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FURTHER INVESTIGATIONS ON SOME RESULTS OF YU

ABHIJIT BANERJEE AND MOLLA BASIR AHAMED

Abstract. With the help of weighted sharing of values, we investigate the uniqueness of rational function of a meromorphic functions sharing a small function with its differential polynomial. Our results will extend and improve a number of result in the direction of Yu [18]. Specifically we stress on the improvement of two recent results of Charak and Lal [8] and Li, Yang and Liu [14]. We have exhibited several examples to justify our certain claims.

1. Introduction, definitions and results

Let f and g be two non-constant meromorphic functions defined in the open complex plane $\mathbb{C}$. If for some $a \in \mathbb{C} \cup \{\infty\}$, $f-a$ and $g-a$ have the same set of zeros with the same multiplicities, we say that f and g share the value a CM (counting multiplicities), and if we do not consider the multiplicities then f and g are said to share the value a IM (ignoring multiplicities).

Throughout the paper the standard notations from Nevanlinna's theory of value distribution of meromorphic functions is used, as in [10]. We recall that $T(r,f)$ denotes the Nevanlinna characteristic function of the non-constant meromorphic function and $N(r,a;f)$ ($\overline{N}(r,a;f)$) denotes the counting function (reduced counting function) of a-points of meromorphic functions f.

A meromorphic function a is said to be a small function of f provided that $T(r,a) = S(r,f)$, that is $T(r,a) = o(T(r,f))$ as $r \to \infty$, outside of a possible exceptional set of finite linear measure.

We use I to denote any set of infinite linear measure of $0 < r < \infty$.

We also recall that if $a \in \mathbb{C} \cup \{\infty\}$, the quantity

$$\delta(a;f) = 1 - \limsup_{r\to\infty} \frac{N(r,a;f)}{T(r,f)}$$

is called Nevanlinna deficiency of the value a and by ramification index we mean

$$\Theta(a;f) = 1 - \limsup_{r\to\infty} \frac{\overline{N}(r,a;f)}{T(r,f)}.$$

Throughout this paper, we use the symbol, $\chi_m = \begin{cases} 0, \text{ if } m = 0, \\ 1, \text{ if } m \geqslant 1. \end{cases}$ We start the discussion on the following result of R. Brück [6] who first considered the uniqueness problem of an entire function sharing one value with its derivative.

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THEOREM A. [6] *Let f be a non-constant entire function. If f and f' share the value 1 CM and if $N(r,0;f') = S(r,f)$ then $\dfrac{f'-1}{f-1}$ is a nonzero constant.*

In fact, Brück obtained the above result to justify his famous conjecture, corresponding to the uniqueness for one *CM* shared value of entire function with its first derivative [6]:

CONJECTURE 1. Let f be a non-constant entire function such that the hyper order $\rho_2(f)$ of f is not a positive integer or infinite. If f and f' share a finite value a CM, then $\dfrac{f'-a}{f-a} = c$, where c is a non zero constant.

Later many authors like Zhang [19], Yang [16], Gundersen and Yang [9] et al. ponder over the different aspect of the conjecture and obtained different results. Next we recall the following definition known as weighted sharing of values which has a remarkable influence on the subsequent results of Brück conjecture.

DEFINITION 1. [11, 12] Let k be a nonnegative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a;f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m \leqslant k$ and $k+1$ times if $m > k$. If $E_k(a;f) = E_k(a;g)$, we say that f,g share the value a with weight k.

The definition implies that if f, g share a value a with weight k then z_0 is an a-point of f with multiplicity $m\ (\leqslant k)$ if and only if it is an a-point of g with multiplicity $m\ (\leqslant k)$ and z_0 is an a-point of f with multiplicity $m\ (> k)$ if and only if it is an a-point of g with multiplicity $n\ (> k)$, where m is not necessarily equal to n.

We write f, g share (a,k) to mean that f, g share the value a with weight k. Clearly if f, g share (a,k), then f, g share (a,p) for any integer p, $0 \leqslant p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a,0)$ or (a,∞) respectively.

If a is a small function we define that f and g share a IM or a CM or with weight l according as $f-a$ and $g-a$ share $(0,0)$ or $(0,\infty)$ or $(0,l)$ respectively.

Though we use the standard notations and definitions of the value distribution theory available in [10], we explain some definitions and notations which are used in the paper.

DEFINITION 2. [13] Let p be a positive integer and $a \in \mathbb{C} \cup \{\infty\}$.

(i) $N(r,a;f \mid\geqslant p)$ ($\overline{N}(r,a;f \mid\geqslant p)$)denotes the counting function (reduced counting function) of those a-points of f whose multiplicities are not less than p.

(ii) $N(r,a;f \mid\leqslant p)$ ($\overline{N}(r,a;f \mid\leqslant p)$)denotes the counting function (reduced counting function) of those a-points of f whose multiplicities are not greater than p.

DEFINITION 3. [17] For $a \in \mathbb{C} \cup \{\infty\}$ and a positive integer p we denote by $N_p(r,a;f)$ the sum $\overline{N}(r,a;f) + \overline{N}(r,a;f \mid\geqslant 2) + \ldots + \overline{N}(r,a;f \mid\geqslant p)$. Clearly $N_1(r,a;f) = \overline{N}(r,a;f)$.

DEFINITION 4. [20] For a positive integer p and $a \in \mathbb{C} \cup \{\infty\}$ we put

$$\delta_p(a;f) = 1 - \limsup_{r \longrightarrow \infty} \frac{N_p(r,a;f)}{T(r,f)}$$

Clearly $0 \leqslant \delta(a;f) \leqslant \delta_p(a;f) \leqslant \delta_{p-1}(a;f) \leqslant \ldots \leqslant \delta_2(a;f) \leqslant \delta_1(a;f) = \Theta(a;f)$

DEFINITION 5. [1] Let f and g be two non-constant meromorphic functions such that f and g share the value a *IM*. Let z_0 be a a-point of f with multiplicity p, a a-point of g with multiplicity q. We denote by $\overline{N}_L(r,a;f)$ the counting function of those a-points of f and g where $p > q$, by $N_E^{1)}(r,a;f)$ the counting function of those a-points of f and g where $p = q = 1$ and by $\overline{N}_E^{(2}(r,a;f)$ the counting function of those a-points of f and g where $p = q \geqslant 2$, each point in these counting functions is counted only once. In the same way we can define $\overline{N}_L(r,a;g)$, $N_E^{1)}(r,a;g)$, $\overline{N}_E^{(2}(r,a;g)$.

DEFINITION 6. [11, 12] Let f, g share a value a IM. We denote by $\overline{N}_*(r,a;f,g)$ the reduced counting function of those a-points of f whose multiplicities differ from the multiplicities of the corresponding a-points of g.

Clearly $\overline{N}_*(r,a;f,g) \equiv \overline{N}_*(r,a;g,f)$ and $\overline{N}_*(r,a;f,g) = \overline{N}_L(r,a;f) + \overline{N}_L(r,a;g)$.

In 2003, Yu [18] tried to solve the conjecture in a different way than that was done earlier. He considered the uniqueness problem of an entire or meromorphic functions with its derivative sharing a small function a and obtained the following two results.

THEOREM B. [18] *Let f be a non-constant entire function, $a \in S(f)$ and $a \not\equiv 0, \infty$. If $f - a$ and $f^{(k)} - a$ share 0 CM and $\delta(0;f) > \frac{3}{4}$ then $f \equiv f^{(k)}$.*

THEOREM C. [18] *Let f be a non-constant non-entire meromorphic function, $a \in S(f)$ and $a \not\equiv 0$, ∞. If*

i) f and a have no common poles.

ii) $f - a$ and $f^{(k)} - a$ share the value 0 CM.

iii) $4\delta(0;f) + 2(8+k)\Theta(\infty;f) > 19 + 2k$

then $f \equiv f^{(k)}$, where k is a positive integer.

Recently in connection with the Yu's [18] result, Zhang and Lü [21] considered the uniqueness of the n-th power of a meromorphic function sharing a small function with its k-th derivative and proved the following theorem.

THEOREM D. [21] *Let $k(\geqslant 1)$, $n(\geqslant 1)$ be integers and f be a non-constant meromorphic function. Also let $a(z)(\not\equiv 0, \infty)$ be a small function with respect to f. Suppose $f^n - a$ and $f^{(k)} - a$ share $(0,s)$. If $s = \infty$ and*

$$(3+k)\,\Theta(\infty;f) + 2\,\Theta(0;f) + \delta_{2+k}(0;f) > 6 + k - n \tag{1.1}$$

or $s=0$ *and*

$$(6+2k)\,\Theta(\infty;f)+4\,\Theta(0;f)+2\delta_{2+k}(0;f)>12+2k-n \tag{1.2}$$

then $f^n\equiv f^{(k)}$

Next we give the following definition.

DEFINITION 7. [2, 3] Let $n_{0j},n_{1j},\ldots,n_{kj}$ be non negative integers.

The expression $M_j[f]=(f)^{n_{0j}}(f^{(1)})^{n_{1j}}\ldots(f^{(k)})^{n_{kj}}$ is called a differential monomial generated by f of degree $d(M_j)=\sum\limits_{i=0}^{k}n_{ij}$ and weight $\Gamma_{M_j}=\sum\limits_{i=0}^{k}(i+1)n_{ij}$.

The sum $P[f]=\sum\limits_{j=1}^{t}b_jM_j[f]$ is called a differential polynomial generated by f of degree $\overline{d}(P)=\max\{d(M_j):1\leqslant j\leqslant t\}$ and weight $\Gamma_P=\max\{\Gamma_{M_j}:1\leqslant j\leqslant t\}$, where $T(r,b_j)=S(r,f)$ for $j=1,2,\ldots,t$.

The numbers $\underline{d}(P)=\min\{d(M_j):1\leqslant j\leqslant t\}$ and k(the highest order of the derivative of f in $P[f]$) are called respectively the lower degree and order of $P[f]$. $P[f]$ is said to be homogeneous if $\overline{d}(P)$=$\underline{d}(P)=d$ (say). $P[f]$ is called a Linear Differential Polynomial generated by f if $\overline{d}(P)=1$. Otherwise $P[f]$ is called Non-linear Differential Polynomial. We denote by $Q=\max\{\Gamma_{M_j}-d(M_j):1\leqslant j\leqslant t\}=\max\{n_{1j}+2n_{2j}+\ldots+kn_{kj}:1\leqslant j\leqslant t\}$.

DEFINITION 8. [4] For any two positive integers n and $r\leqslant 3$,

$$\mu_r=\min\{r,n\}\qquad\text{and}\qquad\mu_r^*=(r+1)-\mu_r.$$

In [5], Bhoosnurmath and Kabbur considered the uniqueness of a meromorphic function f and its differential polynomial $\mathscr{P}[f]$. Motivated by such an uniqueness result later Charak and Lal [7] asked the following natural question

Is it true that $p(f)\equiv\mathscr{P}[f]$, *when* $p(f)$ *with* $p(0)=0$ *and* $\mathscr{P}[f]$ *share* (a,s), *where* a *is a small meromorphic function of* f *?*

In [7], Charak and Lal investigated and answered the above question affirmatively as follows.

THEOREM E. *Let* f *be a non-constant meromorphic function,* $a(\not\equiv 0,\infty)$ *be a meromorphic small function and* $p(z)$ *be of degree* $n\geqslant 1$ *with* $p(0)=0$. *Let* $\mathscr{P}[f]$ *be a differential polynomial of* f. *Suppose* $p(f)-a$ *and* $\mathscr{P}[f]-a$ *share* $(0,s)$. *If* $s\geqslant 2$ *and*

$$\begin{aligned}&(Q+3)\Theta(\infty;f)+2n\Theta(0;p(f))+\overline{d}(\mathscr{P})\delta(0,f)\\&>Q+3+2\overline{d}(\mathscr{P})-\underline{d}(\mathscr{P})+n,\end{aligned} \tag{1.3}$$

or, if $s=1$ *and*

$$\begin{aligned}&\left(Q+\frac{7}{2}\right)\Theta(\infty;f)+\frac{5n}{2}\Theta(0;p(f))+\overline{d}(\mathscr{P})\delta(0,f)\\&>Q+\frac{7}{2}+2\overline{d}(\mathscr{P})-\underline{d}(\mathscr{P})+\frac{3n}{2},\end{aligned} \tag{1.4}$$

or, if $s=0$ *and*

$$\begin{aligned}&(2Q+6)\Theta(\infty;f)+4n\Theta(0;p(f))+2\overline{d}(\mathscr{P})\delta(0,f)\\ >\ &2Q+6+4\overline{d}(\mathscr{P})-2\underline{d}(\mathscr{P})+3n,\end{aligned} \tag{1.5}$$

then $p(f)\equiv\mathscr{P}[f]$.

Very recently, in this direction, for homogeneous differential polynomial, Li, Yang and Liu [14] obtained the following result.

THEOREM F. [14] *Let* f *be a non-constant meromorphic function and* $\mathscr{P}[f]$ *be a non-constant homogeneous differential polynomial of degree* d *and weight* Γ *satisfying* $\Gamma\geqslant(k+2)d-2$. *Let* $a(z)$ *be a small meromorphic function of* f *such that* $a(z)\not\equiv 0,\infty$. *Suppose that* $f-a$ *and* $\mathscr{P}[f]-a$ *share* $(0,s)$. *If* $s\geqslant 2$ *and*

$$3\Theta(\infty;f)+d\delta_{2+\Gamma-d}(0,f^d)+\delta_2(0,f)+\delta(a,f)>4 \tag{1.6}$$

or, $s=1$ *and*

$$\begin{aligned}&\frac{7+\Gamma-d}{2}\Theta(\infty;f)+\frac{d}{2}\delta_{1+\Gamma-d}(0;f^d)+d\delta_{2+\Gamma-d}(0;f^d)+\delta_2(0;f)+\delta(a;f)\\ >\ &\frac{\Gamma+9}{2}\end{aligned} \tag{1.7}$$

or, $s=0$ *and*

$$\begin{aligned}&[2(\Gamma-d)+6]\Theta(\infty;f)+d\delta_{1+\Gamma-d}(0,f^d)+d\delta_{2+\Gamma-d}(0;f^d)+\delta_2(0;f)\\ &+\Theta(0;f)+\delta(a;f)>2\Gamma+8\end{aligned} \tag{1.8}$$

then $\dfrac{\mathscr{P}[f]-a}{f-a}=C$, *where* C *is a non-zero constant.*

Specially when $s=0$ *i.e., when* f *and* $\mathscr{P}[f]$ *share* $(a,0)$, *then* $f\equiv\mathscr{P}[f]$.

From the above discussions, we observe that till date in the above theorems, the factor f^n can not be extended up to a n-th degree polynomial expression of a meromorphic function $P_n(f)$ without any restriction. Now since f^n or $P_n(f)$ with $P_n(0)=0$ are nothing but a part of a rational function $\mathscr{R}(f)$, it is quite natural to extend the above mentioned theorems up to a relation between a $\mathscr{R}(f)$ of a meromorphic function f and a general differential polynomial $\mathscr{P}[f]$ generated by f.

Henceforth we denote by $\mathscr{R}(f)$ as defined in Lemma 1 and so we mean $\lambda=\max\{m,n\}$, p_i $(1\leqslant i\leqslant u)$ and q_j $(1\leqslant j\leqslant l)$ are positive integers. Let $P_n(f)=a_n\prod_{i=1}^{u}(f-d_i)^{p_i}$, $1\leqslant u\leqslant n$ and $P_m(z)=b_m\prod_{j=1}^{l}(f-c_j)^{q_j}$, $1\leqslant l\leqslant m$ respectively, where u and l are two positive integers. Let $c_0\neq c_j\,(j=1,\dots,l)$ be a complex constant.

We now define for any positive integer $r\leqslant 3$,

$$\mu_r^i=\min\{p_i,r\}\quad\text{ and }\quad\mu_r^{*i}=(r+1)-\mu_r^i,\quad\forall\quad i=1,\dots,u.$$

Clearly, for $i=1$, this coincides with the Definition 8.

Let us define $l^{*}=\begin{cases}\chi_m, & \text{if m=0},\\ l\chi_m, & \text{if } m\geqslant 1.\end{cases}$

Now it is natural to consider the above mentioned problems in more general setting in a compact and convenient way. A question arises as follows:

QUESTION 1. Is it possible $\mathscr{R}(f)\equiv\mathscr{P}[f]$, when $\mathscr{R}(f)-a$ and $\mathscr{P}[f]-a$ share $(0,s)$, where a is a small function of a meromorphic function f ?

To find out the possible answer of the above question is the main motivation of writing this paper. The following is the main result of this paper.

THEOREM 1. (Main) *Let f be a non-constant meromorphic function and $m(\geqslant 0)$, $n(\geqslant 1)$ are integers and $a\equiv a(z)(\not\equiv 0,\infty)$ be a meromorphic small function of f. Let $\mathscr{P}[f]$ be a differential polynomial containing at least one derivative. Suppose that $\mathscr{R}(f)-a$ and $\mathscr{P}[f]-a$ share $(0,s)$. If $2\leqslant s<\infty$ and*

$$(Q+3)\Theta(\infty;f)+\overline{d}(\mathscr{P})\delta(0;f)+\sum_{j=0}^{l^*}\chi_j\left\{\Theta_{(2}(c_j;f)+\Theta(c_j;f)\right\}+\sum_{i=1}^{u}\mu_2^i\delta_{\mu_2^{*i}}(d_i;f)$$
$$>Q+3+2\overline{d}(\mathscr{P})-\underline{d}(\mathscr{P})+2l^*+\sum_{i=1}^{u}\mu_2^i-\lambda, \tag{1.9}$$

or, if $s=1$ and

$$\left(Q+\frac{7}{2}\right)\Theta(\infty;f)+\overline{d}(\mathscr{P})\delta(0;f)+\sum_{j=0}^{l^*}\chi_j\left\{\Theta_{(2}(c_j;f)+\frac{3}{2}\Theta(c_j;f)\right\}+\sum_{i=1}^{u}\mu_2^i\delta_{\mu_2^{*i}}(d_i;f)$$
$$+\frac{1}{2}\sum_{i=1}^{u}\Theta(d_i;f)>Q+\frac{7}{2}+2\overline{d}(\mathscr{P})-\underline{d}(\mathscr{P})+\frac{5l^*}{2}+\sum_{i=1}^{u}\mu_2^i+\frac{1}{2}u-\lambda, \tag{1.10}$$

or, if $s=0$ and

$$(2Q+6)\Theta(\infty;f)+2\overline{d}(\mathscr{P})\delta(0;f)+\sum_{j=0}^{l^*}\chi_j\left\{\Theta_{(2}(c_j;f)+3\Theta(c_j;f)\right\}+\sum_{i=1}^{u}\mu_2^i\delta_{\mu_2^{*i}}(d_i;f)$$
$$+2\sum_{i=1}^{u}\Theta(d_i;f)>2Q+6+4\overline{d}(\mathscr{P})-2\underline{d}(\mathscr{P})+4l^*+\sum_{i=1}^{u}\mu_2^i+2u-\lambda, \tag{1.11}$$

then $\mathscr{R}(f)\equiv\mathscr{P}[f]$.

Now putting $m=0$ in Theorem 1, we can immediately deduce the following corollary which improve the result of Charak and Lal [8].

COROLLARY 1. *Let f be a non-constant meromorphic function and $n(\geqslant 1)$ be an integer and $a\equiv a(z)(\not\equiv 0,\infty)$ be a meromorphic small function of f. Let $\mathscr{P}[f]$ be* *a differential polynomial containing atleast one derivative. Suppose that* $p(f)-a$ *and* $\mathscr{P}[f]-a$ *share* $(0,s)$. *If* $2\leqslant s<\infty$ *and*

$$(Q+3)\Theta(\infty;f)+\overline{d}(\mathscr{P})\delta(0;f)+\sum_{i=1}^{u}\mu_2^i\delta_{\mu_2^{*i}}(d_i;f) \tag{1.12}$$
$$>Q+3+2\overline{d}(\mathscr{P})-\underline{d}(\mathscr{P})+\sum_{i=1}^{u}\mu_2^i-n,$$

or, if $s=1$ *and*

$$\left(Q+\frac{7}{2}\right)\Theta(\infty;f)+\overline{d}(\mathscr{P})\delta(0;f)+\sum_{i=1}^{u}\mu_2^i\delta_{\mu_2^{*i}}(d_i;f)+\frac{1}{2}\sum_{i=1}^{u}\Theta(d_i;f) \tag{1.13}$$
$$>Q+\frac{7}{2}+2\overline{d}(\mathscr{P})-\underline{d}(\mathscr{P})+\sum_{i=1}^{u}\mu_2^i+\frac{1}{2}u-n,$$

or, if $s=0$ *and*

$$(2Q+6)\Theta(\infty;f)+2\overline{d}(\mathscr{P})\delta(0;f)+\sum_{i=1}^{u}\mu_2^i\delta_{\mu_2^{*i}}(d_i;f)+2\sum_{i=1}^{u}\Theta(d_i;f) \tag{1.14}$$
$$>2Q+6+4\overline{d}(\mathscr{P})-2\underline{d}(\mathscr{P})+\sum_{i=1}^{u}\mu_2^i+2u-n,$$

then $p(f)\equiv\mathscr{P}[f]$.

If we consider a meromorphic function f and its homogeneous differential polynomial $\mathscr{P}[f]$ sharing a small function then we get the following result from Corollary 1.

COROLLARY 2. *Let* f *be a non-constant meromorphic function and* $a\equiv a(z)(\not\equiv 0,\infty)$ *be a meromorphic small function of* f. *Let* $\mathscr{P}[f]$ *be a homogeneous differential polynomial containing atleast one derivative. Suppose that* $f-a$ *and* $\mathscr{P}[f]-a$ *share* $(0,s)$. *If* $2\leqslant s<\infty$ *and*

$$(\Gamma+3-d)\Theta(\infty;f)+d\,\delta(0;f)+\delta_2(0;f)>\Gamma+3,$$

or, if $s=1$ *and*

$$\left(\Gamma+\frac{7}{2}-d\right)\Theta(\infty;f)+d\,\delta(0;f)+\delta_2(0;f)+\frac{1}{2}\Theta(0;f)>\Gamma+4,$$

or, if $s=0$ *and*

$$(2(\Gamma-d)+6)\Theta(\infty;f)+2d\,\delta(0;f)+\delta_2(0;f)+2\Theta(0;f)>2\Gamma+8,$$

then $f\equiv\mathscr{P}[f]$.

REMARK 1. If we compare the conditions of Corollary 2 and Theorem F, we see that

$$(\Gamma+3-d)\Theta(\infty;f)+d\,\delta(0;f)+\delta_2(0;f)>\Gamma+3$$

implies

$$3\Theta(\infty;f)+d\,\delta(0;f)+\delta_2(0;f)>3+d\geqslant 4$$

and

$$\begin{aligned}&3\Theta(\infty;f)+d\,\delta(0;f)+\delta_2(0;f)\\ <\;&3\Theta(\infty;f)+d\delta_{2+\Gamma-d}(0,f^d)+\delta_2(0,f)+\delta(a,f).\end{aligned}$$

Again

$$\left(\Gamma+\frac{7}{2}-d\right)\Theta(\infty;f)+d\,\delta(0;f)+\delta_2(0;f)+\frac{1}{2}\Theta(0;f)>\Gamma+4$$

implies

$$\frac{7+\Gamma-d}{2}\Theta(\infty;f)+d\,\delta(0;f)+\delta_2(0;f)+\frac{1}{2}\Theta(0;f)>\frac{\Gamma}{2}+4+\frac{d}{2}\geqslant\frac{\Gamma+9}{2}$$

and

$$\begin{aligned}&\frac{7+\Gamma-d}{2}\Theta(\infty;f)+d\,\delta(0;f)+\delta_2(0;f)+\frac{1}{2}\Theta(0;f)\\ <\;&\frac{7+\Gamma-d}{2}\Theta(\infty;f)+\frac{d}{2}\delta_{1+\Gamma-d}(0;f^d)+d\delta_{2+\Gamma-d}(0;f^d)+\delta_2(0;f)+\delta(a;f).\end{aligned}$$

Also we see that for the case of *IM* sharing

$$\begin{aligned}&(2(\Gamma-d)+6)\Theta(\infty;f)+2d\,\delta(0;f)+\delta_2(0;f)+2\Theta(0;f)\\ <\;&[2(\Gamma-d)+6]\Theta(\infty;f)+d\delta_{1+\Gamma-d}(0,f^d)+d\delta_{2+\Gamma-d}(0;f^d)+\delta_2(0;f)+\Theta(0;f)+\delta(a;f).\end{aligned}$$

It is clear from Remark 1.1 that Corollary 2 is a direct extension and improvement of Theorem F.

The following examples show that the condition $a\not\equiv 0$ is necessary in Theorem 1.

EXAMPLE 1. Let $\mathscr{R}(f)=\dfrac{f^n}{f^2-1}$ with $n\geqslant 14$ and $\mathscr{P}[f]=-ff'$, where $f=\dfrac{e^z}{e^z-1}$. It is clear that $\mathscr{R}(f)$ and $\mathscr{P}[f]$ share $(0,\infty)$ and all the conditions (1.9)–(1.11) in Theorem 1 are satisfied but $\mathscr{R}(f)\not\equiv\mathscr{P}[f]$.

EXAMPLE 2. Let $\mathscr{R}(f)=\dfrac{f^n}{f^m-a}$, where $n>m+8$, $a\in\mathbb{C}-\{0\}$ and $\mathscr{P}[f]=f^2+ff'$, where $f=\dfrac{1}{e^z-1}$. It is clear that $\mathscr{R}(f)$ and $\mathscr{P}[f]$ share $(0,\infty)$ and all the conditions (1.9)–(1.11) in Theorem 1 are satisfied but $\mathscr{R}(f)\not\equiv\mathscr{P}[f]$.

The following examples show that the deficiency conditions (1.9)–(1.11) in Theorem 1 can't be removed when $\mathscr{P}[f]$ is a homogeneous differential polynomial.

EXAMPLE 3. Let $\mathscr{R}(f)=\dfrac{f^2+5f-4}{5(f^2+1)}$ and $\mathscr{P}[f]=\dfrac{1}{5}f'$, where $f=e^z$. It is clear that $\mathscr{R}(f)-\dfrac{1}{5}$ and $\mathscr{P}[f]-\dfrac{1}{5}$ share $(0,\infty)$ but none of the deficiency conditions (1.9)–(1.11) in Theorem 1 is satisfied and $\mathscr{R}(f)\not\equiv\mathscr{P}[f]$.

EXAMPLE 4. Let $\mathscr{R}(f)=\dfrac{2f+1}{f}$ and $\mathscr{P}[f]=f-f'$, where $f=\dfrac{e^z-1}{e^z+1}$. It is clear that $\mathscr{R}(f)-1$ and $\mathscr{P}[f]-1$ share $(0,\infty)$ but none of the deficiency conditions (1.9)–(1.11) in Theorem 1 is satisfied and $\mathscr{R}(f)\not\equiv\mathscr{P}[f]$.

EXAMPLE 5. For $c\neq 0$, let $\mathscr{R}(f)=\dfrac{(c+1)f}{f-1}$ and $\mathscr{P}[f]=f'$, where $f=e^{-z}$. It is clear that $\mathscr{R}(f)-c$ and $\mathscr{P}[f]-c$ share $(0,\infty)$ but none of the deficiency conditions (1.9)–(1.11) in Theorem 1 is satisfied and $\mathscr{R}(f)\not\equiv\mathscr{P}[f]$.

EXAMPLE 6. Let $\mathscr{R}(f)=f^3+c-1$, where $c\in\mathbb{C}-\{0\}$ and $\mathscr{P}[f]=\dfrac{c}{\beta}f^2f'$, where $f=e^{\beta z}$, $\beta\in\mathbb{C}-\{0\}$. It is clear that $\mathscr{R}(f)-c=e^{3\beta z}-1$ and $\mathscr{P}[f]-c=c(e^{3\beta z}-1)$ share $(0,\infty)$ but none of the deficiency conditions (1.9)–(1.11) in Theorem 1 is satisfied and $\mathscr{R}(f)\not\equiv\mathscr{P}[f]$.

EXAMPLE 7. Let $f(z)=-\sin(\alpha z)+a-\dfrac{a}{\alpha^{4k}}$, $k\in\mathbb{N}$; where $\alpha\neq 0, \alpha^{4k}\neq 1$ and $a\in\mathbb{C}-\{0\}$. Let $\mathscr{R}(f)=f$ and $\mathscr{P}[f]=f^{(4k)}$. Then $\mathscr{P}[f]=-\alpha^{4k}\sin(\alpha z)$. Here $\delta(0;f)=0$. Also it is clear that $\mathscr{R}(f)$ and $\mathscr{P}[f]$ share (a,∞) but none of the deficiency conditions (1.9)–(1.11) in Theorem 1 is satisfied, hence $\mathscr{R}(f)\not\equiv\mathscr{P}[f]$.

EXAMPLE 8. Let $f(z)=e^{Nz}$, where N is a non-zero integer. For a natural number n we define

$$\mathscr{R}(f)=-N^{2n}\sum_{r=0}^{2n-1}(-1)^r\binom{2n}{r}f^{2n-r}\quad\text{and}\quad\mathscr{P}[f]=f^{(2n)}.$$

Then it is clear that

$$\mathscr{R}(f)-N^{2n}=-N^{2n}(e^{Nz}-1)^{2n}\quad\text{and}\quad\mathscr{P}[f]-N^{2n}=N^{2n}(e^{Nz}-1).$$

Thus we see that $\mathscr{R}(f)$ and $\mathscr{P}[f]$ share $(N^{2n},0)$. Here $\Theta(\infty;f)=1$ and $\delta_q(0;f)=1, \forall q\in\mathbb{N}$.

Thus the condition (1.11) in Theorem 1 is not satisfied and $\mathscr{R}(f)\not\equiv\mathscr{P}[f]$.

The following examples show that the deficiency conditions (1.9)–(1.11) in Theorem 1 can't be removed when $\mathscr{P}[f]$ is non homogeneous differential polynomial.

EXAMPLE 9. Let $\mathscr{R}(f)=\dfrac{(f+1)^3-(af^2+b)}{af^2+b}$, where $a,b\in\mathbb{C}$ with $a\neq 0$, $b\neq 1$ and $\mathscr{P}[f]=f^2f'+3ff'+3f$, where $f=e^z$. It is clear that $\mathscr{R}(f)+1=\dfrac{(e^z+1)^3}{af^2+b}$ and $\mathscr{P}[f]+1=(e^z+1)^3$ share $(0,\infty)$ but none of the deficiency conditions (1.9)–(1.11) in Theorem 1 is satisfied, hence $\mathscr{R}(f)\not\equiv\mathscr{P}[f]$.

EXAMPLE 10. Let $\mathscr{R}(f)=\dfrac{f(f+ac+2)}{af+1}$, where $a,c\in\mathbb{C}-\{0\}$ and $\mathscr{P}[f]=ff'+2f$, where $f=e^z$. It is clear that $\mathscr{R}(f)-c=\dfrac{e^{2z}+2e^z-c}{ae^z+1}$ and $\mathscr{P}[f]-c=e^{2z}+2e^z-c$ share $(0,\infty)$ but none of the deficiency conditions (1.9)–(1.11) in Theorem 1 is satisfied, hence $\mathscr{R}(f)\not\equiv\mathscr{P}[f]$.

EXAMPLE 11. Let $\mathscr{R}(f)=f^3+2f^2+f-1$ and $\mathscr{P}[f]=f^2+2f'$, where $f=e^z$. It is clear that $\mathscr{R}(f)+1=e^z(e^z+1)^2$ and $\mathscr{P}[f]+1=(e^z+1)^2$ share $(0,\infty)$ but none of the deficiency conditions (1.9)–(1.11) in Theorem 1 is satisfied, hence $\mathscr{R}(f)\not\equiv\mathscr{P}[f]$.

2. Useful lemmas

In this section we present some lemmas which will be needed in the sequel. Let $\mathscr{F}$, $\mathscr{G}$ be two non-constant meromorphic functions. Henceforth we shall denote by $\mathscr{H}$ the following function.

$$\mathscr{H}=\left(\frac{\mathscr{F}''}{\mathscr{F}'}-\frac{2\,\mathscr{F}'}{\mathscr{F}-1}\right)-\left(\frac{\mathscr{G}''}{\mathscr{G}'}-\frac{2\,\mathscr{G}'}{\mathscr{G}-1}\right). \tag{2.1}$$

LEMMA 1. [15] *Let f be a non-constant meromorphic function and let*

$$\mathscr{R}(f)=\frac{\sum\limits_{k=0}^{n}a_kf^k}{\sum\limits_{j=0}^{m}b_jf^j}$$

be an irreducible rational function in f with constant coefficients $\{a_k\}$ and $\{b_j\}$ where $a_n\neq 0$ and $b_m\neq 0$. Then

$$T(r,\mathscr{R}(f))=\lambda T(r,f)+S(r,f),$$

where $\lambda=\max\{n,m\}$.

LEMMA 2. [5] *Let f be a meromorphic function and $\mathscr{P}[f]$ be a differential polynomial. Then*

$$m\left(r,\frac{\mathscr{P}[f]}{f^{\overline{d}(\mathscr{P})}}\right)\leqslant(\overline{d}(\mathscr{P})-\underline{d}(\mathscr{P}))m\left(r,\frac{1}{f}\right)+S(r,f). \tag{2.2}$$

$$N\left(r,\frac{\mathscr{P}[f]}{f^{\overline{d}(\mathscr{P})}}\right)\leqslant(\overline{d}(\mathscr{P})-\underline{d}(\mathscr{P}))N\left(r,\frac{1}{f}\right)+Q\left[\overline{N}(r,\infty;f)+\overline{N}(r,0;f)\right]+S(r,f). \tag{2.3}$$

$$N\left(r,\frac{1}{\mathscr{P}[f]}\right)\leqslant Q\,\overline{N}(r,\infty;f)+(\overline{d}(\mathscr{P})-\underline{d}(\mathscr{P}))\,m\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{f^{\overline{d}(\mathscr{P})}}\right)+S(r,f). \tag{2.4}$$

3. Proof of the theorem

Proof of Theorem 1. Let $\mathscr{F}=\dfrac{\mathscr{R}(f)}{a}$ and $\mathscr{G}=\dfrac{\mathscr{P}[f]}{a}$. Then $\mathscr{F}-1=\dfrac{\mathscr{R}(f)-a}{a}$ and $\mathscr{G}-1=\dfrac{\mathscr{P}[f]-a}{a}$. Since $\mathscr{R}(f)-a$ and $\mathscr{P}[f]-a$ share $(0,s)$ it follows that $\mathscr{F}$, $\mathscr{G}$ share $(1,s)$ except the zeros and poles of $a(z)$. Now we consider the following cases.

Case 1. Let $\mathscr{H}\not\equiv 0$.

Then from (2.1) we get

$$\begin{aligned}&N(r,\infty;\mathscr{H}) \\ \leqslant\ &\overline{N}(r,\infty;f)+\sum_{j=0}^{l^*}\chi_j\overline{N}(r,c_j;f\mid\geqslant 2)+\overline{N}_*(r,1;\mathscr{F},\mathscr{G})+\overline{N}(r,0;\mathscr{F}\mid\geqslant 2)+\overline{N}(r,0;\mathscr{G}\mid\geqslant 2)\\ &+\overline{N}_0(r,0;\mathscr{F}')+\overline{N}_0(r,0;\mathscr{G}')+\overline{N}(r,0;a)+\overline{N}(r,\infty;a).\end{aligned} \tag{3.1}$$

where $\overline{N}_0(r,0;\mathscr{F}')$ is the reduced counting function of those zeros of $\mathscr{F}'$ which are not the zeros of $\mathscr{F}(\mathscr{F}-1)$ and $\overline{N}_0(r,0;\mathscr{G}')$ is similarly defined. Let z_0 be a simple zero of $\mathscr{F}-1$.

By the *Second Fundamental Theorem*, we get

$$\begin{aligned}&T(r,\mathscr{F})+T(r,\mathscr{G}) \\ \leqslant\ &2\overline{N}(r,\infty;f)+\sum_{j=0}^{l^*}\chi_j\overline{N}(r,c_j;f)+\overline{N}(r,0;\mathscr{F})+\overline{N}(r,0;\mathscr{G})+\overline{N}(r,1;\mathscr{F})+\overline{N}(r,1;\mathscr{G})\\ &-N_0(r,0;\mathscr{F}')-N_0(r,0;\mathscr{F}')+S(r,f),\end{aligned} \tag{3.2}$$

where $N_0(r,0;\mathscr{F}')$ is the counting function of those zeros of $\mathscr{F}'$ which are not the zeros of $\mathscr{F}(\mathscr{F}-1)$ and $N_0(r,0;\mathscr{G}')$ is similarly defined.

Subcase 1.1. Let $s\geqslant 1$. Then from (2.1), we get

$$\begin{aligned}&N_E^{1)}(r,1;\mathscr{F}) \\ \leqslant\ &N(r,\mathscr{H})+S(r,f)\\ \leqslant\ &\overline{N}(r,\infty;f)+\sum_{j=0}^{l^*}\chi_j\overline{N}(r,c_j;f\mid\geqslant 2)+\overline{N}_*(r,1;\mathscr{F},\mathscr{G})+\overline{N}(r,0;\mathscr{F}\mid\geqslant 2)+\overline{N}(r,0;\mathscr{G}\mid\geqslant 2)\\ &+\overline{N}_0(r,0;\mathscr{F}')+\overline{N}_0(r,0;\mathscr{G}')+S(r,f)\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}
&\overline{N}(r.1;\mathscr{F})+\overline{N}(r,1;\mathscr{G}) \qquad (3.4)\\
&= N_E^{1)}(r,1;\mathscr{F})+\overline{N}_E^{(2}(r,1;\mathscr{F})+\overline{N}_L(r,1;\mathscr{F})+\overline{N}_L(r,1;\mathscr{G})+\overline{N}(r,1;\mathscr{G})+S(r,f)\\
&\leqslant \overline{N}(r,\infty;f)+\sum_{j=0}^{l^*}\chi_j\overline{N}(r,c_j;f\mid\geqslant 2)+\overline{N}_*(r,1;\mathscr{F},\mathscr{G})+\overline{N}(r,0;\mathscr{F}\mid\geqslant 2)\\
&\quad +\overline{N}(r,0;\mathscr{G}\mid\geqslant 2)+\overline{N}_0(r,0;\mathscr{F}')+\overline{N}_0(r,0;\mathscr{G}')+\overline{N}_E^{(2}(r,1;\mathscr{F})+\overline{N}_L(r,1;\mathscr{F})\\
&\quad +\overline{N}_L(r,1;\mathscr{G})+\overline{N}(r,1;\mathscr{G})+S(r,f).
\end{aligned}$$

Subcase 1.1.*a*. Let $s\geqslant 2$. In this case proceeding same way as in [7, Subcase 1.2, Proof of Theorem 1] we get,

$$\begin{aligned}
&T(r,\mathscr{F}) \qquad (3.5)\\
&\leqslant 3\overline{N}(r,\infty;f)+\sum_{j=0}^{l^*}\chi_j\overline{N}(r,c_j;f)+\sum_{j=0}^{l^*}\chi_j\overline{N}(r,c_j;f\mid\geqslant 2)+N_2(r,0;\mathscr{F})\\
&\quad +N(r,0;\mathscr{P}[f])+S(r,f).
\end{aligned}$$

Using (2.4) of Lemma 2 in (3.5), we obtained

$$\begin{aligned}
&T(r,F)\\
&\leqslant (Q+3)\overline{N}(r,\infty;f)+\sum_{j=0}^{l^*}\chi_j\overline{N}(r,c_j;f)+\sum_{j=0}^{l^*}\chi_j\overline{N}(r,c_j;f\mid\geqslant 2)+\sum_{i=1}^{u}\mu_2^i N_{\mu_2^{*i}}(r,d_i;f)\\
&\quad +(\overline{d}(\mathscr{P})-\underline{d}(\mathscr{P}))T(r,f)+\overline{d}(\mathscr{P})N(r,0;f)+S(r,f)
\end{aligned}$$

i.e., for any $\varepsilon>0$, we get

$$\begin{aligned}
&\Bigg\{(Q+3)\,\Theta(\infty;f)+\sum_{j=0}^{l^*}\chi_j\Big\{\Theta(c_j;f)+\Theta_{(2}(c_j;f)\Big\}+\overline{d}(\mathscr{P})\delta(0;f)\\
&\quad +\sum_{i=1}^{u}\mu_2^i\delta_{\mu_2^{*i}}(d_i;f)\Bigg\}T(r,f)\\
&\leqslant \Bigg\{Q+3+2\overline{d}(\mathscr{P})-\underline{d}(\mathscr{P})+2l^*+\sum_{i=1}^{u}\mu_2^i-\lambda+\varepsilon\Bigg\}T(r,f)+S(r,f),
\end{aligned}$$

which contradicts (1.9).

Subcase 1.1.*b*. Let $s=1$

Then proceeding same way as in [7, Subcase 1.1, Proof of Theorem 1] we get,

$$\begin{aligned}
&\overline{N}(r,1;\mathscr{F})+\overline{N}(r,1;\mathscr{G}) \qquad (3.6)\\
&\leqslant \frac{3}{2}\overline{N}(r,\infty;f)+\frac{1}{2}\sum_{j=0}^{l^*}\chi_j\overline{N}(r,c_j;f)+\sum_{j=0}^{l^*}\chi_j\overline{N}(r,c_j;f\mid\geqslant 2)+\frac{1}{2}\overline{N}(r,0;F)\\
&\quad +\overline{N}(r,0;\mathscr{F}\mid\geqslant 2)+\overline{N}(r,0;\mathscr{G}\mid\geqslant 2)+T(r,\mathscr{G})+\overline{N}_0(r,0;\mathscr{F}')+\overline{N}(r,0;\mathscr{G}')+S(r,f).
\end{aligned}$$

and then from (3.2), we get as above

$$\begin{aligned} &T(r,\mathscr{F}) \qquad (3.7)\\ \leqslant\; &\frac{7}{2}\overline{N}(r,\infty;f)+\frac{3}{2}\sum_{j=0}^{l^*}\chi_j\overline{N}(r,c_j;f)+\sum_{j=0}^{l^*}\chi_j\overline{N}(r,c_j;f\mid\geqslant 2)+N_2(r,0;\mathscr{F})+\frac{1}{2}\overline{N}(r,0;\mathscr{F})\\ &+N(r,0;\mathscr{P}[f])+S(r,f)\end{aligned}$$

Now using (2.4) in Lemma 2, we get from (3.7)

$$\begin{aligned} &T(r,\mathscr{F})\\ \leqslant\; &\left(Q+\frac{7}{2}\right)\overline{N}(r,\infty;f)+\frac{3}{2}\sum_{j=0}^{l^*}\chi_j\overline{N}(r,c_j;f)+\sum_{j=0}^{l^*}\overline{N}(r,c_j;f\mid\geqslant 2)+\sum_{i=1}^{u}\mu_2^iN_{\mu_2^{*i}}(r,d_i;f)\\ &+\frac{1}{2}\sum_{i=1}^{u}\overline{N}(r,d_i;f)+(\overline{d}(\mathscr{P})-\underline{d}(\mathscr{P}))T(r,f)+\overline{d}(\mathscr{P})N(r,0;f)+S(r,f)\end{aligned}$$

i.e., for any $\varepsilon>0$, we get

$$\begin{aligned} &\left\{\left(Q+\frac{7}{2}\right)\Theta(\infty;f)+\sum_{j=0}^{l^*}\chi_j\left\{\Theta_{(2}(c_j;f)+\frac{3}{2}\Theta(c_j;f)\right\}+\sum_{i=1}^{u}\mu_2^i\delta_{\mu_2^{*i}}(d_i;f)\right.\\ &\left.+\frac{1}{2}\sum_{i=1}^{u}\Theta(d_i;f)+\overline{d}(\mathscr{P}\delta(0;f))\right\}T(r,f)\\ \leqslant\; &\left\{Q+\frac{7}{2}+2\overline{d}(\mathscr{P})-\underline{d}(\mathscr{P})+\frac{5l^*}{2}+\sum_{i=1}^{u}\mu_2^i+\frac{1}{2}u-\lambda+\varepsilon\right\}T(r,f)+S(r,f),\end{aligned}$$

which contradicts (1.10).

Subcase 1.2. Let $s=0$.

In this case also proceeding same way as in [7, Subcase 1.2, Proof of Theorem 1] we get,

$$\begin{aligned} &T(r,\mathscr{F}) \qquad (3.8)\\ \leqslant\; &6\overline{N}(r,\infty;f)+3\sum_{j=0}^{l^*}\chi_j\overline{N}(r,c_j;f)+\sum_{j=0}^{l^*}\chi_j\overline{N}(r,c_j;f\mid\geqslant 2)+N_2(r,0;\mathscr{F})+2\overline{N}(r,0;\mathscr{F})\\ &+2N(r,0;\mathscr{P}[f])+S(r,f).\end{aligned}$$

Using (2.4) of Lemma 2 in (3.8), we obtained

$$\begin{aligned} &T(r,\mathscr{F})\\ \leqslant\; &(2Q+6)\overline{N}(r,\infty;f)+3\sum_{j=0}^{l^*}\chi_j\overline{N}(r,c_j;f)+\sum_{j=0}^{l^*}\chi_j\overline{N}(r,c_j;f\mid\geqslant 2)+\sum_{i=1}^{u}\mu_2^iN_{\mu_2^{*i}}(r,d_i;f)\\ &+2\sum_{i=1}^{u}\overline{N}(r,d_i;f)+2(\overline{d}(\mathscr{P})-\underline{d}(\mathscr{P}))T(r,f)+2\overline{d}(\mathscr{P})\overline{N}(r,0;f)+S(r,f)\end{aligned}$$

i.e., for any $\varepsilon > 0$, we get

$$\begin{aligned}&\Big\{(2Q+6)\Theta(\infty;f)+\sum_{j=0}^{l^*}\chi_j\Big\{\Theta_{(2}(c_j;f)+3\Theta(c_j;f)\Big\}+\sum_{i=1}^{u}\mu_2^i\delta_{\mu_2^{*i}}(d_i;f)\\&+2\sum_{i=1}^{u}\Theta(d_i;f)+2\overline{d}(\mathscr{P})\delta(0;f)\Big\}T(r,f)\\ \leqslant&\Big\{2Q+6+4\overline{d}(\mathscr{P})-2\underline{d}(\mathscr{P})+4l^*+\sum_{i=1}^{u}\mu_2^i+2u-\lambda+\varepsilon\Big\}T(r,f)+S(r,f),\end{aligned}$$

which contradicts (1.11).

Case 2. Let $\mathscr{H}\equiv 0$.

On integration we get from (2.1)

$$\frac{1}{\mathscr{F}-1}\equiv\frac{\mathscr{C}}{\mathscr{G}-1}+\mathscr{D}, \tag{3.9}$$

where $\mathscr{C}$, $\mathscr{D}$ are constants and $\mathscr{C}\neq 0$.

Subcase 2.1. Suppose $\mathscr{D}\neq 0,-1$.

Following the same procedure as in [7, Case (i), Proof of Theorem 1] and applying *Second Fundamental Theorem* we obtained,

$$\lambda T(r,f)\leqslant 2\overline{N}(r,\infty;f)+\sum_{j=0}^{l^*}\chi_j\overline{N}(r,c_j;f)+\sum_{i=1}^{u}\overline{N}(r,d_i;f)+S(r,f),$$

which implies

$$2\Theta(\infty;f)+\sum_{j=0}^{l^*}\chi_j\Theta(c_j;f)+\sum_{i=1}^{u}\Theta(d_i;f)\leqslant 2+l^*+u-\lambda,$$

which contradicts (1.9), (1.10) and (1.11).

Subcase 2.2. Let $\mathscr{D}=0$, then proceeding exactly same way as in [7, Case (ii), Proof of Theorem 1] and applying *Second Fundamental Theorem* we obtained,

$$\begin{aligned}T(r,\mathscr{F})\leqslant&(Q+1)\overline{N}(r,\infty;f)+\sum_{j=0}^{l^*}\chi_j\overline{N}(r,c_j;f)+\sum_{i=1}^{u}\overline{N}(r,d_i;f)\\&+(\overline{d}(\mathscr{P})-\underline{d}(\mathscr{P}))T(r,f)+\overline{d}(\mathscr{P})N(r,0;f)+S(r,f),\end{aligned}$$

which implies

$$\begin{aligned}&(Q+1)\Theta(\infty;f)+\sum_{j=0}^{l^*}\chi_j\Theta(c_j;f)+\sum_{i=1}^{u}\Theta(d_i;f)+\overline{d}(\mathscr{P})\delta(0;f)\\ \leqslant&\ Q+1+2\overline{d}(\mathscr{P})-\underline{d}(\mathscr{P})+l^*+u-\lambda,\end{aligned}$$

which contradicts (1.9), (1.10) and (1.11).

In this case we get, $\mathscr{F} \equiv \mathscr{G}$ i.e., $\mathscr{R}(f) \equiv \mathscr{P}[f]$.

Subcase 2.3. Let $\mathscr{D} = -1$ and if $\mathscr{C} \neq -1$, then proceeding exactly same as in [7, Case (iii), Proof of Theorem 1] and applying *Second Fundamental Theorem*, this case follows exactly the Subcase 2.2.

So let $\mathscr{C} = -1$, then we get $\mathscr{F}\mathscr{G} \equiv 1$ i.e., $\mathscr{R}(f)\mathscr{P}[f] \equiv a^2$.

From above we have $N(r,0;f) = S(r,f)$ and $N(r,\infty;f) = S(r,f)$.

In view of the *First Fundamental Theorem* and (2.2) in Lemma 2, we get from above

$$\begin{aligned}
&(\lambda + \overline{d}(\mathscr{P}))T(r,f)\\
&= T\left(r, \frac{a^2}{\mathscr{R}(f) f^{\overline{d}(P)}}\right) + S(r,f)\\
&\leqslant T\left(r, \frac{\mathscr{P}[f]}{f^{\overline{d}(\mathscr{P})}}\right) + S(r,f)\\
&= m\left(r, \frac{\mathscr{P}[f]}{f^{\overline{d}(\mathscr{P})}}\right) + N\left(r,\infty; \frac{\mathscr{P}[f]}{f^{\overline{d}(\mathscr{P})}}\right) + S(r,f)\\
&\leqslant (\overline{d}(\mathscr{P}) - \underline{d}(\mathscr{P}))m\left(r, \frac{1}{f}\right) + N(r,\infty;\mathscr{P}[f]) + \overline{d}(\mathscr{P})N(r,0;f) + S(r,f)\\
&= (\overline{d}(\mathscr{P}) - \underline{d}(\mathscr{P}))(T(r,f) - N(r,0;f)) + S(r,f),
\end{aligned}$$

i.e., $(\lambda + \underline{d}(\mathscr{P}))T(r,f) \leqslant S(r,f)$, which is impossible.

This completes the proof. □

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Abhijit Banerjee
Department of Mathematics
University of Kalyani
West Bengal, 741235, India
e-mail: abanerjee_kal@yahoo.co.in,
abanerjeekal@gmail.com

Molla Basir Ahamed
Department of Mathematics
Kalipada Ghosh Tarai Mahavidyalaya
West Bengal, 734014, India
e-mail: bsrhmd117@gmail.com,
bsrhmd2014@gmail.com