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ON SOME GENERALIZATIONS OF ENESTRÖM–KAKEYA THEOREM

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Abstract. In this paper, we obtain some generalizations of a well-known result of Eneström-Kakeya concerning the bounds for the moduli of the zeros of polynomials with complex coefficients which improve some known results.

1. Introduction

If $P(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n with real coefficients satisfying

$$a_n \geqslant a_{n-1} \geqslant \cdots \geqslant a_1 \geqslant a_0 > 0$$

then according to a well-known result of Eneström-Kakeya (see [21, 22, 23]) all the zeros of $P(z)$ lie in $|z| \leqslant 1$.
We may apply this result to $P(tz)$ to obtain following more general result.

THEOREM A. *If $P(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n with real coefficients satisfying*

$$t^n a_n \geqslant t^{n-1} a_{n-1} \geqslant \cdots \geqslant t a_1 \geqslant a_0 > 0,$$

for some $t > 0$. Then all the zeros of $P(z)$ lie in $|z| \leqslant t$.

In literature [1,2,3,4,5,6,7,8,9,10,11,12,13,14,16,17,18,19,20,21,22,23,24,25] there exits several extensions of Eneström-Kakeya theorem. An exhaustive survey on the Eneström-Kakeya theorem and some of its generalizations is given in [15] by Gardner and Govil. For the polynomials with complex coefficients, A. Aziz and Q. G. Mohammad [5] used matrix method and proved among others, the following generalization of Theorem A.

THEOREM B. *If $P(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n with complex coefficients such that for $k = 0, 1, 2, \cdots, n$ and for some $t > 0$,*

$$t^n |a_n| \leqslant t^{n-1} |a_{n-1}| \leqslant \cdots \leqslant t^k |a_k| \geqslant t^{k-1} |a_{k-1}| \geqslant \cdots \geqslant t|a_1| \geqslant |a_0|,$$

then $P(z)$ has all its zeros in the circle

$$|z| \leqslant t \left\{ \frac{2t^k |a_k|}{t^n |a_n|} - 1 \right\} + 2 \sum_{j=0}^{n} \frac{\big|a_j - |a_j|\big|}{|a_n| t^{n-j-1}}. \tag{1}$$

A. Aziz and Q. G. Mohammad [5] also proved the following result.

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THEOREM C. *Let* $P(z)=\sum_{j=0}^{n}a_jz^j$ *be a polynomial of degree* n *with complex coefficients. If* $\operatorname{Re}a_j=\alpha_j$ *and* $\operatorname{Im}a_j=\beta_j$ *for* $j=0,1,2,\cdots,n$. *If* $t>0$ *can be found such that*

$$t^n\alpha_n\leqslant t^{n-1}\alpha_{n-1}\leqslant\cdots\leqslant t^{k+1}\alpha_{k+1}\leqslant t^k\alpha_k,\quad t^k\alpha_k\geqslant t^{k-1}\alpha_{k-1}\geqslant\cdots\geqslant t\alpha_1\geqslant\alpha_0$$

and

$$t^n\beta_n\leqslant t^{n-1}\beta_{n-1}\leqslant\cdots\leqslant t^{m+1}\beta_{m+1}\leqslant t^m\beta_m,\quad t^m\beta_m\geqslant t^{m-1}\beta_{m-1}\geqslant\cdots\geqslant t\beta_1\geqslant\beta_0,$$

$0\leqslant k,m\leqslant n,\ a_{-1}=a_{n+1}=0,\ \alpha_n>0$, *then all the zero of* $P(z)$ *lie in circle*

$$|z|\leqslant\frac{2}{|a_n|t^n}\left\{\alpha_kt^{k+1}+\beta_mt^{m+1}\right\}-\frac{t(\alpha_n+\beta_n)}{|a_n|}. \tag{2}$$

In this paper, we first present the following result which among other things provides a refinement of Theorem B for $0\leqslant k\leqslant n-1$.

THEOREM 1. *Let* $P(z)=\sum_{j=0}^{n}a_jz^j$ *be a polynomial of degree* n *with complex coefficients. If* $t>0$ *can be found such that*

$$t^n|a_n|\leqslant t^{n-1}|a_{n-1}|\leqslant\cdots\leqslant t^{k+1}|a_{k+1}|\leqslant t^k|a_k|,\quad t^k|a_k|\geqslant t^{k-1}|a_{k-1}|\geqslant\cdots\geqslant t|a_1|\geqslant|a_0|,$$

$0\leqslant k\leqslant n-1$, *then all the zero of* $P(z)$ *lie in*

$$\left|z+\frac{a_{n-1}}{a_n}-t\right|\leqslant\frac{2t^{k+1}|a_k|}{t^n|a_n|}+2\sum_{\nu=0}^{n}\frac{|a_\nu-|a_\nu||}{t^{n-\nu-1}|a_n|}-\frac{|a_{n-1}|+\big|a_{n-1}-|a_{n-1}|\big|+t\big|a_n-|a_n|\big|}{|a_n|}. \tag{3}$$

Proof. Consider the polynomial

$$\begin{aligned}F(z)=&(t-z)P(z)=(t-z)\left(a_nz^n+a_{n-1}z^{n-1}+\cdots+a_1z+a_0\right)\\=&-a_nz^{n+1}+(ta_n-a_{n-1})z^n+\sum_{\nu=0}^{n-1}(ta_\nu-a_{\nu-1})z^\nu\quad(a_{-1}=0).\end{aligned}$$

Let $|z|>t$, then

$$\begin{aligned}|F(z)|\geqslant&|a_n||z|^n\left[\left|z+\frac{a_{n-1}}{a_n}-t\right|-\sum_{\nu=0}^{n-1}\left|\frac{ta_\nu-a_{\nu-1}}{a_n}\right|\frac{1}{|z|^{n-\nu}}\right]\\>&|a_n||z|^n\left[\left|z+\frac{a_{n-1}}{a_n}-t\right|-\frac{1}{|a_n|}\sum_{\nu=0}^{n-1}\frac{|ta_\nu-a_{\nu-1}|}{t^{n-\nu}}\right].\end{aligned} \tag{4}$$

Now,

$$\begin{aligned}\sum_{\nu=0}^{n-1}\frac{|ta_\nu-a_{\nu-1}|}{t^{n-\nu}} &\leqslant \sum_{\nu=0}^{n-1}\frac{\big|t|a_\nu|-|a_{\nu-1}|\big|}{t^{n-\nu}}+\sum_{\nu=0}^{n-1}\frac{t\big|a_\nu-|a_\nu|\big|+\big|a_{\nu-1}-|a_{\nu-1}|\big|}{t^{n-\nu}}\\ &=\sum_{\nu=0}^{k}\frac{\big|t|a_\nu|-|a_{\nu-1}|\big|}{t^{n-\nu}}+\sum_{\nu=k+1}^{n-1}\frac{\big|t|a_\nu|-|a_{\nu-1}|\big|}{t^{n-\nu}}\\ &\quad+\sum_{\nu=0}^{n-1}\frac{t\big|a_\nu-|a_\nu|\big|+\big|a_{\nu-1}-|a_{\nu-1}|\big|}{t^{n-\nu}}\\ &\leqslant \frac{2|a_k|}{t^{n-k-1}}+2\sum_{\nu=0}^{n}\frac{\big|a_\nu-|a_\nu|\big|}{t^{n-\nu-1}}-\Big(|a_{n-1}|+\big|a_{n-1}-|a_{n-1}|\big|+t\big|a_n-|a_n|\big|\Big).\end{aligned}$$

Using this in (4), we have

$$\begin{aligned}|F(z)|\geqslant |a_n||z|^{n+1}\Bigg[\Big|z+\frac{a_{n-1}}{a_n}-t\Big|-\frac{2|a_k|}{t^{n-k-1}|a_n|}-2\sum_{\nu=0}^{n}\frac{\big|a_\nu-|a_\nu|\big|}{t^{n-\nu-1}|a_n|}\\ +\frac{|a_{n-1}|+\big|a_{n-1}-|a_{n-1}|\big|+t\big|a_n-|a_n|\big|}{|a_n|}\Bigg]>0,\end{aligned}$$

if

$$\Big|z+\frac{a_{n-1}}{a_n}-t\Big|>\frac{2|a_k|}{t^{n-k-1}|a_n|}+2\sum_{\nu=0}^{n}\frac{\big|a_\nu-|a_\nu|\big|}{t^{n-\nu-1}|a_n|}-\frac{|a_{n-1}|+\big|a_{n-1}-|a_{n-1}|\big|+t\big|a_n-|a_n|\big|}{|a_n|}.$$

Therefore, it follows that all the zeros of $F(z)$ whose modulus is greater than t lie in circle

$$\Big|z+\frac{a_{n-1}}{a_n}-t\Big|\leqslant\frac{2|a_k|}{t^{n-k-1}|a_n|}+2\sum_{\nu=0}^{n}\frac{\big|a_\nu-|a_\nu|\big|}{t^{n-\nu-1}|a_n|}-\frac{|a_{n-1}|+\big|a_{n-1}-|a_{n-1}|\big|+t\big|a_n-|a_n|\big|}{|a_n|}. \tag{5}$$

We now show that all those zeros of $F(z)$ whose modulus is less than or equal to t also satisfy (5). Let $|z|\leqslant t$,

$$\begin{aligned}\Big|z+\frac{a_{n-1}}{a_n}-t\Big| &\leqslant t+\Big|\frac{a_{n-1}}{a_n}-t\Big|\leqslant t+\frac{\big|t|a_n|-|a_{n-1}|\big|}{|a_n|}+\frac{t\big|a_n-|a_n|\big|+\big|a_{n-1}-|a_{n-1}|\big|}{|a_n|}\\ &=t+\frac{|a_{n-1}|-t|a_n|}{|a_n|}+\frac{\big|a_n-|a_n|\big|t+\big|a_{n-1}-|a_{n-1}|\big|}{|a_n|}\\ &=\frac{2|a_{n-1}|}{|a_n|}+\frac{\big|a_{n-1}-|a_{n-1}|\big|}{|a_n|}+\frac{t\big|a_n-|a_n|\big|}{|a_n|}-\frac{|a_{n-1}|}{|a_n|}.\end{aligned} \tag{6}$$

Now, by hypothesis,

$$\Big|\frac{a_{n-1}}{a_n}\Big|\leqslant\frac{|a_k|}{|a_n|t^{n-k-1}},\qquad \text{for}\quad 0\leqslant k\leqslant n-1. \tag{7}$$

Using (7) in (6), we obtain for $0 \leqslant k \leqslant n-1$,

$$\left|z+\frac{a_{n-1}}{a_n}-t\right| \leqslant \frac{2|a_k|}{|a_n|t^{n-k-1}}+\frac{\left|a_{n-1}-|a_{n-1}|\right|}{|a_n|}+\frac{t\left|a_n-|a_n|\right|}{|a_n|}-\frac{|a_{n-1}|}{|a_n|}$$
$$\leqslant \frac{2|a_k|}{t^{n-k-1}|a_n|}+2\sum_{\nu=0}^{n}\frac{\left|a_\nu-|a_\nu|\right|}{t^{n-\nu-1}|a_n|}-\frac{|a_{n-1}|+\left|a_{n-1}-|a_{n-1}|\right|+t\left|a_n-|a_n|\right|}{|a_n|}.$$

Thus, we have shown that if $|z| \leqslant t$, then for $0 \leqslant k \leqslant n-1$,

$$\left|z+\frac{a_{n-1}}{a_n}-t\right| \leqslant \frac{2|a_k|}{t^{n-k-1}|a_n|}+2\sum_{\nu=0}^{n}\frac{\left|a_\nu-|a_\nu|\right|}{t^{n-\nu-1}|a_n|}-\frac{|a_{n-1}|+\left|a_{n-1}-|a_{n-1}|\right|+t\left|a_n-|a_n|\right|}{|a_n|}.$$

This shows that all the zeros of $F(z)$ whose moduli is less than or equal to t also lie in the circle defined by (5). But all the zeros of $P(z)$ are also the zeros of $F(z)$, therefore, we conclude that all the zeros of $P(z)$ lie in the circle defined by (5). This completes the proof of Theorem 1. □

REMARK 1. In general Theorem 1 gives much better result than Theorem B for $0 \leqslant k \leqslant n-1$. To see this, we show that the circle defined by (3) is contained in the circle defined by (1). Let z be any point belonging to the circle defined by (3), then

$$\left|z+\frac{a_{n-1}}{a_n}-t\right| \leqslant \frac{2|a_k|}{t^{n-k-1}|a_n|}+2\sum_{\nu=0}^{n}\frac{\left|a_\nu-|a_\nu|\right|}{t^{n-\nu-1}|a_n|}-\left(\frac{|a_{n-1}|+\left|a_{n-1}-|a_{n-1}|\right|+t\left|a_n-|a_n|\right|}{|a_n|}\right).$$

This implies

$$\begin{aligned}
|z| &= \left|z+\frac{a_{n-1}}{a_n}-t+t-\frac{a_{n-1}}{a_n}\right| \leqslant \left|z+\frac{a_{n-1}}{a_n}-t\right|+\left|t-\frac{a_{n-1}}{a_n}\right| \\
&\leqslant \frac{2|a_k|}{t^{n-k-1}|a_n|}+2\sum_{\nu=0}^{n}\frac{\left|a_\nu-|a_\nu|\right|}{t^{n-\nu-1}|a_n|}-\left(\frac{|a_{n-1}|+\left|a_{n-1}-|a_{n-1}|\right|+t\left|a_n-|a_n|\right|}{|a_n|}\right) \\
&\quad +\frac{\left||a_n|t-|a_{n-1}|\right|}{|a_n|}+\frac{\left|a_n-|a_n|\right|t+\left|a_{n-1}-|a_{n-1}|\right|}{|a_n|} \\
&= \frac{2|a_k|}{t^{n-k-1}|a_n|}+2\sum_{\nu=0}^{n}\frac{\left|a_\nu-|a_\nu|\right|}{t^{n-\nu-1}|a_n|}-\left(\frac{|a_{n-1}|+t\left|a_n-|a_n|\right|}{|a_n|}\right) \\
&\quad +\frac{|a_{n-1}|-|a_n|t}{|a_n|}+\frac{\left|a_n-|a_n|\right|t}{|a_n|} \\
&\leqslant \frac{2t|a_k|}{t^{n-k}|a_n|}+\frac{2}{|a_n|}\sum_{\nu=0}^{n}\frac{\left|a_\nu-|a_\nu|\right|}{t^{n-\nu-1}}-t = t\left\{\frac{2t^k|a_k|}{t^n|a_n|}-1\right\}+\frac{2}{|a_n|}\sum_{\nu=0}^{n}\frac{\left|a_\nu-|a_\nu|\right|}{t^{n-\nu-1}}.
\end{aligned}$$

This shows that the point z belongs to the circle defined by (1). Hence the circle defined by (3) is contained in the circle defined by (1).

For polynomial $P(z)=\sum_{j=0}^{n} a_j z^j$ with real and positive coefficients, we obtain the following result.

COROLLARY 1. *Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with real and positive coefficients. If $t>0$ can be found such that*

$$t^n a_n \leqslant t^{n-1}a_{n-1} \leqslant \cdots \leqslant t^{k+1}a_{k+1} \leqslant t^k a_k, \quad t^k a_k \geqslant t^{k-1}a_{k-1} \geqslant \cdots \geqslant ta_1 \geqslant a_0,$$

$0 \leqslant k \leqslant n-1$, *then all the zero of $P(z)$ lie in*

$$\left|z+\frac{a_{n-1}}{a_n}-t\right| \leqslant \frac{2t^{k+1}a_k}{t^n a_n}-\frac{a_{n-1}}{a_n}.$$

Next, we present the following result which improves the bound of Theorem C for $0 \leqslant k,m \leqslant n-1$.

THEOREM 2. *Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with complex coefficients. If $\operatorname{Re} a_j=\alpha_j$ and $\operatorname{Im} a_j=\beta_j$ for $j=0,1,2,\cdots,n$. If $t>0$ can be found such that*

$$0<t^n\alpha_n \leqslant t^{n-1}\alpha_{n-1} \leqslant \cdots \leqslant t^{k+1}\alpha_{k+1} \leqslant t^k\alpha_k, \quad t^k\alpha_k \geqslant t^{k-1}\alpha_{k-1} \geqslant \cdots \geqslant t\alpha_1 \geqslant \alpha_0>0$$

and

$$\begin{aligned} 0<t^n\beta_n &\leqslant t^{n-1}\beta_{n-1} \leqslant \cdots \leqslant t^{m+1}\beta_{m+1} \leqslant t^m\beta_m, \\ &t^m\beta_m \geqslant t^{m-1}\beta_{m-1} \geqslant \cdots \geqslant t\beta_1 \geqslant \beta_0>0, \end{aligned}$$

$0 \leqslant k,m \leqslant n-1,\ a_{-1}=a_{n+1}=0$, *then all the zero of $P(z)$ lie in*

$$\left|z+\frac{a_{n-1}-ta_n}{a_n}\right| \leqslant \frac{2}{|a_n|t^n}\left\{\alpha_k t^{k+1}+\beta_m t^{m+1}\right\}-\frac{\alpha_{n-1}+\beta_{n-1}}{|a_n|}. \tag{8}$$

Proof. Consider the polynomial

$$\begin{aligned} F(z) =&(t-z)P(z)=(t-z)\left(a_n z^n+a_{n-1}z^{n-1}+\cdots+a_1 z+a_0\right) \\ =&-a_n z^{n+1}+(ta_n-a_{n-1})z^n+\sum_{\nu=0}^{n-1}(ta_\nu-a_{\nu-1})z^\nu \quad (a_{-1}=0). \end{aligned}$$

Let $|z|>t$, then

$$\begin{aligned} |F(z)| \geqslant& |z|^n\left[|a_n z+a_{n-1}-ta_n|-\sum_{\nu=0}^{n-1}|ta_\nu-a_{\nu-1}|\frac{1}{|z|^{n-\nu}}\right] \\ >& |z|^n\left[|a_n z+a_{n-1}-ta_n|-\frac{1}{t^n}\sum_{\nu=0}^{n-1}|ta_\nu-a_{\nu-1}|t^\nu\right]. \end{aligned} \tag{9}$$

Now by hypothesis

$$
\begin{aligned}
\sum_{\nu=0}^{n-1}|ta_{\nu}-a_{\nu-1}|t^{\nu} \leqslant & \sum_{\nu=0}^{n-1}|t\alpha_{\nu}-\alpha_{\nu-1}|t^{\nu}+\sum_{\nu=0}^{n-1}|t\beta_{\nu}-\beta_{\nu-1}|t^{\nu} \\
&= \sum_{\nu=0}^{k}|t\alpha_{\nu}-\alpha_{\nu-1}|t^{\nu}+\sum_{\nu=k+1}^{n-1}|t\alpha_{\nu}-\alpha_{\nu-1}|t^{\nu}+\sum_{\nu=0}^{m}|t\beta_{\nu}-\beta_{\nu-1}|t^{\nu} \\
&\quad +\sum_{\nu=m+1}^{n-1}|t\beta_{\nu}-\beta_{\nu-1}|t^{\nu} \\
&= 2\alpha_k t^{k+1}+2\beta_m t^{m+1}-(\alpha_{n-1}+\beta_{n-1})t^{n}.
\end{aligned}
$$

Using this in (9), we obtain

$$
|F(z)| \geqslant |z|^{n+1}\left\{|a_n z+a_{n-1}-ta_n|-2\alpha_k\frac{t^{k+1}}{t^n}-2\beta_m\frac{t^{m+1}}{t^n}+(\alpha_{n-1}+\beta_{n-1})\right\}>0,
$$

if

$$
\left|z+\frac{a_{n-1}-ta_n}{a_n}\right| > \frac{2}{|a_n|t^n}\left\{\alpha_k t^{k+1}+\beta_m t^{m+1}\right\}-\frac{\alpha_{n-1}+\beta_{n-1}}{|a_n|}.
$$

Hence all the zeros of $F(z)$ whose modulus is greater than t lie in the circle

$$
\left|z+\frac{a_{n-1}-ta_n}{a_n}\right| \leqslant \frac{2}{|a_n|t^n}\left\{\alpha_k t^{k+1}+\beta_m t^{m+1}\right\}-\frac{\alpha_{n-1}+\beta_{n-1}}{|a_n|}. \tag{10}
$$

Now, if $|z|\leqslant t$, then we have

$$
\begin{aligned}
\left|a_n z+a_{n-1}-ta_n\right| &\leqslant |a_n|t+|a_{n-1}-ta_n| \leqslant t\alpha_n+t\beta_n+(\alpha_{n-1}-t\alpha_n)+(\beta_{n-1}-t\beta_n) \\
&= \alpha_{n-1}+\beta_{n-1},
\end{aligned}
$$

this gives,

$$
\left|z+\frac{a_{n-1}}{a_n}-t\right| \leqslant \frac{\alpha_{n-1}+\beta_{n-1}}{|a_n|}. \tag{11}
$$

By hypothesis for $0\leqslant k,m\leqslant n-1$,

$$
t^{n-1}\alpha_{n-1}\leqslant \alpha_k t^k \quad \text{and} \quad t^{n-1}\beta_{n-1}\leqslant \beta_m t^m.
$$

Therefore,

$$
2(t^{n-1}\alpha_{n-1}+t^{n-1}\beta_{n-1})\leqslant 2(\alpha_k t^k+\beta_m t^m).
$$

Equivalently,

$$
\alpha_{n-1}+\beta_{n-1}\leqslant \frac{2}{t^n}(\alpha_k t^{k+1}+\beta_m t^{m+1})-(\alpha_{n-1}+\beta_{n-1}). \tag{12}
$$

Using (12) in (11), we obtain for $0 \leqslant k \leqslant n-1$,

$$\left|z+\frac{a_{n-1}-ta_n}{a_n}\right| \leqslant \frac{2}{|a_n|t^n}\left\{\alpha_k t^{k+1}+\beta_m t^{m+1}\right\}-\frac{\alpha_{n-1}+\beta_{n-1}}{|a_n|}.$$

This shows that all the zeros of $F(z)$ whose modulus is less than or equal to t also satisfy the inequality (10). Thus we conclude that all the zeros of $F(z)$ and hence that of $P(z)$ lie in the circle defined by (10). This completes the proof of Theorem 2. □

REMARK 2. In general Theorem 2 also gives much better result than Theorem B for $0 \leqslant k \leqslant n-1$. To see this, we show that the circle defined by (8) is contained in the circle defined by (2). Let z be any point belonging to the circle defined by (8), then

$$\left|z+\frac{a_{n-1}-ta_n}{a_n}\right| \leqslant \frac{2}{|a_n|t^n}\left\{\alpha_k t^{k+1}+\beta_m t^{m+1}\right\}-\frac{\alpha_{n-1}+\beta_{n-1}}{|a_n|}.$$

This implies

$$\begin{aligned}|z| &= \left|z+\frac{a_{n-1}-ta_n}{a_n}-\frac{a_{n-1}-ta_n}{a_n}\right| \leqslant \left|z+\frac{a_{n-1}-ta_n}{a_n}\right|+\left|\frac{a_{n-1}-ta_n}{a_n}\right| \\ &\leqslant \frac{2}{|a_n|t^n}\left\{\alpha_k t^{k+1}+\beta_m t^{m+1}\right\}-\frac{\alpha_{n-1}+\beta_{n-1}}{|a_n|}+\frac{\alpha_{n-1}-t\alpha_n}{|a_n|}+\frac{\beta_{n-1}-t\beta_n}{|a_n|} \\ &= \frac{2}{|a_n|t^n}\left\{\alpha_k t^{k+1}+\beta_m t^{m+1}\right\}-\frac{t(\alpha_n+\beta_n)}{|a_n|},\end{aligned}$$

which shows that the point z belongs to the circle defined by (2). Hence the circle defined by (8) is contained in the circle defined (2).

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