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BOHR RADIUS FOR CERTAIN CLASSES OF ANALYTIC FUNCTIONS

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Abstract. In this paper, we discuss Bohr's inequality for certain classes of analytic functions associated with q-function theory for $q \in (0,1)$. Interestingly, in particular cases when $q \to 1$, we obtain very fundamental theorems of univalent function theory such as covering and growth theorems for starlike and convex functions. Subsequently, we obtain the Bohr radius for the classes of starlike and convex functions.

1. Introduction

Let $\mathscr{A}$ denote the class of all functions analytic in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. Given a power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \tag{1}$$

its majorant series is given by

$$M_f(r) = \sum_{n=0}^{\infty} |a_n| r^n \quad (r = |z|). \tag{2}$$

Note that both the series (1) and (2) converge or diverge together in open subsets of $\mathbb{D}$. However, the values of $f(z)$ and $M_f(r)$ and also the values of certain norms of these two functions may differ. In this regard, function theorists started comparing the sup norms of these two functions. The first step in this setting was taken by Bohr [14] in 1914. He proved that "if $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathscr{A}$ and $|f(z)| < 1$ for all $z \in \mathbb{D}$, then

$$M_f(r) \leqslant 1 \tag{3}$$

for all $z \in \mathbb{D}$ with $|z| \leqslant 1/6$." Later on, Wiener, Riesz, and Schur independently proved that inequality (3) holds true for $|z| \leqslant 1/3$ and the constant $1/3$ is the best possible. Obtaining the largest r such that (3) holds for all $z \in \mathbb{D}$ with $|z| < r$ is called the Bohr radius.

Initially, Bohr radius was obtained for the classes of mappings from unit disk onto itself. Later, the notion of Bohr radius was generalized to the classes of mappings from $\mathbb{D}$ into some other domain $G \subset \mathbb{C}$ (see, [8, 1, 2]). One way of the generalization is to rewrite Bohr's inequality in the equivalent form $\sum_{n=1}^{\infty} |a_n z^n| \leqslant 1 - |a_0|$. The right hand side then can be written as the distance from $f(0)$ to the boundary $\partial\mathbb{D}$. In this form,

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the notion of Bohr's radius can be generalized to the class of functions $f(z)$ analytic in $\mathbb{D}$, which take values in a given domain $G \subset \mathbb{C}$ as follows:

For a given domain $G \subset \mathbb{C}$, find the largest radius $r_G > 0$ such that

$$\operatorname{dist}\left(\sum_{n=1}^{\infty} |a_n z^n|, |f(0)|\right) \leqslant \operatorname{dist}(f(0), \partial G) \tag{4}$$

for all $|z| < r_G$ and for all functions $f(z)$ analytic in $\mathbb{D}$ such that $f(\mathbb{D}) \subset G$.

In [8], it is proved that if G is a convex domain, then the inequality (4) holds true for $|z| < 1/3$ and the radius $r_G = 1/3$ is the best possible. When G is any proper simply connected domain and $f(z)$ is analytic in $\mathbb{D}$ having values in G, then (4) holds true for $|z| < 3 - 2\sqrt{2}$ and the radius is sharp [1]. Bohr radius for wedge domains, alternating series, and even analytic functions are obtained in [11]. In addition, in [11], the authors have obtained an upper and lower bound of Bohr's radius for odd analytic functions and posed an open problem regarding Bohr's radius for odd analytic functions which is recently settled by Kayumov and Ponnusamy in [20]. An improved version of Bohr's inequality is obtained by Kayumov and Ponnusamy in [21]. Bohr radius for lacunary series and for locally univalent harmonic mappings are obtained respectively in [23] and [22]. In recent years, many results related to Bohr's theorem are obtained in the setting of several complex variables. For example, Boas and Khavinson [13] obtained some multidimensional generalizations of Bohr's theorem and Aizenberg [5] extended it for further studies on the topic. For recent developements on Bohr's inequality we refer to [6, 7, 8, 9, 24, 25, 26] and to the survey article by Ali et al. [10] and references therein.

In this paper, we are interested to establish a connection between Bohr's phenomenon with the q-function theory. More precisely, we are interested in estimating the Bohr inequality for q-starlike and q-convex functions which are generalizations of the class of starlike and convex functions in terms of q. For definitions of starlike and convex function we refer the classic book by Duren [15]. Recall that for $0 < q < 1$, *the q-difference operator* (see [18]), denoted as $D_q f$, is defined by the equation

$$(D_q f)(z) = \frac{f(z) - f(qz)}{z(1-q)}, \quad z \neq 0, \quad (D_q f)(0) = f'(0).$$

The class of *q-starlike functions*, denoted by $\mathscr{S}_q^*$, is defined as follows:

DEFINITION 1. [18, Definition 1.3] A function $f \in \mathscr{A}$ is said to be in the class $\mathscr{S}_q^*$, if

$$\left| \frac{z(D_q f)(z)}{f(z)} - \frac{1}{1-q} \right| \leqslant \frac{1}{1-q}, \quad z \in \mathbb{D}.$$

In the light of the well-known Alexander's theorem [15, Theorem 2.12], Baricz and Swaminathan in [12] defined a q-analog of convex functions, denoted by $\mathscr{C}_q$, in the following way.

DEFINITION 2. [12, Definition 3.1] A function $f \in \mathscr{A}$ is said to belong to $\mathscr{C}_q$ if and only if $z(D_q f)(z) \in \mathscr{S}_q^*$.

We call the functions of the class $\mathscr{C}_q$ as *q-convex functions*. Note that the class $\mathscr{C}_q$ is non-empty as shown in [12, Theorem 3.2] and as $q \to 1$, the classes $\mathscr{S}_q^*$ and $\mathscr{C}_q$ reduce to $\mathscr{S}^*$ (the class of starlike functions) and $\mathscr{C}$ (the class of convex functions) respectively.

We now state our main theorems. The Bohr inequality for the class $\mathscr{S}_q^*$ is stated as follows:

THEOREM 1. *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n = z\exp[\phi(z)] \in \mathscr{S}_q^*$. Then*

$$|z|\exp\left[\sum_{n=1}^{\infty}|\phi_n||z|^n\right] \leqslant d(0, \partial f(\mathbb{D}))$$

for $|z| \leqslant r_f$, where $\phi(z) = \sum_{n=1}^{\infty}\phi_n z^n$ and r_f is the unique root of

$$r\exp[F_q(r)] = \exp[F_q(-1)].$$

The radius is sharp and attained by suitable rotation of $G_q(z)$.

Here $G_q(z)$ is the function defined in Lemma 2. Similarly, the Bohr inequality for the class $\mathscr{C}_q$ is stated as follows:

THEOREM 2. *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathscr{C}_q$. Then*

$$r + \sum_{n=2}^{\infty}|a_n|r^n \leqslant d(0, \partial f(\mathbb{D}))$$

for $|z| \leqslant r_f$, where r_f is the unique root of

$$I_q(\exp[F_q(r)]) = I_q(\exp[F_q(-1)]).$$

The radius is sharp and attained by suitable rotation of $E_q(z)$.

Here $E_q(z)$ is the function defined in Lemma 4.

2. Prerequisites

This section is devoted to the required preliminaries to prove our main theorems. We begin this section with the Herglotz representation for the class $\mathscr{S}_q^*$.

LEMMA 1. [18, Theorem 1.15] *Let $f \in \mathscr{A}$. Then $f \in \mathscr{S}_q^*$ if and only if there exists a probability measure μ supported on the unit circle such that*

$$\frac{zf'(z)}{f(z)} = 1 + \int_{|\sigma|=1} \sigma z F_q'(\sigma z)\,\mathrm{d}\mu(\sigma)$$

where

$$F_q(z) = \sum_{n=1}^{\infty}\frac{-2\ln q}{1-q^n}z^n, \quad z \in \mathbb{D}. \tag{5}$$

The Bieberbach-type theorem for the class $\mathscr{S}_q^*$ is states as follows:

LEMMA 2. [18, Theorem 1.18] *Let*

$$G_q(z) := z\exp[F_q(z)] = z + \sum_{n=2}^{\infty} c_n z^n. \tag{6}$$

Then $G_q \in \mathscr{S}_q^*$. Moreover, if $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathscr{S}_q^*$, then $|a_n| \leqslant c_n$ with equality holding for all n if and only if f is a rotation of G_q.

The Herglotz representation theorem and the Bieberbach-type theorem for the class $\mathscr{C}_q(\alpha)$, $0 \leqslant \alpha < 1$, are respectively stated in [3, Theorem 2.8] and [3, Theorem 2.9]. The substitution $\alpha = 0$ in [3, Theorem 2.8] gives the Herglotz representation theorem for the class $\mathscr{C}_q$ stated as follows:

LEMMA 3. *Let $f \in \mathscr{A}$. Then $f \in \mathscr{C}_q$ if and only if there exists a probability measure μ supported on the unit circle such that*

$$\frac{z(D_q f)'(z)}{(D_q f)(z)} = \int_{|\sigma|=1} \sigma z F_q'(\sigma z)\mathrm{d}\mu(\sigma).$$

The substitution $\alpha = 0$ in [3, Theorem 2.9] gives the Bieberbach-type theorem for the class $\mathscr{C}_q$ which is stated as follows:

LEMMA 4. *Let*

$$E_q(z) := I_q\left(\exp[F_q(z)]\right) = z + \sum_{n=2}^{\infty}\left(\frac{1-q}{1-q^n}\right) c_n z^n \tag{7}$$

where c_n is the n-th coefficient of the function $z\exp[F_q(z)]$. Then $E_q \in \mathscr{C}_q$. Moreover, if $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathscr{C}_q$, then $|a_n| \leqslant ((1-q)/(1-q^n))c_n$ with equality holding for all n if and only if f is a rotation of E_q.

Here I_q is called the q-integral. Recall that Thomae introduced the q-integral [27]

$$\int_0^1 f(t)\,d_q t = (1-q)\sum_{n=0}^{\infty} q^n f(q^n),$$

provided the q-series converges. In 1910, Jackson defined the general q-integral [19] (see also [16, 27]) in the following manner:

$$\int_a^b f(t)\,d_q t := \int_0^b f(t)\,d_q t - \int_0^a f(t)\,d_q t,$$

where

$$I_q(f(x)) := \int_0^x f(t)\,d_q t = x(1-q)\sum_{n=0}^{\infty} q^n f(xq^n),$$

provided the q-series converges. Observe that

$$D_q I_q f(x) = f(x) \quad \text{and} \quad I_q D_q f(x) = f(x) - f(0),$$

where the second equality holds if f is continuous at $x = 0$.

We now prove the main result of this section which is a key result to obtain Bohr's inequality for the classes $\mathscr{S}_q^*$ and $\mathscr{C}_q$. We call it as a distortion theorem for the class $\mathscr{S}_q^*$.

LEMMA 5. (Distortion theorem) *Let $f(z) = zh(z) \in \mathscr{S}_q^*$. Then*

$$\exp[F_q(-r)] \leqslant |h(z)| \leqslant \exp[F_q(r)].$$

Equality occurs if and only if $h(z)$ is a suitable rotations of $\exp[F_q(z)]$.

Proof. Since $f(z) \in \mathscr{S}_q^*$, by Lemma 1, there exists a probability measure μ supported on the unit circle such that

$$\frac{zf'(z)}{f(z)} = 1 + \int_{|\sigma|=1} \sigma z F_q'(\sigma z) \mathrm{d}\mu(\sigma).$$

Simple computation yields

$$f(z) = z\exp\left[\int_{|\sigma|=1} F_q(\sigma z) \mathrm{d}\mu(\sigma)\right]$$

or,

$$|h(z)| = \exp\left[\operatorname{Re} \int_{|\sigma|=1} F_q(\sigma z) \mathrm{d}\mu(\sigma)\right].$$

Hence,

$$\begin{aligned}
\ln|h(z)| &= \operatorname{Re} \int_{|\sigma|=1} F_q(\sigma z) \mathrm{d}\mu(\sigma) \\
&= \operatorname{Re} \int_{|\sigma|=1} \frac{-2\ln q}{1-q} [\sigma z \Phi(q,q,q^2,q,\sigma z)] \mathrm{d}\mu(\sigma) \\
&= \frac{-2\ln q}{1-q} \operatorname{Re} \int_0^{2\pi} [(e^{i\theta} z)\Phi(q,q,q^2,q,(e^{i\theta} z))] \mathrm{d}\mu(\theta) \\
&= \frac{-2\ln q}{1-q} \operatorname{Re} \int_0^{2\pi} [w\Phi(q,q,q^2,q,w)] \mathrm{d}\mu(\theta), \quad w = e^{i\theta} z \in \mathbb{D} \\
&= \frac{-2\ln q}{1-q} \operatorname{Re} \int_0^{2\pi} \frac{w\Phi(q^1,q^1,q^2,q,w)}{\Phi(q^0,q^1,q^2,q,w)} \mathrm{d}\mu(\theta),
\end{aligned}$$

where $\Phi(a,b;c;q,z)$ is called the basic hypergeometric function. For the definition of basic hypergeometric function we refer [4]. By [4, Theorem 2.5], we get the integral representation for the ratio of basic hypergeometric as follows:

$$\frac{w\Phi(q^1,q^1,q^2,q,w)}{\Phi(q^0,q^1,q^2,q,w)} = \int_0^1 \frac{w}{1-tw} \mathrm{d}\mu(\mathrm{t}).$$

The minimum of the function $\text{Re}\,(w/(1-tw))$ for $|w|=|z|\leqslant r$ is attained at the point $w=-r$ and maximum is attained at the point $w=r$. So

$$\begin{aligned}\ln|h(z)| &\geqslant \frac{-2\ln q}{1-q}[-r\Phi(q,q,q^2,q,-r)]\\ &= F_q(-r)\end{aligned}$$

and

$$\begin{aligned}\ln|h(z)| &\leqslant \int_{|\sigma|=1} F_q(r)\mathrm{d}\mu(\sigma)\\ &= F_q(r).\end{aligned}$$

Hence,

$$\exp[F_q(-r)] \leqslant |h(z)| \leqslant \exp[F_q(r)].$$

The proof is complete. □

As $q\to 1$, Lemma 5 gives the following well-known theorem for starlike functions.

COROLLARY 1. [17, Theorem 8] *Let $f(z)\in\mathscr{S}^*$. Then*

$$\frac{r}{(1+r)^2} \leqslant |f(z)| \leqslant \frac{r}{(1-r)^2}, \quad |z|=r<1.$$

Equality occurs for a suitable rotation of the Koebe function $z/(1-z)^2$.

Proof. When $q\to 1$, $\exp[F_q(r)]\to 1/(1-r)^2$ and $\exp[F_q(-r)]\to 1/(1+r)^2$ and hence the proof follows from Lemma 5. □

LEMMA 6. *Let $f(z)=zh(z)\in\mathscr{S}_q^*$. Then*

$$\exp[F_q(-1)] \leqslant d(0,\partial f(\mathbb{D})).$$

Equality occurs if and only if $f(z)$ is a suitable rotation of $G_q(z)$.

Proof. By Lemma 5,

$$d(0,\partial f(\mathbb{D})) = \liminf_{|z|\to 1}|f(z)-f(0)| = \liminf_{|z|\to 1}\frac{|f(z)|}{|z|} = \liminf_{|z|\to 1}|h(z)| \geqslant \exp[F_q(-1)].$$

The proof is complete. □

As $q\to 1$, Lemma 6 gives the Koebe One-Quarter theorem for starlike functions.

COROLLARY 2. [15, Theorem 2.3] *The range of every function of class $\mathscr{S}$ contains the disk $\{w : |w|<1/4\}$.*

Proof. As $q\to 1$, $\exp[F_q(-r)]\to 1/(1+r)^2$. Hence $\exp[F_q(-1)]\to 1/4$. □

We obtain similar results related to the class $\mathscr{C}_q$.

LEMMA 7. *Let $f(z)\in\mathscr{C}_q$. Then*

$$I_q\left(\exp[F_q(-r)]\right)\leqslant|f(z)|\leqslant I_q(\exp[F_q(r)]).$$

Equality occurs if and only if $f(z)$ is a suitable rotations of $E_q(z)$.

Proof. Let $f(z)\in\mathscr{C}_q$. By definition, $z(D_qf)(z)\in\mathscr{S}_q^*$. Hence, from Lemma 5, we conclude that

$$\exp[F_q(-r)]\leqslant|(D_qf)(z)|\leqslant\exp[F_q(r)].$$

The required inequalities will follow by taking q-integral of the above inequalities. □

As $q\to 1$, Lemma 7 gives the following well-known theorem for convex functions.

COROLLARY 3. [17, Theorem 9] *Let $f(z)\in\mathscr{C}$. Then*

$$\frac{r}{(1+r)}\leqslant|f(z)|\leqslant\frac{r}{(1-r)},\qquad|z|=r<1.$$

Equality occurs for a suitable rotation of the function $z/(1-z)$.

Proof. When $q\to 1$, $I_q(\exp[F_q(r)])\to r/(1-r)$ and $I_q(\exp[F_q(-r)])\to r/(1+r)$ and hence the proof follows from Lemma 7. □

LEMMA 8. *Let $f(z)\in\mathscr{C}_q$. Then*

$$I_q(\exp[F_q(-1)])\leqslant d(0,\partial f(\mathbb{D})).$$

Equality occurs if and only if $f(z)$ is a suitable rotation of $E_q(z)$.

Proof. By Lemma 7,

$$d(0,\partial f(\mathbb{D}))=\liminf_{|z|\to 1}|f(z)-f(0)|=\liminf_{|z|\to 1}|f(z)|\geqslant I_q(\exp[F_q(-1)])$$

and hence the proof is complete. □

As $q\to 1$, Lemma 8 gives the following covering theorem for the class $\mathscr{C}$.

COROLLARY 4. [15, Theorem 2.15] *The range of every function $f\in\mathscr{C}$ contains the disk $|w|<1/2$. In other language it can be written as for $f\in\mathscr{C}$,*

$$\frac{1}{2}\leqslant d(0,\partial f(\mathbb{D})).$$

Proof. As $q\to 1$, $I_q(\exp[F_q(-r)])\to r/(1+r)$. Hence $I_q(\exp[F_q(-1)])\to 1/2$. □

3. Proof of main theorems

In this section, we prove our main theorems.

Proof of Theorem 1. Suppose that $f \in \mathscr{S}_q^*$. Then by proof of [18, Theorem 1.18], f can be written as

$$f(z) = z\exp[\phi(z)],$$

where, $\phi(z) = \sum_{n=1}^{\infty} \phi_n z^n$ and

$$|\phi_n| \leqslant \frac{-2\ln q}{1-q^n}$$

which is the sharp bound. Now,

$$r\exp\left[\sum_{n=1}^{\infty} |\phi_n| r^n\right] \leqslant r\exp\left[\sum_{n=1}^{\infty} \frac{-2\ln q}{1-q^n} r^n\right] = r\exp[F_q(r)] \leqslant \exp[F_q(-1)] \leqslant d(0, \partial f(\mathbb{D}))$$

if and only if

$$r\exp[F_q(r)] \leqslant \exp[F_q(-1)].$$

Hence, the Bohr radius r_f is the positive root of the equation

$$r\exp[F_q(r)] = \exp[F_q(-1)].$$

This completes the proof of the theorem. □

As $q \to 1$, Theorem 1 leads to the Bohr radius for the class $\mathscr{S}^*$ which is also mentioned in [11, p. 156].

COROLLARY 5. *Let* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathscr{S}^*$. *Then*

$$r + \sum_{n=2}^{\infty} |a_n| r^n \leqslant \frac{1}{4}$$

for $|z| = r \leqslant 3 - 2\sqrt{2}$. *The radius is sharp and attained by suitable rotation of the Koebe function* $z/(1-z)^2$.

Proof. Letting $q \to 1$ in Theorem 1, we get $r\,(0 < r < 1)$ as the Bohr radius for the class $\mathscr{S}^*$, where r is the solution of the equation

$$\frac{r}{(1-r)^2} = \frac{1}{4}.$$

Solving for r we get $r = 3 - 2\sqrt{2}$ and hence the proof is complete. □

Proof of Theorem 2. Suppose that $f \in \mathscr{C}_q$. Then Lemma 4 and Lemma 7 yields

$$\begin{aligned} r + \sum_{n=2}^{\infty} |a_n| r^n &\leqslant r + \sum_{n=2}^{\infty} \left(\frac{1-q}{1-q^n}\right) c_n r^n = I_q(\exp[F_q(r)]) \\ &\leqslant I_q(\exp[F_q(-1)]) \leqslant d(0, \partial f(\mathbb{D})) \end{aligned}$$

if and only if

$$I_q(\exp[F_q(r)]) \leqslant I_q(\exp[F_q(-1)]).$$

Hence, the Bohr radius r_f is the positive root of the equation

$$I_q(\exp[F_q(r)]) = I_q(\exp[F_q(-1)]).$$

The proof is complete. □

Note that when $q \to 1$, Theorem 2 gives the Bohr radius for the class $\mathscr{C}$ (see, [11, p. 156]).

COROLLARY 6. *Let* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathscr{C}$. *Then*

$$r + \sum_{n=2}^{\infty} |a_n| r^n \leqslant \frac{1}{2}$$

for $|z| = r \leqslant 1/3$. *The radius is sharp and attained by suitable rotation of the function* $z/(1-z)$.

Proof. In the limiting sense when $q \to 1$, Theorem 2 leads to the Bohr radius r $(0 < r < 1))$ for the class $\mathscr{C}$, where r is the solution of the equation

$$\frac{r}{(1-r)} = \frac{1}{2}.$$

That is, $r = 1/3$. The proof is complete. □

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