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ON TRIPLE SEQUENCE OF BERNSTEIN OPERATOR OF WEIGHTED ROUGH I_λ–CONVERGENCE

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Abstract. We introduce and study some basic properties of rough I_λ -convergence of weight g, where $g:\mathbb{N}^3\to[0,\infty)$ is a function satisfying $g(m,n,k)\to\infty$ and $\frac{\|(m,n,k)\|}{g(m,n,k)}\not\to 0$ as $m,n,k\to\infty$, of triple sequence of Bernstein polynomials and also study the set of all rough I_λ -convergence of weight g limits of a triple sequence of Bernstein polynomials and relation between analyticness and rough I_λ -convergence of weight g of a triple sequences of Bernstein polynomials.

1. Introduction

The notion of the ideal convergence is the dual (equivalent) to the notion of filter convergence introduced by Cartan et al. [4]. The notion of the filter convergence is a generalization of the classical notion of convergence of a sequence and it has been an important tool in general topology and functional analysis. Nowadays many authors to use an equivalent dual notion of the ideal convergence. Kostyrko et al. [16] and Nuray and Ruckle [18] independently studied in details about the notion of ideal convergence which is based on the structure of the admissible ideal I of subsets of natural numbers $\mathbb{N}$. Later on it was further investigated by many authors, e.g. Šalát et al [25], Hazarika and Mohiuddine [15], and references therein.

The idea of rough convergence was first introduced by Phu [20, 21, 22] in finite dimensional normed spaces. He showed that the set LIM_x^r is bounded, closed and convex; and he introduced the notion of rough Cauchy sequence. He also investigated the relations between rough convergence and other convergence types and the dependence of LIM_x^r on the roughness of degree r. Aytar [1] studied of rough statistical convergence and defined the set of rough statistical limit points of a sequence and obtained to statistical convergence criteria associated with this set and prove that this set is closed and convex. Also, Aytar [2] studied that the r-limit set of the sequence is equal to intersection of these sets and that r-core of the sequence is equal to the union of these sets. Dündar and Cakan [7] investigated of rough ideal convergence and defined the set of rough ideal limit points of a sequence. Dündar [9] introduced rough ideal convergence for double sequences. In [24], Sahiner and Tripathy introduced the notion of I-convergence of a triple sequences, which is based on the structure of the ideal I of subsets of $\mathbb{N}\times\mathbb{N}\times\mathbb{N}$, where $\mathbb{N}$ is the set of all natural numbers, is a natural generalization of the notion of convergence and statistical convergence.

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In this paper we investigate some basic properties of rough I-convergence of a triple sequence of Bernstein polynomials in three dimensional cases which are not earlier. We study the set of all rough I-limits of a triple sequence of Bernstein polynomials and also the relation between analyticness and rough I-convergence of a triple sequence of Bernstein polynomials.

Let K be a subset of the set of positive integers $\mathbb{N}\times\mathbb{N}\times\mathbb{N}$ and let us denote the set $K_{ik\ell}=\{(m,n,k)\in K : m\leqslant i, n\leqslant j, k\leqslant \ell\}$. Then the natural density of K is given by

$$\delta(K)=\lim_{i,j,\ell\to\infty}\frac{|K_{ij\ell}|}{ij\ell},$$

where $|K_{ij\ell}|$ denotes the number of elements in $K_{ij\ell}$.

The Bernstein operator of order (r,s,t) is given by

$$B_{rst}(f,x)=\sum_{m=0}^{r}\sum_{n=0}^{s}\sum_{k=0}^{t} f\left(\frac{mnk}{rst}\right)\binom{r}{m}\binom{s}{n}\binom{t}{k}x^{m+n+k}(1-x)^{(m-r)+(n-s)+(k-t)},$$

where f is a continuous (real or complex valued) function defined on $[0,1]$.

Throughout the paper, $\mathbb{R}^3$ denotes the real of three dimensional space with usual metric. Consider a triple sequence of Bernstein polynomials $(B_{mnk}(f,x))$ such that $(B_{mnk}(f,x))\in\mathbb{R}^3$, $(m,n,k)\in\mathbb{N}^3$.

Let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Bernstein polynomials $(B_{mnk}(f,x))$ is said to be statistically convergent to $0\in\mathbb{R}$, written as $st_3-\lim B_{mnk}(f,x)=0$, provided that the set

$$K_\varepsilon := \left\{(m,n,k)\in\mathbb{N}^3 : |B_{mnk}(f,x)-f(x)|\geqslant\varepsilon\right\}$$

has natural density zero for any $\varepsilon>0$. In this case, 0 is called the statistical limit of the triple sequence of Bernstein polynomials. i.e., $\delta(K_\varepsilon)=0$. That is,

$$\lim_{r,s,t\to\infty}\frac{1}{rst}\left|\{(m,n,k)\leqslant(r,s,t) : |B_{mnk}(f,x)-f(x)|\geqslant\varepsilon\}\right|=0.$$

In this case, we write $st_3-\lim B_{mnk}(f,x)=f(x)$ or $B_{mnk}(f,x)\xrightarrow{st_3}f(x)$.

Throughout the paper, $\mathbb{N}$ denotes the set of all positive integers, χ_A-the characteristic function of $A\subset\mathbb{N},\mathbb{R}$ the set of all real numbers. A subset A of $\mathbb{N}^3$ is said to have asymptotic density $d(A)$ if

$$d(A)=\lim_{i,j,\ell\to\infty}\frac{1}{ij\ell}\sum_{m=1}^{i}\sum_{n=1}^{j}\sum_{k=1}^{\ell}\chi_A(K).$$

A triple sequence (real or complex) can be defined as a function $x:\mathbb{N}\times\mathbb{N}\times\mathbb{N}\to\mathbb{R}(\mathbb{C})$, where $\mathbb{C}$ denote the set of complex numbers. The different types of notions of triple sequence was introduced and investigated by Sahiner et al. [23]. Later on further studied by Esi [10, 14], Esi and Catalbas [11], Esi and Savas [12], Esi et al. [13], Dutta et al. [5], Debnath et al. [6], Malik and Maity [17], Pal et al. [19], Savas and Esi [26], Tripathy and Goswami [27, 30, 31, 32], [28], [29] and many others.

A triple sequence of Bernstein polynomials is said to be triple Bernstein polynomials of analytic if

$$\sup_{m,n,k} |B_{mnk}(f,x) - f(x)|^{\frac{1}{m+n+k}} < \infty.$$

The space of all triple of Bernstein polynomials of analytic sequences are usually denoted by Λ_B^3.

2. Definitions and preliminaries

Throughout the paper, we consider a triple sequence $x = (x_{mnk})$ such that $x_{mnk} \in \mathbb{R}^3; m,n,k \in \mathbb{N}$. We recall the following definitions.

DEFINITION 1. [16] A class I of subsets of a nonempty set X is said to be an ideal in X provided
(i) $\varnothing \in I$
(ii) $A,B \in I$ implies $A \bigcup B \in I$.
(iii) $A \in I, B \subset A$ implies $B \in I$.
I is called a nontrivial ideal if $X \notin I$.

DEFINITION 2. [16] A nonempty class F of subsets of a nonempty set X is said to be a filter in X. Provided
(i) $\varnothing \in F$.
(ii) $A,B \in F$ implies $A \bigcap B \in F$.
(iii) $A \in F, A \subset B$ implies $B \in F$.

DEFINITION 3. I is a non trivial ideal in X, $X \neq \varnothing$, then the class

$$F(I) = \{M \subset X : M = X \backslash A \text{ or some } A \in I\}$$

is a filter on X, called the filter associated with I.

DEFINITION 4. A non trivial ideal I in X is called admissible if $\{x\} \in I$ for each $x \in X$.

DEFINITION 5. [7] A sequence $x = (x_k)$ in a normed space $(X, ||.||)$ is said to be rough I-convergent to x_0 if for every $\varepsilon > 0$,

$$\{k \in \mathbb{N} : ||x_k - x_0|| \geqslant r + \varepsilon\} \in I.$$

It is equivalent that the condition $I - \limsup ||x_k - x_0|| \leqslant r$ is satisfied. In this case we write $x_k \xrightarrow{r-I} x_0$ if and only if $||x_k - x_0|| < r + \varepsilon$ holds for every $\varepsilon > 0$ and almost all k.

REMARK 1. If I is an admissible ideal, then usual convergence in X implies I-convergence in X.

REMARK 2. If I is an admissible ideal, then usual rough convergence implies rough $I-$convergence.

DEFINITION 6. [9] For some given real number $r \geqslant 0$, a double sequence $x = (x_{mn})$ is said to be $r - I_2$-convergent to x_0 with the roughness degree r, denoted by $x_{mn} \xrightarrow{r-I_2} x_0$, provided that

$$\{(m,n) \in \mathbb{N} \times \mathbb{N} : ||x_{mn} - x_0|| \geqslant r + \varepsilon\} \in I_2.$$

It is equivalent that the condition $I_2 - \limsup ||x_{mn} - x_0|| \leqslant r$ is satisfied. In this case we write $x_{mn} \xrightarrow{r-I_2} x_0$ if and only if $||x_{mn} - x_0|| < r + \varepsilon$ holds for every $\varepsilon > 0$ and almost all (m,n).

Now the following definition are obtained:

DEFINITION 7. Let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Bernstein polynomials $(B_{mnk}(f,x))$ is said to be statistically convergent to $f(x)$ denoted by $B_{mnk}(f,x) \xrightarrow{st_3} f(x)$, if for any $\varepsilon > 0$ we have $d(A(\varepsilon)) = 0$, where

$$A(\varepsilon) = \left\{(m,n,k) \in \mathbb{N}^3 : |B_{mnk}(f,x) - f(x)| \geqslant \varepsilon\right\}.$$

In this case, $f(x)$ is called the statistical limit of the sequence of Berstein polynomials.

DEFINITION 8. Let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Bernstein polynomials $(B_{mnk}(f,x))$ in $\left(\mathbb{R}^3, |.,.|\right)$ and r be a non-negative real number, is said to be r-convergent to $f(x)$, denoted by $B_{mnk}(f,x) \xrightarrow{r} f(x)$, if for any $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}$ such that for all $m,n,k \geqslant N_\varepsilon$ we have

$$|B_{mnk}(f,x) - f(x)| < r + \varepsilon.$$

In this case $f(x)$ is called an r-limit of $B_{mnk}(f,x)$.

REMARK 3. We consider r-limit set of $B_{mnk}(f,x)$ which is denoted by $LIM^r_{B_{mnk}(f,x)}$ and is defined by

$$LIM^r_{B_{mnk}(f,x)} = \left\{B_{mnk}(f,x) \in X : B_{mnk}(f,x) \xrightarrow{r} f(x)\right\}.$$

DEFINITION 9. Let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Bernstein polynomials $(B_{mnk}(f,x))$ is said to be r-convergent if $LIM^r_{B_{mnk}(f,x)} \neq \varnothing$ and r is called a rough convergence degree of $B_{mnk}(f,x)$. If $r = 0$ then it is ordinary convergence of triple sequence of Bernstein polynomials.

DEFINITION 10. Let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Bernstein polynomials $(B_{mnk}(f,x))$ in a metric space $(X,|.,.|)$ and r be a non-negative real number is said to be r-statistically convergent to $f(x)$, denoted by $B_{mnk}f(x) \xrightarrow{r-st_3} f(x)$, if for any $\varepsilon > 0$ we have $d(A(\varepsilon)) = 0$, where

$$A(\varepsilon) = \left\{(m,n,k) \in \mathbb{N}^3 : |B_{mnk}f(x) - f(x)| \geqslant r + \varepsilon\right\}.$$

In this case $f(x)$ is called r-statistical limit of $B_{mnk}f(x)$. If $r = 0$ then it is ordinary statistical convergent of triple sequence of Bernstein polynomials.

3. Weighted rough I_λ -convergence

Consider $\omega = \{0,1,2,\ldots\}$, recently Balcerzak et al. [3] introduced the density of the weight g, where $g : \omega \longrightarrow [0,\infty)$ with $g(n) \longrightarrow \infty$ but $\frac{n}{g(n)} \nrightarrow 0$. Based on this concept we introduce the rough ideal convergence of weight g, for triple sequences of Bernstein polynomials of reals. Let $\lambda = (\lambda_{pqj})$ be a non-decreasing triple sequence of positive numbers tending to ∞ such that $\lambda_{111} = 1$, $\lambda_{p+1q+1j+1} \leqslant \lambda_{pqj} + 1$ for all p,q,j.

DEFINITION 11. Let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Bernstein polynomials $(B_{mnk}(f,x))$ in $\left(\mathbb{R}^3,|.,.|\right)$ and r be a non-negative real number is said to be rough ideal convergent of weight g or rI_λ-convergent to $f(x)$ of weight g, denoted by $B_{mnk} \xrightarrow{rI_\lambda^g} f(x)$, if for any $\varepsilon > 0$ we have

$$\left\{(p,q,j) \in \mathbb{N}^3 : \frac{1}{g(\lambda_{pqj})} |B_{mnk}(f,x) - f(x)| \geqslant r + \varepsilon\right\} \in I.$$

In this case $f(x)$ is called rI_λ-limit of $(B_{mnk}(f,x))$ of weight g, and a triple sequence of Bernstein polynomials $(B_{mnk}(f,x))$ is called rough I_λ-convergent weight g to $f(x)$ with r as roughness of degree. If $r = 0$ then it is ordinary I_λ-convergent of weight g.

DEFINITION 12. Let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Bernstein polynomials $(B_{mnk}(f,x))$ is said to be I_λ-convergent $f(x)$ of weight g if

$$\left\{(p,q,j) \in \mathbb{N}^3 : \frac{1}{g(\lambda_{pqj})} |B_{mnk}(f,x) - f(x)| \geqslant \varepsilon\right\} \in I.$$

for some $\varepsilon > 0$.

REMARK 4. It is clear that rI_λ^g-limit of $B_{mnk}(f,x)$ is not necessarily unique.

DEFINITION 13. The rI_{λ}^{g}-limit set is denoted by

$$I_{\lambda}^{g}-LIM_{B_{mnk}(f,x)}^{r}=\left\{f(x)\in[0,1]:B_{mnk}(f,x)\xrightarrow{rI_{\lambda}^{g}}f(x)\right\},$$

then the triple sequence of Bernstein polynomials $(B_{mnk}(f,x))$ is said to be rI_{λ}-convergent of weight g, if $I_{\lambda}^{g}-LIM_{B_{mnk}(f,x)}^{r}\neq\phi$ and r is called a rough I_{λ}-convergence of weight g degree of $B_{mnk}(f,x)$.

DEFINITION 14. Let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Bernstein polynomials $(B_{mnk}(f,x))$ is said to be I_{λ}^{g}-analytic if there exists a positive real number M such that

$$\left\{(p,q,j)\in\mathbb{N}^{3}:\frac{1}{g(\lambda_{pqj})}\left|B_{mnk}(f,x)\right|^{1/m+n+k}\geqslant M\right\}\in I.$$

DEFINITION 15. A point $f(x)\in X$ is said to be an I_{λ}^{g}-accumulation point, where f is a continuous function defined on the closed interval $[0,1]$, of a triple sequence of Bernstein polynomials $(B_{mnk}(f,x))$ if and only if for each $\varepsilon>0$ the set

$$\left\{(p,q,j)\in\mathbb{N}^{3}:\frac{1}{g(\lambda_{pqj})}\left|B_{mnk}(f,x)-f(x)\right|<\varepsilon\right\}\notin I.$$

We denote the set of all I_{λ}^{g}-accumulation points of $(B_{mnk}(f,x))$ by $I_{\lambda}^{g}\left(\Gamma_{B_{mnk}(f,x)}\right)$.

DEFINITION 16. Let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Bernstein polynomials $(B_{mnk}(f,x))$ of real numbers, the notions of ideal limit superior and ideal limit inferior are defined as follows:

$$I_{\lambda}^{g}-\limsup B_{mnk}(f,x)=\left\{\begin{array}{cc}\sup B_{B_{mnk}(f,x)}, & \text{if } B_{B_{mnk}(f,x)}\neq\varnothing,\\ -\infty, & \text{if } B_{B_{mnk}(f,x)}=\varnothing\end{array}\right\},$$

and

$$I_{\lambda}^{g}-\liminf(B_{mnk}(f,x))=\left\{\begin{array}{cc}\inf A_{B_{mnk}(f,x)}, & \text{if } A_{B_{mnk}(f,x)}\neq\varnothing,\\ +\infty, & \text{if } A_{B_{mnk}(f,x)}=\varnothing\end{array}\right\},$$

where $A_{B_{mnk}(f,x)}=\left\{a\in\mathbb{R}:\left\{(m,n,k)\in\mathbb{N}^{3}:B_{mnk}(f,x)<a\right\}\notin I\right\}$ and

$$B_{B_{mnk}(f,x)}=\left\{b\in\mathbb{R}:\left\{(m,n,k)\in\mathbb{N}^{3}:B_{mnk}(f,x)>b\right\}\notin I\right\}.$$

DEFINITION 17. Let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Bernstein polynomials $(B_{mnk}(f,x))$ is said to be rough I_{λ}-convergent of weight g, if $I_{\lambda}^{g}-LIM^{r}B_{mnk}(f,x)\neq\varnothing$. It is clear that if $I_{\lambda}^{g}-LIM^{r}B_{mnk}(f,x)\neq\varnothing$ for a triple sequence of Bernstein polynomials $(B_{mnk}(f,x))$ of real numbers, then we have

$$I_{\lambda}^{g}-LIM^{r}B_{mnk}(f,x)=\left[I_{\lambda}^{g}-\limsup B_{mnk}(f,x)-r,I_{\lambda}^{g}-\liminf B_{mnk}(f,x)+r\right].$$

THEOREM 1. *Let f be a continuous function defined on the closed interval $[0,1]$ and let $(B_{mnk}(f,x))$ be a triple sequence of Bernstein polynomials of real numbers. If $I_\lambda^r - LIM^r B_{mnk}(f,x) \neq \varnothing$ for a triple sequence of Bernstein polynomials of real numbers, and $I_\lambda^g - LIM^r B_{mnk}(f,x) = \left[I_\lambda^g - \limsup B_{mnk}(f,x) - r, I_\lambda^g - \liminf B_{mnk}(f,x) + r\right]$ then $diam(LIM^r B_{mnk}(f,x)) \leqslant diam\left(I_\lambda^g - LIM^r B_{mnk}(f,x)\right)$.*

Proof. We know that $I_\lambda^g - LIM^r B_{mnk}(f,x) = \varnothing$ for an unbounded triple sequence of Bernstein polynomials $(B_{mnk}(f,x))$ of real numbers. But such a sequence might be rough I_λ -convergent of weight g. For instance, let $I = I_d$ of $\mathbb{N}$ and define

$$B_{mnk}(f,x) = \left\{ \begin{array}{ll} \cos(mnk)\pi, & \text{if } (m,n,k) \neq (ij\ell)^3 \, (i,j,\ell \in \mathbb{N}), \\ (mnk), & \text{otherwise} \end{array} \right\},$$

in $\mathbb{R}^3$. Because the set $\{1, 64, 739, \cdots\}$ belong to I, we have

$$I_\lambda^g - LIM^r B_{mnk}(f,x) = \left\{ \begin{array}{ll} \phi, & \text{if } r < 1, \\ [1-r, r-1], & \text{otherwise} \end{array} \right\},$$

and $LIM^r B_{mnk}(f,x) = \varnothing$ for all $r \geqslant 0$. The fact that $I_\lambda^g - LIM^r B_{mnk}(f,x) \neq \varnothing$ does not imply $LIM^r B_{mnk}(f,x) \neq \varnothing$. Because I is a admissible ideal

$$LIM^r B_{mnk}(f,x) \neq \varnothing \Longrightarrow I_\lambda^g LIM^r B_{mnk}(f,x) \neq \varnothing,$$

i.e., if $B_{mnk}(f,x)) \in LIM^r B_{mnk}(f,x)$, then by Remark 3, $B_{mnk}(f,x) \in I_\lambda^g - LIM^r B_{mnk}(f,x)$, for each triple sequences of Bernstein polynomials. Also, if we define all the rough convergence of weight g by LIM^r and rough I_λ -convergence of weight g sequences by $I_\lambda^g - LIM^r$, then we get $LIM^r \subseteq I_\lambda^g - LIM^r$.

$$\{r \geqslant 0 : LIM^r B_{mnk}(f,x) \neq \phi\} \subseteq \left\{r \geqslant 0 : I_\lambda^g - LIM^r B_{mnk}(f,x) \neq \varnothing\right\}.$$

Hence the sets yields immediately

$$\inf\{r \geqslant 0 : LIM^r B_{mnk}(f,x) \neq \phi\} \supseteq \left\{r \geqslant 0 : I_\lambda^g - LIM^r B_{mnk}(f,x) \neq \varnothing\right\},$$

for each triple sequences of Bernstein polynomials of $B_{mnk}(f,x)$. Moreover, it also yield directly

$$diam(LIM^r B_{mnk}(f,x)) \leqslant diam\left(I_\lambda^g - LIM^r B_{mnk}(f,x)\right). \quad \square$$

REMARK 5. The rough I_λ -convergent of weight g, limit of a triple sequence of Bernstein polynomials $(B_{mnk}(f,x))$ is unique for the roughness degree $r > 0$. The following result is related to the this fact.

THEOREM 2. *Let f be a continuous function defined on the closed interval $[0,1]$ and let $(B_{mnk}(f,x))$ be a triple sequence of Bernstein polynomials of real numbers, and $I \subset 2^{\mathbb{N}}$ be an admissible ideal. Then $diam\ (I_\lambda^g$-$LIM^r B_{mnk}(f,x)) \leqslant 2r$. In general, $diam\ \left(I_\lambda^g - LIM^r B_{mnk}(f,x)\right)$ has an upper bound.*

Proof. Assume that diam $(LIM^r B_{mnk}(f,x)) = 2r$. Then $\exists u, v \in LIM^r B_{mnk}(f,x) \ni$: $|u-v| > 2r$. Take $\varepsilon \in \left(0, \frac{|u-v|}{2} - r\right)$. Since $u, v \in I_\lambda^g - LIM^r B_{mnk}(f,x)$, we have $A_1(\varepsilon) \in I$ and $A_2(\varepsilon) \in I$ for every $\varepsilon > 0$, where

$$A_1(\varepsilon) = \left\{(p,q,j) \in \mathbb{N}^3 : \frac{1}{g(\lambda_{pqj})} |B_{mnk}(f,x) - u| \geqslant r + \varepsilon\right\}$$

and

$$A_2(\varepsilon) = \left\{(p,q,j) \in \mathbb{N}^3 : \frac{1}{g(\lambda_{pqj})} |B_{mnk}(f,x) - v| \geqslant r + \varepsilon\right\}$$

for all $(m,n,k) \in \mathbb{N}^3$.

Using the properties $F(I)$, we get

$$\left(A_1(\varepsilon)^c \bigcap A_2(\varepsilon)^c\right) \in F(I).$$

Thus we write

$$|u-v| \leqslant \frac{1}{g(\lambda_{pqj})} |B_{mnk}(f,x) - u| + \frac{1}{g(\lambda_{rst})} |B_{mnk}(f,x) - v|$$
$$< (r+\varepsilon) + (r+\varepsilon) < 2(r+\varepsilon), \text{ for all } (p,q,j) \in A_1(\varepsilon)^c \bigcap A_2(\varepsilon)^c$$

which is a contradiction. Hence diam $(LIM^r B_{mnk}(f,x)) \leqslant 2r$.

Now, consider a triple sequence of Bernstein polynomials of $(B_{mnk}(f,x))$ of real numbers such that $I_\lambda^g - \lim_{mnk \to \infty} B_{mnk}(f,x) = f(x)$.

Let $\varepsilon > 0$. For all $(m,n,k) \in \mathbb{N}^3$, we can write

$$\left\{(p,q,j) \in \mathbb{N}^3 : \frac{1}{g(\lambda_{pqj})} |B_{mnk}(f,x) - f(x)| \geqslant \varepsilon\right\} \in I.$$

Thus we have

$$\frac{1}{g(\lambda_{pqj})} |B_{mnk}(f,x) - t| \leqslant \frac{1}{g(\lambda_{pqj})} |B_{mnk}(f,x) - f(x)| + \frac{1}{g(\lambda_{pqj})} |f(x) - t|$$
$$\leqslant \frac{1}{g(\lambda_{pqj})} |B_{mnk}(f,x) - f(x)| + r \leqslant r + \varepsilon$$

for each $t \in \bar{B}_r(f(x)) := \left\{(p,q,j) \in \mathbb{N}^3 : \frac{1}{g(\lambda_{pqj})} |t - f(x)| \leqslant r\right\}$.

Then, we get

$$\frac{1}{g(\lambda_{pqj})} |B_{mnk}(f,x) - t| < r + \varepsilon$$

for each $(m,n,k) \in \left\{(p,q,j) \in \mathbb{N}^3 : \frac{1}{g(\lambda_{pqj})} |B_{mnk}(f,x) - f(x)| < \varepsilon\right\}$. Because the triple sequence of Bernstein polynomials of $B_{mnk}(f,x)$ is I_λ-convergent of weight g to $f(x)$, we have

$$\left\{(p,q,j) \in \mathbb{N}^3 : \frac{1}{g(\lambda_{pqj})} |B_{mnk}(f,x) - f(x)| < \varepsilon\right\} \in F(I).$$

Therefore, we get $t \in I_\lambda^g - LIM^r B_{mnk}(f,x)$. Consequently, we can write

$$I_\lambda^g - LIM^r B_{mnk}(f,x) = \bar{B}_r(f(x)).$$

Since $diam\left(\bar{B}_r(f(x))\right) = 2r$, this shows that in general, the upper bound $2r$ of the diameter of the set $I_\lambda^g - LIM^r B_{mnk}(f,x)$ is not lower bound. □

THEOREM 3. *Let $I \subset 3^{\mathbb{N}}$ be an admissible ideal and Let f be a continuous function defined on the closed interval $[0,1]$, and $(B_{mnk}(f,x))$ be a triple sequence of Bernstein polynomials is I_λ^g-analytic if and only if there exists a non-negative real number r such that $I_\lambda^g - LIM^r B_{mnk}(f,x) \neq \varnothing$ for all $r > 0$, an I_λ^g-analytic triple sequence of Bernstein polynomials always contains a sub sequence $\left(B_{m_i n_j k_\ell}(f,x)\right)$ with $I_\lambda^g - LIM^r B_{m_i n_j k_\ell}(f,x) \neq \varnothing$.*

Proof. Since the triple sequence of Bernstein polynomials of $B_{mnk}(f,x)$ is I_λ^g-analytic then there exists a positive real number M such that

$$\left\{(p,q,j) \in \mathbb{N}^3 : \frac{1}{g(\lambda_{pqj})} |B_{mnk}(f,x)|^{1/m+n+k} \geqslant M\right\} \in I.$$

Define $r' = \sup\left\{(p,q,j) \in \mathbb{N}^3 : \frac{1}{g(\lambda_{pqj})} |B_{mnk}(f,x)|^{1/m+n+k} \geqslant M : (m,n,k) \in K^c\right\}$, where $K = \left\{(p,q,j) \in \mathbb{N}^3 : \frac{1}{g(\lambda_{pqj})} |B_{mnk}(f,x)|^{1/m+n+k} \geqslant M\right\}$.

Then the set $I_\lambda^g - LIM^{r'} B_{mnk}(f,x)$ contains the origin of $\mathbb{R}^3$. So we have $I_\lambda^g - LIM^{r'} B_{mnk}(f,x) \neq \varnothing$.

If $I_\lambda^g - LIM^r B_{mnk}(f,x) \neq \varnothing$ for some $r \geqslant 0$, then there exists $f(x)$ such that $f(x) \in I_\lambda^g - LIM^r B_{mnk}(f,x)$, i.e.,

$$\left\{(p,q,j) \in \mathbb{N}^3 : \frac{1}{g(\lambda_{pqj})} |B_{mnk}(f,x) - f(x)|^{1/m+n+k} \geqslant r + \varepsilon\right\} \in I$$

for each $\varepsilon > 0$. Then we say that almost all $B_{mnk}(f,x)$ are contained in some ball with any radius greater than r. So the triple sequence space of Bernstein polynomials of $B_{mnk}(f,x)$ is I_λ^g-analytic.

Since $B_{mnk}(f,x)$ is a I_λ^g-analytic triple sequence of Bernstein polynomials in a three-dimensional metric space, it certainly contains a I_λ-convergent of weight g sub sequence $\left(B_{m_i n_j k_\ell}(f,x)\right)$. Let $f(x)$ be its I_λ^g-limit point, then $I_\lambda^g - LIM^r B_{m_i n_j k_\ell}(f,x) = \bar{B}_r(f(x))$ and, for $r > 0$,

$$I_\lambda^g - LIM^r\left(B_{m_i n_j k_\ell}(f,x)\right) \neq \varnothing. \quad \square$$

THEOREM 4. *Let $I \subset 3^{\mathbb{N}}$ be an admissible ideal. If $\left(B_{m_i n_j k_\ell}(f,x)\right)$ is a sub sequence of $(B_{mnk}(f,x))$, then*

$$I_\lambda^g - LIM^r B_{mnk}(f,x) \subseteq I_\lambda^g - LIM^r B_{m_i n_j k_\ell}(f,x).$$

Proof. The proof is trivial (See [8], Proposition 2.3). □

THEOREM 5. *Let f be a continuous function defined on the closed interval $[0,1]$ and $(B_{mnk}(f,x))$ be a triple sequence of Bernstein polynomials, and $I \subset 3^{\mathbb{N}}$ be an admissible ideal. Then $I_\lambda^g - LIM^r B_{mnk}(f,x)$ is closed.*

Proof. The result is true for $I_\lambda^g - LIM^r B_{mnk}(f,x) = \phi$.

Assume that $I_\lambda^g - LIM^r B_{mnk}(f,x) \neq \varnothing$. Then, we can choose a triple sequence of Bernstein polynomials of $B_{mnk}(f,y) \subseteq I_\lambda^g - LIM^r B_{mnk}(f,x)$ such that $B_{mnk}(f,y) \xrightarrow{r} f(y)$ for $m,n,k \to \infty$. To prove $f(x) \in I_\lambda^g - LIM^r B_{mnk}(f,x)$.

Let $\varepsilon > 0$ be given. Because $B_{mnk}(f,y) \to f(y), \exists i,j,\ell = i_{\frac{\varepsilon}{2}}, j_{\frac{\varepsilon}{2}}, \ell_{\frac{\varepsilon}{2}} \in \mathbb{N}$ such that

$$\frac{1}{g(\lambda_{pqj})} |B_{mnk}(f,y) - f(y)| < \frac{\varepsilon}{2}, \forall m \geqslant i_{\frac{\varepsilon}{2}}, n \geqslant j_{\frac{\varepsilon}{2}}, k \geqslant \ell_{\frac{\varepsilon}{2}}.$$

Now choose an $m_0, n_0, k_0 \in \mathbb{N}$ such that $m_0 \geqslant i_{\frac{\varepsilon}{2}}, n_0 \geqslant j_{\frac{\varepsilon}{2}}, k_0 \geqslant \ell_{\frac{\varepsilon}{2}}$. Then we can write

$$\frac{1}{g(\lambda_{pqj})} \left|B_{m_0 n_0 k_0}(f,y) - f(y)\right| < \frac{\varepsilon}{2}.$$

On the other hand, because $B_{mnk}(f,y) \subseteq I_\lambda^g - LIM^r B_{mnk}(f,x)$, we have $B_{m_0 n_0 k_0}(f,y) \in I_\lambda^g - LIM^r B_{mnk}(f,x)$, namely,

$$A\left(\frac{\varepsilon}{2}\right) = \left\{(p,q,j) \in \mathbb{N}^3 : \frac{1}{g(\lambda_{pqj})} \left|B_{mnk}(f,x) - B_{m_0 n_0 k_0}(f,y)\right| \geqslant r + \frac{\varepsilon}{2}\right\} \in I.$$

Now let us prove that the inclusion

$$A^c\left(\frac{\varepsilon}{2}\right) \subseteq A^c(\varepsilon) \tag{1}$$

holds, where $A(\varepsilon) = \left\{(p,q,j) \in \mathbb{N}^3 : \frac{1}{g(\lambda_{pqj})} |B_{mnk}(f,x) - f(x)| \geqslant r + \varepsilon\right\}$. Take $(u,v,w) \in A^c\left(\frac{\varepsilon}{2}\right)$. Then we have

$$\frac{1}{g(\lambda_{pqj})} \left|B_{uvw}(f,x) - B_{m_0 n_0 k_0}(f,y)\right| < r + \frac{\varepsilon}{2}$$

and hence

$$\begin{aligned}
&\frac{1}{g(\lambda_{pqj})} |B_{uvw}(f,x) - f(x)| \\
&\leqslant \frac{1}{g(\lambda_{pqj})} \left|B_{uvw}(f,x) - B_{m_0 n_0 k_0}(f,y)\right| + \left|b_{m_0 n_0 k_0}(f,y) - f(x)\right| < r + \varepsilon,
\end{aligned}$$

i.e., $(u,v,w) \in A^{c}(\varepsilon)$, which proves (1). Thus we get

$$A(\varepsilon) \in I\left(i.e., f(x) \in I_{\lambda}^{g} - LIM^{r} B_{mnk}(f,x)\right). \quad \square$$

THEOREM 6. *Let f be a continuous function defined on the closed interval $[0,1]$ and $(B_{mnk}(f,x))$ be a triple sequence of Bernstein polynomials of real numbers, and $I \subset 2^{\mathbb{N}}$ be an admissible ideal. Then the rough I_{λ}^{g}-limit set of triple sequence of Bernstein polynomials of $B_{mnk}(f,x)$ is convex.*

Proof. Let $y_1, y_2 \in I_{\lambda}^{g} - LIM^{r} B_{mnk}(f,x)$ for triple sequence of Bernstein polynomials of $B_{mnk}(f,x)$ and let $\varepsilon > 0$ be given. Define

$$A_1(\varepsilon) = \left\{(p,q,j) \in \mathbb{N}^3 : \frac{1}{g(\lambda_{pqj})} |B_{mnk}(f,x) - y_1| \geqslant r + \varepsilon\right\}$$

and

$$A_2(\varepsilon) = \left\{(p,q,j) \in \mathbb{N}^3 : \frac{1}{g(\lambda_{pqj})} |B_{mnk}(f,x) - y_2| \geqslant r + \varepsilon\right\},$$

because $y_1, y_2 \in I_{\lambda}^{g} - LIM^{r} B_{mnk}(f,x)$, we have $A_1(\varepsilon)$, $A_2(\varepsilon) \in I$. Thus we have

$$\begin{aligned}
&\frac{1}{g(\lambda_{pqj})} |B_{mnk}(f,x) - [(1-\lambda)y_1 + \lambda y_2]| \\
&= \frac{1}{g(\lambda_{pqj})} |(1-\lambda)(B_{mnk}(f,x) - y_1) + \lambda(B_{mnk}(f,x) - y_2)| \\
&\leqslant (1-\lambda)\frac{1}{g(\lambda_{pqj})} |B_{mnk}(f,x) - y_1| + \lambda \frac{1}{g(\lambda_{pqj})} |B_{mnk}(f,x) - y_2| \\
&< (1-\lambda)(r+\varepsilon) + \lambda(r+\varepsilon) < r + \varepsilon
\end{aligned}$$

for each $(m,n,k) \in A_1^{c}(\varepsilon) \bigcap A_2^{c}(\varepsilon)$ and each $\lambda \in [0,1]$. Because $(A_1^{c}(\varepsilon) \bigcap A_2^{c}(\varepsilon)) \in F(I)$ by definition of $F(I)$, we get

$$\left\{(p,q,j) \in \mathbb{N}^3 : \frac{1}{g(\lambda_{pqj})} |B_{mnk}(f,x) - [(1-\lambda)y_1 + \lambda y_2]| \geqslant r + \varepsilon\right\} \in I,$$

that is

$$[(1-\lambda)y_1 + \lambda y_2] \in I_{\lambda}^{g} - LIM^{r} B_{mnk}(f,x),$$

which proves the convexity of the set $I_{\lambda}^{g} - LIM^{r} B_{mnk}(f,x)$. $\square$

THEOREM 7. *Let f be a continuous function defined on the closed interval $[0,1]$ and let $I \subset 3^{\mathbb{N}}$ be an admissible ideal. Then a triple sequence of Bernstein polynomials of $(B_{mnk}(f,x))$ of reals with $r > 0$ is rough I_{λ}-convergent to weight g of $f(x)$ if and only if there exists a triple sequence of Bernstein polynomials of $B_{mnk}(f,y)$ such that*

$$I_{\lambda}^{g} - \lim B_{mnk}(f,y) = f(x)$$

and

$$\frac{1}{g(\lambda_{pqj})} |B_{mnk}(f,x) - B_{mnk}(f,y)| \leqslant r, \text{ for each } m,n,k,p,q,j \in \mathbb{N}. \tag{2}$$

Proof. Assume that the triple sequence of Bernstein polynomials of $B_{mnk}(f,x)$ is rough I_λ-convergent to weight g of $f(x)$. Then we have

$$I_\lambda^g - \limsup \frac{1}{g(\lambda_{pqj})} |B_{mnk}(f,x) - f(x)| \leqslant r. \quad (3)$$

Define

$$B_{mnk}(f,y) = \begin{cases} f(x), \text{ if } \frac{1}{g(\lambda_{pqj})} |B_{mnk}(f,x) - f(x)| \leqslant r, \\ B_{mnk}(f,x) + r \left(\frac{f(x) - B_{mnk}(f,x)}{\frac{1}{g(\lambda_{pqj})} |B_{mnk}(f,x) - f(x)|} \right), \text{ otherwise.} \end{cases}$$

We write

$$\frac{1}{g(\lambda_{pqj})} |B_{mnk}(f,y) - f(x)|$$
$$= \begin{cases} \frac{1}{g(\lambda_{pqj})} |f(x) - f(x)|, if \frac{1}{g(\lambda_{pqj})} |B_{mnk}(f,x) - f(x)| \leqslant r, \\ \frac{1}{g(\lambda_{pqj})} |B_{mnk}(f,x) - f(x)| + r \left(\frac{\frac{1}{g(\lambda_{pqj})} |f(x) - f(x)| - \frac{1}{g(\lambda_{pqj})} |B_{mnk}(f,x) - f(x)|}{\frac{1}{g(\lambda_{pqj})} |B_{mnk}(f,x) - f(x)|} \right), \text{otherwise,} \end{cases}$$

(i.e)

$$\frac{1}{g(\lambda_{pqj})} |B_{mnk}(f,y) - f(x)|$$
$$= \begin{cases} 0, if \frac{1}{g(\lambda_{pqj})} |B_{mnk}(f,x) - f(x)| \leqslant r, \\ \frac{1}{g(\lambda_{pqj})} |B_{mnk}(f,x) - f(x)| - r \left(\frac{\frac{1}{g(\lambda_{pqj})} |B_{mnk}(f,x) - f(x)|}{\frac{1}{g(\lambda_{pqj})} |B_{mnk}(f,x) - f(x)|} \right), \text{ otherwise,} \end{cases}$$

(i.e)

$$\frac{1}{g(\lambda_{pqj})} |B_{mnk}(f,y) - f(x)| = \begin{cases} 0, if \frac{1}{g(\lambda_{pqj})} |B_{mnk}(f,x) - f(x)| \leqslant r, \\ \frac{1}{g(\lambda_{pqj})} |B_{mnk}(f,x) - f(x)| - r, \text{ otherwise.} \end{cases}$$

We have

$$\frac{1}{g(\lambda_{pqj})} |B_{mnk}(f,y) - f(x)| \geqslant \frac{1}{g(\lambda_{pqj})} |B_{mnk}(f,x) - f(x)| - r$$
$$\Longrightarrow \frac{1}{g(\lambda_{pqj})} |B_{mnk}(f,x) - f(x) - B_{mnk}(f,y) + f(y)| \leqslant r$$

i.e.

$$\frac{1}{g(\lambda_{pqj})} |B_{mnk}(f,x) - B_{mnk}(f,x)| \leqslant r$$

for all $m,n,k,p,q,j \in \mathbb{N}$. By equation (3) and by definition of $B_{mnk}(f,y)$, we get

$$I_\lambda^g - \limsup \frac{1}{g(\lambda_{pqj})} |B_{mnk}(f,y) - f(x)| = 0.$$
$$\Longrightarrow I_\lambda^g - \lim B_{mnk}(f,y) = f(x).$$

Assume that (2) holds. Since $I_\lambda^g - \lim B_{mnk}(f,y) = f(x)$, we have

$$A(\varepsilon) = \left\{(p,q,j) \in \mathbb{N}^3 : \frac{1}{g(\lambda_{pqj})} |B_{mnk}(f,y) - f(x)| \geqslant r + \varepsilon\right\} \in I,$$

for each $\varepsilon > 0$. Now, define the set

$$B(\varepsilon) = \left\{(p,q,j) \in \mathbb{N}^3 : \frac{1}{g(\lambda_{pqj})} |B_{mnk}(f,x) - f(x)| \geqslant r + \varepsilon\right\} \in I.$$

We have $B(\varepsilon) \subseteq A(\varepsilon)$ holds. Since $A(\varepsilon) \in I \Longrightarrow B(\varepsilon) \in I$. Hence $B_{mnk}(f,x)$ is rough I_λ-convergent to weight g of $f(x)$. □

REMARK 6. If we replace the condition $\frac{1}{g(\lambda_{pqj})} |B_{mnk}(f,x) - B_{mnk}(f,y)| \leqslant r$ for all $m,n,k,p,q,j \in \mathbb{N}$ in the hypothesis of the above theorem with the condition

$$\left\{(p,q,j) \in \mathbb{N}^3 : \frac{1}{g(\lambda_{pqj})} |B_{mnk}(f,x) - B_{mnk}(f,y)| > r\right\} \in I$$

then the theorem will also be valid.

THEOREM 8. *Let f be a continuous function defined on the closed interval $[0,1]$, let $I \subset 3^{\mathbb{N}}$ be an admissible ideal, and $(B_{mnk}(f,x))$ be a triple sequence of Bernstein polynomialsof real numbers. For an arbitrary $c \in I_\lambda^g(\Gamma_x)$, we have $\frac{1}{g(\lambda_{pqj})} |f(x) - c| \leqslant r$ for all $f(x) \in I_\lambda^g - LIM^r B_{mnk}(f,x)$.*

Proof. Assume on the contrary that there exist a point $c \in I_\lambda^g(\Gamma_x)$ and $f(x) \in I_\lambda^g - LIM^r B_{mnk}(f,x)$ such that $\frac{1}{g(\lambda_{pqj})} |f(x) - c| > r$. Define

$$\varepsilon := \frac{\frac{1}{g(\lambda_{pqj})} |f(x) - c| - r}{3}.$$

Then

$$\left\{(p,q,j) \in \mathbb{N}^3 : \frac{1}{g(\lambda_{pqj})} |f(x) - c| < \varepsilon\right\}$$
$$\subseteq \left\{(p,q,j) \in \mathbb{N}^3 : \frac{1}{g(\lambda_{pqj})} |B_{mnk}(f,x) - f(x)| \geqslant r + \varepsilon\right\}.$$

Since $c \in I_{\lambda}^{g}(\Gamma_x)$, we have

$$\left\{(p,q,j) \in \mathbb{N}^3 : \frac{1}{g(\lambda_{pqj})} |B_{mnk}(f,x) - c| < \varepsilon\right\} \notin I.$$

From definition of I_λ-convergence of weight g, since

$$\left\{(p,q,j) \in \mathbb{N}^3 : \frac{1}{g(\lambda_{pqj})} |B_{mnk}(f,x) - f(x)| \geqslant r + \varepsilon\right\} \in I,$$

so by (3) we have

$$\left\{(p,q,j) \in \mathbb{N}^3 : \frac{1}{g(\lambda_{pqj})} |B_{mnk}(f,x) - c| < \varepsilon\right\} \in I,$$

which contradicts the fact $c \in I_{\lambda}^{g}(\Gamma_x)$. On the other hand, if $c \in I_{\lambda}^{g}(\Gamma_x)$ i.e.,

$$\left\{(p,q,j) \in \mathbb{N}^3 : \frac{1}{g(\lambda_{pqj})} |B_{mnk}(f,x) - c| < \varepsilon\right\} \notin I,$$

then

$$\left\{(p,q,j) \in \mathbb{N}^3 : \frac{1}{g(\lambda_{pqj})} |B_{mnk}(f,x) - f(x)| \geqslant r + \varepsilon\right\} \notin I,$$

which contradicts the fact $f(x) \in I_{\lambda}^{g} - LIM^{r} B_{mnk}(f,x)$. □

THEOREM 9. *Let f be a continuous function defined on the closed interval $[0,1]$, let $(B_{mnk}(f,x))$ be a triple sequence of Bernstein polynomials of real numbers, and $I \subset 3^{\mathbb{N}}$ be an admissible ideal, $(\mathbb{R}^3, |.,.|)$ be a strictly convex, if there exist y_1, y_2, y_3, y_4, y_5, $y_6 \in I_{\lambda}^{g} - LIM^{r} B_{mnk}(f,x)$ such that*

$$\frac{1}{g(\lambda_{pqj})} |y_1 - y_2| < 2r, \quad \frac{1}{g(\lambda_{pqj})} |y_3 - y_4| < 2r$$

and

$$\frac{1}{g(\lambda_{pqj})} |y_5 - y_6| < 2r,$$

then this triple sequence of Bernstein polynomials is I_λ-convergent to weight g to

$$\frac{1}{6} \frac{1}{g(\lambda_{pqj})} (y_1 + y_2 + y_3 + y_4 + y_5 + y_6).$$

Proof. Let $c \in I_{\lambda}^{g}(\Gamma_x)$. Then since y_1, y_2, y_3, y_4, y_5, $y_6 \in I_{\lambda}^{g} - LIM^{r} B_{mnk}(f,x)$. By Theorem 8, we have

$$\begin{aligned} &\frac{1}{g(\lambda_{pqj})} |y_1 - c| \leqslant r, \quad \frac{1}{g(\lambda_{pqj})} |y_2 - c| \leqslant r, \quad \frac{1}{g(\lambda_{pqj})} |y_3 - c| \leqslant r, \\ &\frac{1}{g(\lambda_{pqj})} |y_4 - c| \leqslant r, \quad \frac{1}{g(\lambda_{pqj})} |y_5 - c| \leqslant r \text{ and } \frac{1}{g(\lambda_{pqj})} |y_6 - c| \leqslant r. \end{aligned} \tag{4}$$

On the other hand, we have

$$\begin{aligned} 6rg((p,q,j)) &= |y_1 - y_6| \\ &\leqslant |y_1 - c| + |y_2 - c| + |y_3 - c| + |y_4 - c| + |y_5 - c| + |y_6 - c|. \end{aligned} \tag{5}$$

Therefore, we get $\frac{1}{g(\lambda_{pqj})}|y_1 - c| = \cdots = \frac{1}{g(\lambda_{pqj})}|y_6 - c| = r$ by inequalities (4) and (5). Since

$$\begin{aligned} &\frac{1}{6}\frac{1}{g(\lambda_{pqj})}(y_6 - y_1) \\ &= \frac{1}{6}\frac{1}{g(\lambda_{pqj})}[(c-y_1)+(c-y_2)(c-y_3)(c-y_4)+(c-y_5)+(c-y_6)] \end{aligned} \tag{6}$$

we get $\frac{1}{g(\lambda_{pqj})}\left|\frac{1}{6}(y_6 - y_1)\right| = r$. By the strict convexity of the space from the equality (6), we get

$$\frac{1}{6}\frac{1}{g(\lambda_{pqj})}(y_6 - y_1) = \frac{1}{g(\lambda_{pqj})}(c - y_1) = \cdots = \frac{1}{g(\lambda_{pqj})}(c - y_6),$$

which implies that

$$c = \frac{1}{6}\frac{1}{g(\lambda_{pqj})}(y_1 + y_2 + y_3 + y_4 + y_5 + y_6).$$

Hence c is the unique I_λ^g- cluster point of the triple sequence of Bernstein polynomials of $(B_{mnk}(f,x))$.

On the other hand, the assumption $y_1, y_2, y_3, y_4, y_5, y_6 \in I_\lambda^g - LIM^r B_{mnk}(f,x) \Longrightarrow I_\lambda^g - LIM^r B_{mnk}(f,x) \neq \phi$. By theorem 3, the triple sequence of Bernstein polynomials of $B_{mnk}(f,x)$ is I_λ^g-analytic. Consequently, the triple sequence space of Bernstein polynomials of $B_{mnk}(f,x)$ must I_λ-convergent to weight g to

$$\frac{1}{6}\frac{1}{g(\lambda_{pqj})}(y_1 + y_2 + y_3 + y_4 + y_5 + y_6),$$

i.e.,

$$I_\lambda^g - lim B_{mnk}(f,x) = \frac{1}{6}\frac{1}{g(\lambda_{pqj})}(y_1 + y_2 + y_3 + y_4 + y_5 + y_6). \qquad \square$$

THEOREM 10. *Let f be a continuous function defined on the closed interval $[0,1]$ and let $(B_{mnk}(f,x))$ be a triple sequence of Bernstein polynomials of real numbers and $I \subset 3^{\mathbb{N}}$ be an admissible ideal.*

(i) If $c \in I_\lambda^g(\Gamma_x)$ then $I_\lambda^g - LIM^r B_{mnk}(f,x) \subseteq \bar{B}_r(c)$.

(ii)

$$I_\lambda^g - LIM^r B_{mnk}(f,x) = \bigcap_{c \in I_\lambda^g(\Gamma_x)} \bar{B}_r(c) = \left\{ f(x) \in \mathbb{R}^3 : I_\lambda^g(\Gamma_x) \subseteq \bar{B}_r(f(x)) \right\}. \tag{7}$$

Proof. (i) If $c \in I_\lambda^g(\Gamma_x)$ then by Theorem 8, we have $\frac{1}{g(\lambda_{pqj})}|f(x)-c| \leqslant r$ for all $f(x) \in I_\lambda^g - LIM^r B_{mnk}(f,x)$, other wise we get

$$\left\{(p,q,j) \in \mathbb{N}^3 : \frac{1}{g(\lambda_{pqj})}|B_{mnk}(f,x)-f(x)| \geqslant r+\varepsilon\right\} \notin I,$$

for $\varepsilon := \frac{\frac{1}{g(\lambda_{pqj})}|f(x)-c|-r}{3}$. Because c is an I_λ^g-cluster point of $B_{mnk}(f,x)$, this contradicts with the fact that $f(x) \in I_\lambda^g - LIM^r B_{mnk}(f,x)$.

(ii) From (7) we have

$$I_\lambda^g - LIM^r B_{mnk}(f,x) \subseteq \bigcap_{c \in I_\lambda^g(\Gamma_x)} \bar{B}_r(c). \tag{8}$$

Now, let $B_{mnk}(f,x) \in \bigcap_{c \in I_\lambda^g(\Gamma_x)} \bar{B}_r(c)$. Then we have

$$\frac{1}{g(\lambda_{pqj})}|B_{mnk}(f,x)-c| \leqslant r$$

for all $c \in I_\lambda^g(\Gamma_x)$, which is equivalent to $I_\lambda^g(\Gamma_x) \subseteq \bar{B}_r(B_{mnk}(f,y))$, i.e.,

$$\bigcap_{c \in I_\lambda^g(\Gamma_x)} \bar{B}_r(c) = \left\{f(x) \in \mathbb{R}^3 : I_\lambda^g(\Gamma_x) \subseteq \bar{B}_r(f(x))\right\}. \tag{9}$$

Now, let $B_{mnk}(f,y) \notin I_\lambda^g - LIM^r B_{mnk}(f,x)$. Then, there exists an $\varepsilon > 0$ such that

$$\left\{(p,q,j) \in \mathbb{N}^3 : \frac{1}{g(\lambda_{pqj})}|B_{mnk}(f,x)-B_{mnk}(f,y)| \geqslant r+\varepsilon\right\} \notin I,$$

$\Longrightarrow$ the existence of an I_λ^g-cluster point c of the sequence $B_{mnk}(f,x)$ with

$$\frac{1}{g(\lambda_{pqj})}|B_{mnk}(f,y)-c| \geqslant r+\varepsilon,$$

i.e.,

$$I_\lambda^g(\Gamma_x) \nsubseteq \bar{B}_r(B_{mnk}(f,y))$$

and

$$B_{mnk}(f,y) \notin \left\{f(x) \in \mathbb{R}^3 : I_\lambda^g(\Gamma_x) \subseteq \bar{B}_r(f(x))\right\}.$$

Hence $B_{mnk}(f,y) \in I_\lambda^g - LIM^r B_{mnk}(f,x)$ follows from

$$B_{mnk}(f,y) \in \left\{f(x) \in \mathbb{R}^3 : I_\lambda^g(\Gamma_x) \subseteq \bar{B}_r(f(x))\right\},$$

i.e.,

$$\left\{f(x) \in \mathbb{R}^3 : I_\lambda^g(\Gamma_x) \subseteq \bar{B}_r(f(x))\right\} \subseteq I_\lambda^g - LIM^r B_{mnk}(f,x). \tag{10}$$

Therefore, the inclusions (8)-(10) ensure that (7) holds i.e.,

$$I_\lambda^g - LIM^r B_{mnk}(f,x) \bigcap_{c \in I_\lambda^g(\Gamma_x)} \bar{B}_r(c) = \left\{ f(x) \in \mathbb{R}^3 : I_\lambda^g(\Gamma_x) \subseteq \bar{B}_r(f(x)) \right\}. \quad \square$$

THEOREM 11. *Let $I \subset 3^{\mathbb{N}}$ be an admissible ideal and Let f be a continuous function defined on the closed interval $[0,1]$. If $(B_{mnk}(f,x))$ is an I_λ^g-analytic sequence of Bernstein polynomials of real numbers with $r \geqslant diam\left(I_\lambda^g(\Gamma_x)\right)$, then $I_\lambda^g(\Gamma_x) \subseteq I_\lambda^g - LIM^r B_{mnk}(f,x)$.*

Proof. Let $c \notin I_\lambda^g - LIM^r B_{mnk}(f,x)$. Then there exists an $\varepsilon > 0$ such that

$$\left\{ (p,q,j) \in \mathbb{N}^3 : \frac{1}{g(\lambda_{pqj})} \left| B_{mnk}(f,x) - c \right|^{1/m+n+k} \geqslant r + \varepsilon \right\} \notin I. \tag{11}$$

Since $B_{mnk}(f,x)$ is I_λ^g-analytic and from the inequality (11), there exists an I_λ^g-cluster point c_1 such that

$$\frac{1}{g(\lambda_{pqj})} |c - c_1| > r + \varepsilon_1, \text{ where } \varepsilon_1 := \frac{\varepsilon}{2}.$$

So we get $diam\left(I_\lambda^g(\Gamma_x)\right) > r + \varepsilon_1$.

The converse of this theorem is also true, i.e., if $I_\lambda^g(\Gamma_x) \subseteq I_\lambda^g - LIM^r B_{mnk}(f,x)$, then we have $r \geqslant diam\left(I_\lambda^g(\Gamma_x)\right)$. $\square$

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