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REMARKS ON VĂLEAN'S MASTER THEOREM OF SERIES

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Abstract. We generalize Vălean's Master Theorem of Series proved in [A master theorem of series and evaluation of a cubic harmonic series, J. Classical Analysis, 2(10), 2017, 97-107].

1. Introduction and main results

In [1] Vălean proved a theorem on series, calling it "a Master Theorem of Series", which enabled him to evaluate many interesting series involving harmonic numbers. More precisely, he proved the following theorem.

THEOREM 1. *If k is a positive integer with $M(k) = m(1)+m(2)+\cdots+m(k)$, and $m(k)$ are real numbers, where $\lim_{k\to\infty} m(k) = 0$, then the following identity holds*

$$\sum_{k=1}^{\infty} \frac{M(k)}{(k+1)(k+n+1)} = m(1)\left(\frac{H_n}{n} - \frac{1}{n+1}\right) + \frac{1}{n}\sum_{j=1}^{n}\sum_{k=1}^{\infty}\frac{m(k+1)}{j+k+1}$$
$$= \frac{1}{n}\sum_{j=1}^{n}\sum_{k=1}^{\infty}\frac{m(k)}{j+k}.$$

Using this theorem Vălean provided elementary proofs of the following interesting known harmonic sums:

$$\sum_{n=1}^{\infty}\frac{H_n^2}{n^2} = \frac{17}{4}\zeta(4), \quad \sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^4} = \zeta^2(3) - \frac{\zeta(6)}{3}$$

$$\sum_{n=1}^{\infty}\frac{H_n^2}{n^4} = \frac{97}{24}\zeta(6) - 2\zeta^2(3), \quad \text{and} \quad \sum_{n=1}^{\infty}\left(\frac{H_n}{n}\right)^3 = \frac{93}{16}\zeta(6) - \frac{5}{2}\zeta^2(3),$$

where $H_n^{(m)}$ are generalized harmonic numbers defined by

$$H_n^{(1)} = H_n \quad \text{and} \quad H_n^{(m)} = \sum_{k=1}^{n}\frac{1}{k^m},$$

H_n is the ussual n th harmonic number, and ζ is the Riemann zeta function defined by

$$\zeta(m) = \sum_{k=1}^{\infty}\frac{1}{k^m}, \quad m = 2,3\dots.$$

Our aim in this short note is to generalize Vălean's Master Theorem of Series. We need the following Lemma for the proof of our main result.

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LEMMA 1. *Let $(a_k)_{k\geqslant 1}$ and $(b_k)_{k\geqslant 1}$ be any two sequences of real or complex numbers. Then we have*

$$\sum_{k=1}^{n} a_k \sum_{j=1}^{k} b_j = \sum_{p=0}^{n-1} \sum_{k=1}^{n-p} b_k a_{p+k}. \tag{1}$$

Proof. We prove the lemma by induction on n. Clearly (1) is true for $n=1$. We assume that (1) is true for n and prove that it is also true for $n+1$. We have

$$\begin{aligned}\sum_{k=1}^{n+1} a_k \sum_{j=1}^{k} b_j &= a_{n+1} \sum_{j=1}^{n+1} b_j + \sum_{k=1}^{n} a_k \sum_{j=1}^{k} b_j \\ &= a_{n+1} \sum_{j=1}^{n+1} b_j + \sum_{k=1}^{n} a_k b_k + \sum_{p=1}^{n-1} \sum_{j=1}^{n-p} b_j a_{p+j} \\ &= \sum_{j=1}^{n+1} a_j b_j + a_{n+1} \sum_{j=1}^{n} b_j + \sum_{p=1}^{n-1} \sum_{j=1}^{n-p} b_j a_{p+j}.\end{aligned}$$

Since

$$\begin{aligned}\sum_{p=1}^{n} \sum_{k=1}^{n+1-p} b_k a_{p+k} &= \sum_{p=1}^{n} \left(b_{n+1-p} a_{n+1} + \sum_{k=1}^{n-p} b_k a_{p+k} \right) \\ &= a_{n+1} \sum_{p=1}^{n} b_{n+1-p} + \sum_{p=1}^{n} \sum_{k=1}^{n-p} b_k a_{p+k} \\ &= a_{n+1} \sum_{k=1}^{n} b_k + \sum_{p=1}^{n} \sum_{k=1}^{n-p} b_k a_{p+k},\end{aligned}$$

we have

$$\sum_{k=1}^{n+1} a_k \sum_{j=1}^{k} b_j = \sum_{j=1}^{n+1} a_j b_j + \sum_{p=1}^{n} \sum_{k=1}^{n+1-p} b_k a_{p+k} = \sum_{p=0}^{n} \sum_{k=1}^{n+1-p} b_k a_{p+k}.$$

This shows that (1) is valid for $n+1$, which completes the proof of the lemma. □

If we let $n \to \infty$ in (1), we get the following theorem.

THEOREM 2. *Let $(a_n)_{n\geqslant 1}$ and $(b_n)_{n\geqslant 1}$ be any two sequences of real or complex numbers. Then we have*

$$\sum_{k=1}^{\infty} a_k \sum_{j=1}^{k} b_j = \sum_{k=1}^{\infty} a_k b_k + \sum_{p=1}^{\infty} \sum_{k=1}^{\infty} b_k a_{p+k} = \sum_{p=0}^{\infty} \sum_{k=1}^{\infty} b_k a_{p+k}, \tag{2}$$

provided that the series involved are convergent.

If we let

$$a_k = \frac{1}{(k+1)(k+n+1)}, \quad b_k = m(k) \quad \text{and} \quad M(k) = \sum_{j=1}^{k} m(j)$$

in (2), we get

$$\begin{aligned}
&\sum_{k=1}^{\infty} \frac{M(k)}{(k+1)(n+k+1)} \\
&= \sum_{k=1}^{\infty} \frac{m(k)}{(k+1)(k+n+1)} + \sum_{p=1}^{\infty}\sum_{k=1}^{\infty} \frac{m(k)}{(p+k+1)(p+k+n+1)} \\
&= \sum_{k=1}^{\infty} \frac{m(k)}{n}\left[\frac{1}{k+1} - \frac{1}{k+n+1}\right] \\
&\quad + \sum_{k=1}^{\infty} \frac{m(k)}{n} \sum_{p=1}^{\infty}\left[\frac{1}{p+k+1} - \frac{1}{p+k+n+1}\right] \\
&= \sum_{k=1}^{\infty} \frac{m(k)}{n}\left[\frac{1}{k+1} - \frac{1}{k+n+1}\right] \\
&\quad - \sum_{k=1}^{\infty} \frac{m(k)}{n}\left[\sum_{p=1}^{\infty}\left(\frac{1}{p} - \frac{1}{p+k+1}\right) - \sum_{p=1}^{\infty}\left(\frac{1}{p} - \frac{1}{p+n+k+1}\right)\right].
\end{aligned} \tag{3}$$

Since

$$\sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{1}{n+k}\right) = H_n,$$

we conclude from (3) that

$$\begin{aligned}
&\sum_{k=1}^{\infty} \frac{M(k)}{(k+1)(n+k+1)} \\
&= \frac{1}{n}\sum_{k=1}^{\infty}\left[\frac{m(k)}{k+1} - \frac{m(k)}{k+n+1}\right] - \frac{1}{n}\sum_{k=1}^{\infty} m(k)(H_{k+1} - H_{n+k+1}).
\end{aligned} \tag{4}$$

We have

$$\sum_{k=1}^{\infty} m(k)(H_{k+1} - H_{n+k+1}) = -\sum_{k=1}^{\infty} m(k) \sum_{j=2}^{n+1} \frac{1}{k+j} = -\sum_{j=2}^{n+1}\sum_{k=1}^{\infty} \frac{m(k)}{k+j}. \tag{5}$$

Employing (5) in (4), we obtain

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{M(k)}{(k+1)(n+k+1)} &= \frac{1}{n}\sum_{k=1}^{\infty}\left[\frac{m(k)}{k+1} - \frac{m(k)}{k+n+1}\right] \\
&\quad + \frac{1}{n}\sum_{j=2}^{n+1}\sum_{k=1}^{\infty} \frac{m(k)}{j+k} = \frac{1}{n}\sum_{j=1}^{n}\sum_{k=1}^{\infty} \frac{m(k)}{j+k},
\end{aligned}$$

which proves Vălean's Master Theorem of Series.

If the series $\sum_{k=1}^{\infty} b_k$ is convergent, we have

$$\sum_{n=1}^{\infty} a_n \sum_{k=1}^{n} b_k = \sum_{n=1}^{\infty} a_n \left(\sum_{k=1}^{\infty} b_k - \sum_{k=n+1}^{\infty} b_k \right)$$
$$= \left(\sum_{n=1}^{\infty} a_n \right) \left(\sum_{n=1}^{\infty} b_n \right) - \sum_{k=1}^{\infty} a_k \sum_{k=1}^{\infty} b_{n+k}. \tag{6}$$

By Theorem 2 we have

$$\sum_{k=1}^{\infty} a_k \sum_{k=1}^{\infty} b_{n+k} = \sum_{n=1}^{\infty} b_n \sum_{k=1}^{n} a_k - \sum_{k=1}^{\infty} a_k b_k. \tag{7}$$

Thus we conclude from (6) and (7) the following corollary

COROLLARY 1. *Let $(a_n)_{n \geqslant 1}$ and $(b_n)_{n \geqslant 1}$ be any two sequences of real or complex numbers such that the series involved in the following expressions are convergent. Then we have*

$$\sum_{n=1}^{\infty} a_n \sum_{k=1}^{n} b_k + \sum_{n=1}^{\infty} b_n \sum_{k=1}^{n} a_k = \left(\sum_{n=1}^{\infty} a_n \right) \left(\sum_{n=1}^{\infty} b_n \right) + \sum_{k=1}^{\infty} a_k b_k.$$

If we take

$$a_n = \frac{1}{n^p} \quad \text{and} \quad b_n = \frac{1}{n^q}$$

in Corollary 1 we get for $p, q > 1$, both series converge when p and q are strictly greater than 1, that

$$\sum_{n=1}^{\infty} \frac{H_n^{(q)}}{n^p} + \sum_{n=1}^{\infty} \frac{H_n^{(p)}}{n^q} = \zeta(p+q) + \zeta(p)\zeta(q),$$

which is a known result.

REFERENCES

[1] C. I. VĂLEAN, *A master theorem of series and evaluation of a cubic harmonic series*, J. Classical Analysis **2** (10), 2017, 97–107.

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