Journal of Classical Analysis
Volume 13, Number 2 (2018), 123–131 doi:10.7153/jca-2018-13-08

ON k–ANALOGUES OF DIGAMMA AND POLYGAMMA FUNCTIONS

EMRAH YILDIRIM AND İNCI EGE

Abstract. In this work, we obtain some integral representations of k-analogue of classical digamma function $\psi(x)$. Then by using the concepts of neutrix and neutrix limit, we generalize the k-digamma function $\psi_k(x)$ and the k-polygamma function $\psi_k^{(r)}(x)$ for all real values of x, $r \in \mathbb{N}$ and $k > 0$. Also further results are given.

1. Introduction

The gamma function, which was introduced by Euler, is defined by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt$$

for $x > 0$ [1]. The logarithmic derivative of gamma function is known as digamma (or psi) function and it is given by

$$\psi(x) = \frac{d}{dx}\log\Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}$$

and its some integral representations are defined by

$$\psi(x) = -\gamma + \int_0^1 \frac{1-t^{x-1}}{1-t}\, dt = \int_0^\infty \frac{e^{-t}-(1+t)^{-x}}{t}\, dt = \int_0^\infty \left(\frac{e^{-t}}{t} - \frac{e^{-tx}}{1-e^{-t}}\right) dt \quad (1)$$

for positive real values of x, where γ denotes Euler-Mascheroni constant such that

$$\gamma = \lim_{n\to\infty}\left(1 + \ldots + \frac{1}{n} - \log n\right),$$

in [1].

Mathematics subject classification (2010): 33B15, 33B99.
Keywords and phrases: Digamma function, polygamma function, k-digamma function, k-polygamma function, neutrix.
This research is supported by Aydın Adnan Menderes University Scientific Research Projects Coordination Unit (BAP) with Project No: FEF-17011.

Paper JCA-13-08

1.1. Neutrix calculus

In [2], van der Corput introduced the concepts of neutrix and neutrix limit as below:

DEFINITION 1. (Neutrix) Let N' be a nonempty set and let $\mathscr{N}$ be a commutative, additive group of functions mapping from N' into a commutative, additive group N''. The group $\mathscr{N}$ is called neutrix if the function which is identically equal to zero is the only constant function occurring $\mathscr{N}$. The function which belongs to $\mathscr{N}$ is called "negligible function" in $\mathscr{N}$.

Let N' be a domain lying in a topological space with a limit point b not belonging to N' and $\mathscr{N}$ be a commutative additive group of functions defined on N' with the following property:

$$\text{“}f \in \mathscr{N},\ \lim_{\varepsilon \to b} f(\varepsilon) = c \text{ (constant) for } \varepsilon \in N' \text{ then } c = 0\text{”.}$$

Then this group $\mathscr{N}$ is a neutrix.

DEFINITION 2. (Neutrix limit) Let f be a real valued function defined on N' and suppose that it is possible to find a constant c such that $f(x) - c$ is negligible in $\mathscr{N}$. Then c is called the neutrix limit of $f(x)$ as x tends to y and denoted by

$$\mathrm{N}-\lim_{x \to y} f(x) = c. \tag{2}$$

In this study, we let $\mathscr{N}$ be the neutrix having domain $N' = (0,\infty)$, range $N'' = \mathbb{R}$ and neglible functions finite linear sums of the functions

$$\varepsilon^{\lambda} \ln^{r-1} \varepsilon,\ \ln^{r} \varepsilon \quad (\lambda < 0,\ r = 1,2,\ldots)$$

and all functions $o(\varepsilon)$ which converge to zero in the normal sense as ε tends to zero.

In [9], authors proved that the digamma function can be defined by

$$\psi(x) = -\gamma + \mathrm{N}-\lim_{\varepsilon \to 0} \int_{\varepsilon}^{1} \frac{1-t^{x-1}}{1-t} dt \tag{3}$$

for all real values of x. Also they obtained that

$$\psi(-n) = -\gamma + \sum_{i=1}^{n} \frac{1}{i}$$

for $n = 0,1,2,\ldots$.

In [16], Salem and Kılıçman showed the following properties:

LEMMA 1. *The neutrix limit as ε tends to zero of the integral*

$$\int_{\varepsilon}^{1} t^{\alpha-1} \ln^{r} t\, dt$$

exists for all real values of α *and* $r \in \mathbb{Z}^+$ *and*

$$\underset{\varepsilon\to 0}{\mathrm{N-lim}} \int_\varepsilon^1 t^{\alpha-1} \ln^r t\,dt = \begin{cases} \dfrac{(-1)^r r!}{\alpha^{r+1}}, & \alpha \neq 0, \\ 0, & \alpha = 0 \end{cases}$$

By using Lemma 1, author showed the existence of neutrix limit of the integral part of classical polygamma function for non-positive real values of x and $r \in \mathbb{Z}^+$. Then they gave the following definition.

DEFINITION 3. [16] The polygamma function $\psi^{(r)}(x)$ can be defined by

$$\psi^{(r)}(x) = -\left(\underset{\varepsilon\to 0}{\mathrm{N-lim}} \int_\varepsilon^1 \frac{t^{x-1} \ln^r t}{1-t} dt \right)$$

for all $x \in \mathbb{R}$ and $r \in \mathbb{Z}^+$.

We refer the reader to [4, 5, 6, 7, 14, 17, 18, 19] and references therein for more applications of neutrix and neutrix limit to the special functions.

1.2. k-Analogue of classical gamma function and its related function

Díaz and Pariguan define k-analogue of gamma function by the following Riemann integral

$$\Gamma_k(x) = \int_0^\infty t^{x-1} e^{-\frac{t^k}{k}} dt \tag{4}$$

for $x \in \mathbb{C}$, $Re(x) > 0$ in [3]. Also, they obtain several results which are generalizations of the classical gamma function. Some of them are given by the followings:

$$\Gamma_k(x+k) = x\Gamma_k(x), \tag{5}$$

$$\Gamma_k(x) = k^{\frac{x}{k}-1} \Gamma\left(\frac{x}{k}\right), \tag{6}$$

$$\frac{1}{\Gamma_k(x)} = x k^{-\frac{x}{k}} e^{\frac{x}{k}\gamma} \prod_{n=1}^\infty \left(\left(1+\frac{x}{nk}\right) e^{-\frac{x}{nk}} \right). \tag{7}$$

By using the equation (7), Krasniqi in [10] obtain the following series representation of k-digamma function

$$\Psi_k(x) = \frac{\ln k - \gamma}{k} - \frac{1}{x} + \sum_{i=1}^\infty \frac{x}{ik(x+ik)}, \tag{8}$$

for $x > 0$. The reader can find more properties on k-gamma and its related functions in [10, 11, 12, 13, 15, 20].

2. Some integral representations of k-digamma function

In this section we seek to redefine k-digamma function as integrals for positive real values of x. At first, we need the following lemma.

LEMMA 2. [8] *Assume $a,\ b > 0$. Then*

$$\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x} dx = \ln \frac{b}{a}.$$

THEOREM 1. *For $x > 0$, a k-analogue of digamma function can also be expressed by*

$$\psi_k(x) = \frac{1}{k} \int_0^\infty \frac{1}{t} \left(e^{-t} - (1+tk)^{-\frac{x}{k}} \right) dt. \tag{9}$$

Proof. By differentiating the equality (4) with respect to x, we have

$$\Gamma_k'(x) = \int_0^\infty u^{x-1} e^{-\frac{u^k}{k}} \ln u du = \frac{1}{k} \int_0^\infty u^{x-1} e^{-\frac{u^k}{k}} \ln u^k du,$$

and then by using Lemma 2, we get

$$\begin{aligned} \Gamma_k'(x) &= \frac{1}{k} \int_0^\infty u^{x-1} e^{-\frac{u^k}{k}} \int_0^\infty \frac{e^{-t} - e^{-u^k t}}{t} dt\, du \\ &= \frac{1}{k} \int_0^\infty \frac{1}{t} \left(e^{-t} \int_0^\infty u^{x-1} e^{-\frac{u^k}{k}} du - \int_0^\infty u^{x-1} e^{-u^k(\frac{1}{k}+t)} du \right) dt. \end{aligned}$$

The first integral of the integrand on the right side of the last equation is equal to $\Gamma_k(x)$. For the second one, let us change variable as $u^k(\frac{1}{k}+t) = \frac{z^k}{k}$. Then we obtain

$$\begin{aligned} \int_0^\infty u^{x-1} e^{-u^k(\frac{1}{k}+t)} du &= \int_0^\infty \frac{z^{x-1}}{(1+tk)^{\frac{x-1}{k}}} e^{-\frac{z^k}{k}} \frac{dz}{(1+tk)^{\frac{1}{k}}} \\ &= \frac{1}{(1+tk)^{\frac{x}{k}}} \int_0^\infty z^{x-1} e^{-\frac{z^k}{k}} dz = \frac{\Gamma_k(x)}{(1+tk)^{\frac{x}{k}}}. \end{aligned}$$

Hence we get

$$\Gamma_k'(x) = \frac{\Gamma_k(x)}{k} \int_0^\infty \frac{1}{t} \left(e^{-t} - (1+tk)^{-\frac{x}{k}} \right) dt$$

which gives the equation (9). □

By using previous theorem, we get the following results.

COROLLARY 1. *k-Digamma function can also be given by*

$$\psi_k(x) = \frac{1}{k}\int_0^1 \frac{e^{\frac{1}{k}\left(1-\frac{1}{t}\right)} - t^{\frac{x}{k}}}{t(1-t)}dt \tag{10}$$

for $x > 0$.

REMARK 1. The equation (9) shows that there is a relation between k-digamma and classical digamma functions as

$$\psi_k(x) = \frac{\ln k}{k} + \frac{1}{k}\psi\left(\frac{x}{k}\right) \tag{11}$$

for $x > 0$.

We want to note that the equation (11) can be obtained by taking logarithmic derivative of the equation (6).

COROLLARY 2. *The k-digamma function can also be defined by*

$$\psi_k(x) = \frac{\ln k}{k} + \frac{1}{k}\int_0^\infty \left(\frac{e^{-t}}{t} - \frac{e^{-\frac{x}{k}t}}{1-e^{-t}}\right)dt \tag{12}$$

for $x > 0$.

In this section, we already obtain some integral representations of the k-digamma function by using some properties of integrals and the definition of k-gamma function. In the next theorem, we obtain another representation of it by using the series expression (8).

THEOREM 2. *The k-digamma function is defined by*

$$\psi_k(x) = \frac{\ln k - \gamma}{k} + \int_0^1 \frac{t^{k-1} - t^{x-1}}{1-t^k}dt \tag{13}$$

for $x > 0$.

Proof. By using the equation (8), we get

$$\begin{aligned}
\psi_k(x) &= \frac{\ln k - \gamma}{k} - \frac{1}{x} + \sum_{i=1}^{\infty}\left(\frac{1}{ik} - \frac{1}{x+ik}\right) = \frac{\ln k - \gamma}{k} + \sum_{i=1}^{\infty}\frac{1}{ik} - \sum_{i=0}^{\infty}\frac{1}{x+ik} \\
&= \frac{\ln k - \gamma}{k} + \sum_{i=1}^{\infty}\int_0^1 t^{ik-1}dt - \sum_{i=0}^{\infty}\int_0^1 t^{x+ik-1}dt \\
&= \frac{\ln k - \gamma}{k} + \int_0^1 \sum_{i=1}^{\infty} t^{ik-1}dt - \int_0^1 \sum_{i=0}^{\infty} t^{x+ik-1}dt \\
&= \frac{\ln k - \gamma}{k} + \int_0^1 t^{-1}\sum_{i=1}^{\infty}\left(t^k\right)^i dt - \int_0^1 t^{x-1}\sum_{i=0}^{\infty}\left(t^k\right)^i dt.
\end{aligned}$$

Since $t \in (0,1)$ and $k > 0$, both geometric series are convergent. Taking into account that the first series starts with t^k and the second with 1, we obtain

$$\psi_k(x) = \frac{\ln k - \gamma}{k} + \int_0^1 t^{-1}\frac{t^k}{1-t^k}dt - \int_0^1 t^{x-1}\frac{1}{1-t^k}dt = \frac{\ln k - \gamma}{k} + \int_0^1 \frac{t^{k-1}-t^{x-1}}{1-t^k}dt$$

as desired. □

Using logarithmic derivative of the equation (5) leads us

$$\psi_k(x+k) = \frac{1}{x} + \psi_k(x) \tag{14}$$

for $x > 0$. We want to note that the formula can also be obtained by using its integral representation (13).

3. Expanding the domain of the k-digamma function

The equation (14) helps us to define k-digamma function as

$$\psi_k(x) = \psi_k(x+k) - \frac{1}{x}$$

for $-k < x < 0$. So by mathematical induction we see that k-digamma function can be defined as

$$\psi_k(x) = \psi_k(x+nk) - \sum_{j=1}^{n}\frac{1}{x+jk-k}$$

for $-nk < x < (-n+1)k$, $k > 0$ and $n \in \mathbb{Z}^+$.

Now, we will give a definition for $\psi_k(x)$ on the real line. For this, we need the following theorem.

THEOREM 3. *The neutrix limit as ε tends to zero of the integral*

$$\int_\varepsilon^1 \frac{t^{k-1}-t^{x-1}}{1-t^k}dt$$

exists for all real values of x and $k > 0$.

Proof. We have

$$\int_\varepsilon^1 \frac{t^{k-1}-t^{-1}}{1-t^k}dt = -\int_\varepsilon^1 \frac{t^{-1}\left(1-t^k\right)}{1-t^k}dt = -\int_\varepsilon^1 \frac{dt}{t} = \ln\varepsilon.$$

Since $\ln\varepsilon$ is negligible function, the proof is completed for $x = 0$.

Now let us take $-nk < x < (-n+1)k$. Then we have

$$\begin{aligned}\int_\varepsilon^1 \frac{t^{k-1}-t^{x-1}}{1-t^k}dt &= \int_\varepsilon^1 \frac{t^{k-1}-t^{x+nk-1}}{1-t^k}dt - \int_\varepsilon^1 t^{x-1}\frac{1-\left(t^k\right)^n}{1-t^k}dt \\ &= \int_\varepsilon^1 \frac{t^{k-1}-t^{x+nk-1}}{1-t^k}dt - \int_\varepsilon^1 t^{x-1}\sum_{j=0}^{n-1} t^{jk}dt \\ &= \int_\varepsilon^1 \frac{t^{k-1}-t^{x+nk-1}}{1-t^k}dt - \sum_{j=0}^{n-1}\left(\frac{1}{x+jk}-\frac{\varepsilon^{x+jk}}{x+jk}\right).\end{aligned}$$

Taking neutrix limit on both sides leads us that

$$\mathrm{N}-\lim_{\varepsilon\to 0}\int_\varepsilon^1 \frac{t^{k-1}-t^{x-1}}{1-t^k}dt = \mathrm{N}-\lim_{\varepsilon\to 0}\int_\varepsilon^1 \frac{t^{k-1}-t^{x+nk-1}}{1-t^k}dt - \mathrm{N}-\lim_{\varepsilon\to 0}\sum_{j=0}^{n-1}\left(\frac{1}{x+jk}-\frac{\varepsilon^{x+jk}}{x+jk}\right).$$

Since the neutrix limit of the integral on the right side of the equation collides with normal one and the last term on the right side is negligible function, we get desired result.

For $x=-nk$ and $n\in\mathbb{Z}^+$, we have

$$\begin{aligned}\int_\varepsilon^1 \frac{t^{k-1}-t^{-nk-1}}{1-t^k}dt &= -\int_\varepsilon^1 \frac{t^{-nk-1}\left(\left(t^k\right)^{n+1}-1\right)}{t^k-1}dt \\ &= -\int_\varepsilon^1 t^{-nk-1}\sum_{j=0}^{n} t^{jk}dt = -\sum_{j=0}^{n-1}\int_\varepsilon^1 t^{(-n+j)k-1}dt - \int_\varepsilon^1 \frac{dt}{t} \\ &= -\sum_{j=0}^{n-1}\left[\frac{1}{(-n+j)k}-\frac{\varepsilon^{(-n+j)k}}{(-n+j)k}\right] + \ln\varepsilon.\end{aligned}$$

If we take the neutrix limit on both sides of the equation, we obtain

$$\begin{aligned}\mathrm{N}-\lim_{\varepsilon\to 0}\int_\varepsilon^1 \frac{t^{k-1}-t^{-nk-1}}{1-t^k}dt &= -\sum_{j=0}^{n-1}\frac{1}{(-n+j)k} \\ &= -\left(-\frac{1}{nk}-\frac{1}{(n-1)k}-\dots-\frac{1}{k}\right) = \sum_{j=1}^{n}\frac{1}{jk}\end{aligned}$$

as desired. □

By Theorem 3 and the fact that for $x>0$, the neutrix limit of the integral collides with ordinary one as $\varepsilon\to 0$, we can give the following definition:

DEFINITION 4. The k-digamma function is defined by

$$\psi_k(x) = \frac{\ln k-\gamma}{k} + \mathrm{N}-\lim_{\varepsilon\to 0}\int_\varepsilon^1 \frac{t^{k-1}-t^{x-1}}{1-t^k}dt \tag{15}$$

for all real values x and $k>0$.

EXAMPLE 1. Let $x=-nk$ $(n\in\mathbb{Z}^+, k>0)$ in the equation (15). Then

$$\psi_k(-nk)=\frac{\ln k-\gamma}{k}+\sum_{j=1}^{n}\frac{1}{jk}. \tag{16}$$

Note that all the results on k-digamma function in this paper tend to classical ones in [9] as k tends to 1.

The rth derivative of the equation (13) leads us to the following definition.

DEFINITION 5. The k-polygamma function is defined by

$$\psi_k^{(r)}(x)=-\int_0^1\frac{t^{x-1}\ln^r t}{1-t^k}dt \tag{17}$$

for $x,k>0$ and $r\in\mathbb{Z}^+$.

By using similar method on k-digamma function, we can define the k-polygamma function via neutrix limit as below.

DEFINITION 6. The k-polygamma function can be defined by

$$\psi_k^{(r)}(x)=-\operatorname{N-}\lim_{\varepsilon\to 0}\int_\varepsilon^1\frac{t^{x-1}\ln^r t}{1-t^k}dt \tag{18}$$

for all real values of x and $r\in\mathbb{Z}^+$.

EXAMPLE 2. The k-polygamma function has values at zero and $-nk$ for $n=1,2,\dots$ and $k>0$ as the following:

$$\psi_k^{(r)}(0)=-\psi_k^{(r)}(k),$$
$$\psi_k^{(r)}(-nk)=\psi_k^{(r)}(k)+\sum_{j=1}^{n}\frac{r!}{(jk)^{r+1}}.$$

All this results on k-polygamma function tends to classical ones in [16] as $k\to 1$.

Acknowledgement. The author thanks to referee to their valuable suggestions which led to the improvement of this paper.

REFERENCES

[1] M. ABRAMOWITZ AND I. A. STEGUN (Eds), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, National Bureau of Standards, Applied Mathematics Series **55**, Washington, 1970.

[2] J. G. VAN DER CORPUT, *Introduction to the neutrix calculus*, Journal d'Analyse Mathématique **7**, 1 (1959), 281–398.

[3] R. DIAZ AND E. PARIGUAN, *On hypergeometric functions and Pochhammer k-symbol*, Divulgaciones Matemáticas **15**, 2 (2007), 179–192.

[4] İ. EGE AND E. YILDIRIM, *Some generalized equalities for the q-gamma function*, Filomat **26**, 6 (2012), 1227–1232.

[5] B. FISHER AND Y. KURIBAYASHI, *Neutrices and the gamma function*, J. Fac. Ed. Tottori Univ. Math. Sci. **36**, 1-2 (1987), 1–7.

[6] B. FISHER AND Y. KURIBAYASHI, *Some results on the gamma function*, The Journal of Faculty of Education Tottori University **37**, 2 (1988).

[7] B. FISHER AND A. KILIÇMAN, *Some results on the gamma function for negative integers*, Appl. Math. Inform. Sci. **6**, 2 (2012), 173–176.

[8] I. S. GRADSHTEYN AND I. M. RYZHIK, *Table of integrals, series, and products*, Academic press, 2014.

[9] B. JOLEVSKA-TUNESKA AND I. JOLEVSKI, *Some results on the digamma function*, Appl. Math. Inform. Sci. **7**, 1 (2013), 167–170.

[10] V. KRASNIQI, *Inequalities and monotonicity for the ration of k-gamma function*, Scientia Magna **6**, 1 (2010), 40–45.

[11] C. G. KOKOLOGIANNAKI, *Properties and inequalities of generalized k-gamma, beta and zeta functions*, Int. J. Contemp. Math. Sciences **5**, 14 (2010), 653–660.

[12] C. G. KOKOLOGIANNAKI AND V. KRASNIQI, *Some properties of the k-gamma function*, Le Matematiche **68**, 1 (2013), 13–22.

[13] M. MANSOUR, *Determining the k-generalized gamma function $\Gamma_k(x)$ by functional equations*, Int. J. Contemp. Math. Sciences **4**, 21 (2009), 1037–1042.

[14] E. ÖZÇAĞ, *Results on compositions involving Dirac-delta function*, In AIP Conference Proceedings **1895**, 1 (2017), 050007.

[15] A. REHMAN, S. MUBEEN, N. SADIQ AND F. SHAHEEN, *Some inequalities involving k-gamma and k-beta functions with applications*, Journal of Inequalities and Applications **2014**, 1 (2014), 224.

[16] A. SALEM AND A. KILIÇMAN, *Estimating the polygamma functions for negative integers*, Journal of Inequalities and applications **2013**, 1 (2013), 523.

[17] A. SALEM, *The neutrix limit of the q-Gamma function and its derivatives*, Applied Mathematics Letters **23**, 10 (2010), 1262–1268.

[18] A. SALEM, *Generalized the q-Digamma and the q-Polygamma Functions via Neutrices*, Filomat **31**, 5 (2017), 1475–1481.

[19] Z. SUN, A. LI AND H. QIN, *The Neutrix Limit of the Hurwitz Zeta Function and Its Application*, IAENG Int. J. Appl. Math. **47**, 1 (2017), 56–65.

[20] J. ZHANG AND H. N. SHI, *Two double inequalities for k-gamma and k-Riemann zeta functions*, Journal of Inequalities and Applications **2014**, 1 (2014), 191.

(Received July 9, 2018)

Emrah Yıldırım
Aydın Adnan Menderes University
Department of Mathematics
09100, Aydın, Turkey
e-mail: emrahyildirim@adu.edu.tr

İnci Ege
Aydın Adnan Menderes University
Department of Mathematics
09100, Aydın, Turkey
e-mail: iege@adu.edu.tr