulations. Also, LerchPhi can be evaluated to arbitrary numerical precision.

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