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HYPERSTABILITY OF A MIXED TYPE CUBIC–QUARTIC FUNCTIONAL EQUATION IN ULTRAMETRIC SPACES

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Abstract. In this paper, we present the hyperstability results of a mixed type cubic- quartic functional equations in ultrametric Banach spaces.

1. Introduction

The starting point of studying the stability of functional equations seems to be the famous talk of Ulam [25] in 1940, in which he discussed a number of important unsolved problems. Among those was the question concerning the stability of group homomorphisms.

Let G_1 be a group and let G_2 be a metric group with a metric $d(.,.)$. Given $\varepsilon > 0$, does there exists a $\delta > 0$ such that if a mapping $h : G_1 \longrightarrow G_2$ satisfies the inequality $d\big(h(xy), h(x)h(y)\big) < \delta$, for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \longrightarrow G_2$ with $d\big(h(x), H(x)\big) < \varepsilon$, for all $x \in G_1$?

The first partial answer, in the case of Cauchy equation in Banach spaces, to Ulam question was given by Hyers [18]. Later, the result of Hyers was first generalized by Aoki. And only much later by Rassias [23] and Găvruţa [16]. Since then, the stability problems of several functional equations have been extensively investigated.

We say a functional equation is *hyperstable* if any function f satisfying the equation approximately (in some sense) must be actually a solution to it. It seems that the first hyperstability result was published in [8] and concerned the ring homomorphisms. However, the term *hyperstability* has been used for the first time in [21]. Quite often the hyperstability is confused with superstability, which admits also bounded functions. Numerous papers on this subject have been published and we refer to [4], [11]-[14], [17], [21], [22], [24].

Throughout this paper, $\mathbb{N}$ stands for the set of all positive integers, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, N_{m_0} the set of integers $\geqslant m_0$, $\mathbb{R}_+ := [0, \infty)$ and we use the notation X_0 for the set $X \setminus \{0\}$.

Let us recall (see, for instance, [20]) some basic definitions and facts concerning non-Archimedean normed spaces.

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DEFINITION 1. By a *non-Archimedean* field we mean a field $\mathbb{K}$ equipped with a function (*valuation*) $|\cdot|:\mathbb{K}\longrightarrow[0,\infty)$ such that for all $r,s\in\mathbb{K}$, the following conditions hold:

1. $|r|=0$ if and only if $r=0$;
2. $|rs|=|r||s|$;
3. $|r+s|\leqslant\max\{|r|,|s|\}$.

The pair $(\mathbb{K},|.|)$ is called a *valued field*.

In any non-Archimedean field we have $|1|=|-1|=1$ and $|n|\leqslant 1$ for $n\in\mathbb{N}_0$. In any field $\mathbb{K}$ the function $|\cdot|:\mathbb{K}\longrightarrow\mathbb{R}_+$ given by

$$|x|:=\begin{cases}0, & x=0,\\ 1, & x\neq 0,\end{cases}$$

is a valuation which is called *trivial*, but the most important examples of non-Archimedean fields are p-adic numbers which have gained the interest of physicists for their research in some problems coming from quantum physics, p-adic strings and superstrings.

DEFINITION 2. Let X be a vector space over a scalar field $\mathbb{K}$ with a non-Archimedean non-trivial valuation $|\cdot|$. A function $||\cdot||_*:X\to\mathbb{R}$ is a *non-Archimedean norm (valuation)* if it satisfies the following conditions:

1. $||x||_*=0$ if and only if $x=0$;
2. $||rx||_*=|r|\ ||x||_*$ $(r\in\mathbb{K},x\in X)$;
3. the strong triangle inequality (ultrametric), namely

$$||x+y||_*\leqslant\max\{||x||_*,||y||_*\},\ \ x,y\in X.$$

Then $(X,||\cdot||_*)$ is called *a non-Archimedean normed space* or *an ultrametric normed space*.

DEFINITION 3. Let $\{x_n\}$ be a sequence in a non-Archimedean normed space X.

1. A sequence $\{x_n\}_{n=1}^{\infty}$ in a non-Archimedean space is *a Cauchy sequence* iff the sequence $\{x_{n+1}-x_n\}_{n=1}^{\infty}$ converges to zero.
2. The sequence $\{x_n\}$ is said to be *convergent* if, there exists $x\in X$ such that, for any $\varepsilon>0$, there is a positive integer N such that $||x_n-x||_*\leqslant\varepsilon$, for all $n\geqslant N$. Then the point $x\in X$ is called the *limit* of the sequence $\{x_n\}$, which is denoted by $lim_{n\to\infty}x_n=x$.
3. If every Cauchy sequence in X converges, then the non-Archimedean normed space X is called a *non-Archimedean Banach space* or an *ultrametric Banach space*.

In 2013, A. Bahyrycz and al. [3] used the fixed point theorem from [9, Theorem 1] to prove the stability results for a generalization of p-Wright affine equation in ultrametric spaces. Recently, corresponding results for more general functional equations (in classical spaces) have been proved in [5], [6], [26] and [27].

Let X, Y be normed spaces. A function $f: X \longrightarrow Y$ is mixed type cubic-quartic provided it satisfies the functional equation

$$4f(x+y)+4f(x-y)+3f(2y)=24f(y)+6f(x)+f(x+2y)+f(x-2y), \text{ for all } x,y\in X, \tag{1}$$

and we can say that $f: X \longrightarrow Y$ is mixed type cubic-quartic on X_0 if it satisfies (1) for all $x,y \in X_0$ such that $x+y \neq 0$ and $x-y \neq 0$.

In 2009 Eshaghi Godji et al.[15] proved the solution and stability of a mixed type cubic and quartic functional equation in quasi-Banach spaces. The stability of general form of (1) has been studied by A. Bodaghi et al. [15] in 2017.

In this paper, by using the fixed point method derived from [4] and [11] , we present some hyperstability results for the equation (1) in ultrametric Banach spaces. Before proceeding to the main results, we state Theorem 1 which is useful for our purpose. To present it, we introduce the following three hypotheses.

(**H1**) X is a nonempty set, Y is an ultrametric Banach space over a non-Archimedean field, $f_1,\ldots,f_k : X \longrightarrow X$ and $L_1,\ldots,L_k : X \longrightarrow \mathbb{R}_+$ are given.

(**H2**) $\mathscr{T} : Y^X \longrightarrow Y^X$ is an operator satisfying the inequality

$$\|\mathscr{T}\xi(x)-\mathscr{T}\mu(x)\|_* \leqslant \max_{1\leqslant i\leqslant k}\{L_i(x)\|\xi(f_i(x))-\mu(f_i(x))\|_*\}, \xi,\mu\in Y^X, x\in X.$$

(**H3**) $\Lambda : \mathbb{R}_+^X \longrightarrow \mathbb{R}_+^X$ is a linear operator defined by

$$\Lambda\delta(x) := \max_{1\leqslant i\leqslant k}\{L_i(x)\delta(f_i(x))\},\ \delta\in\mathbb{R}_+^X, x\in X.$$

THEOREM 1. *Let hypotheses (**H1**)-(**H3**) be valid and functions $\varepsilon : X \longrightarrow \mathbb{R}_+$ and $\varphi : X \longrightarrow Y$ fulfill the following two conditions*

$$\|\mathscr{T}\varphi(x)-\varphi(x)\|_* \leqslant \varepsilon(x),\ x\in X,$$
$$\lim_{n\to\infty}\Lambda^n\varepsilon(x)=0,\ x\in X.$$

Then there exists a unique fixed point $\psi\in Y^X$ of $\mathscr{T}$ with

$$\|\varphi(x)-\psi(x)\|_* \leqslant \sup_{n\in\mathbb{N}_0}\Lambda^n\varepsilon(x),\ x\in X.$$

Moreover

$$\psi(x) := \lim_{n\to\infty}\mathscr{T}^n\varphi(x),\ x\in X.$$

2. Main results

In this section, using Theorem 1 as a basic tool to prove the hyperstability results of the cubic functional equation in ultrametric Banach spaces.

THEOREM 2. *Let $(X,\|\cdot\|)$ and $(Y,\|\cdot\|_*)$ be normed space and ultrametric Banach space, respectively, $c \geqslant 0$, $p,q \in \mathbb{R}$, $p+q \neq 0$ and let $f : X \longrightarrow Y$ satisfy*

$$\begin{aligned} &\|4f(x+y)+4f(x-y)+3f(2y)-24f(y)-6f(x)-f(x+2y)-f(x-2y)\|_* \\ &\leqslant c\,\|x\|^p\,\|y\|^q, \end{aligned} \tag{2}$$

for all $x,y \in X_0$. Then, f is mixed type cubic-quartic on X_0.

Proof. First case: $p+q<0$.
Take $m \in \mathbb{N}$ such that

$$\alpha_m := \left(\frac{m-1}{4}\right)^{p+q} < 1 \text{ and } m \geqslant 6.$$

Since $p+q<0$, one of p,q must be negative. Assume that $p<0$ and replace y by $\left(\frac{m-1}{4}\right)x$ and x by $\left(\frac{m+1}{2}\right)x$ in (2). Thus,

$$\begin{aligned} &\left\| 4f\left(\left(\frac{3m+1}{4}\right)x\right)+4f\left(\left(\frac{m+3}{4}\right)x\right)+3f\left(\left(\frac{m-1}{2}\right)x\right)-24f\left(\left(\frac{m-1}{4}\right)x\right)\right. \\ &\left. -6f\left(\left(\frac{m+1}{2}x\right)\right)-f(mx)-f(x)\right\|_* \leqslant c\left(\frac{m+1}{2}\right)^p\left(\frac{m-1}{4}\right)^q\|x\|^{p+q}. \end{aligned} \tag{3}$$

Define operators $\mathscr{T}_m : Y^{X_0} \longrightarrow Y^{X_0}$ and $\Lambda_m : \mathbb{R}_+^{X_0} \longrightarrow \mathbb{R}_+^{X_0}$ by

$$\begin{aligned} \mathscr{T}_m\xi(x) :=&4\xi\left(\left(\frac{3m+1}{4}\right)x\right)+4\xi\left(\left(\frac{m+3}{4}\right)x\right)+3\xi\left(\left(\frac{m-1}{2}\right)x\right) \\ &-24\xi\left(\left(\frac{m-1}{4}\right)x\right)-6\xi\left(\left(\frac{m+1}{2}\right)x\right)-\xi(mx),\ \xi \in Y^{X_0},\ x \in X_0, \end{aligned}$$

$$\begin{aligned} \Lambda_m\delta(x) :=\max\Bigg\{&\delta\left(\left(\frac{3m+1}{4}\right)x\right), \delta\left(\left(\frac{m+3}{4}\right)x\right), \delta\left(\left(\frac{m-1}{2}\right)x\right), \\ &\delta\left(\left(\frac{m-1}{4}\right)x\right), \delta\left(\left(\frac{m+1}{2}\right)x\right), \delta(mx)\Bigg\},\ \delta \in \mathbb{R}_+^{X_0},\ x \in X_0. \end{aligned}$$

And write

$$\varepsilon_m(x) := c\left(\frac{m-1}{4}\right)^q\left(\frac{m+1}{2}\right)^p\|x\|^{p+q},\ x \in X_0. \tag{4}$$

It is easily seen that Λ_m has the form described in (**H3**) with $k=6$,

$$f_1(x)=\left(\frac{3m+1}{4}\right)x,\ f_2(x)=\left(\frac{m+3}{4}\right)x,\ f_3(x)=\left(\frac{m-1}{2}\right)x,$$
$$f_4(x)=\left(\frac{m-1}{4}\right)x,\ f_5(x)=\left(\frac{m+1}{2}\right)x,\ f_6(x)=mx$$

and

$$L_1(x)=L_2(x)=L_3(x)=L_4(x)=L_5(x)=L_6(x)=1.$$

Further, (3) can be written in the following way

$$\|\mathscr{T}_m f(x)-f(x)\|_*\leqslant \varepsilon_m(x),\ x\in X_0.$$

Moreover, for every $\xi,\mu\in Y^{X_0}$, $x\in X_0$

$$\begin{aligned}
&\|\mathscr{T}_m\xi(x)-\mathscr{T}_m\mu(x)\|_*\\
&=\Bigg\|4\xi\left(\left(\frac{3m+1}{4}\right)x\right)+4\xi\left(\left(\frac{m+3}{4}\right)x\right)+3\xi\left(\left(\frac{m-1}{2}\right)x\right)-24\xi\left(\left(\frac{m-1}{4}\right)x\right)\\
&\quad-6\xi\left(\left(\frac{m+1}{2}\right)x\right)-\xi(mx)-4\mu\left(\left(\frac{3m+1}{4}\right)x\right)-4\mu\left(\left(\frac{m+3}{4}\right)x\right)\\
&\quad-3\mu\left(\left(\frac{m-1}{2}\right)x\right)+24\mu\left(\left(\frac{m-1}{4}\right)x\right)+6\mu\left(\left(\frac{m+1}{2}\right)x\right)+\mu(mx)\Bigg\|_*\\
&\leqslant\max\Bigg\{\left\|\xi\left(\left(\frac{3m+1}{4}\right)x\right)-\mu\left(\left(\frac{3m+1}{4}\right)x\right)\right\|_*,\\
&\qquad\left\|\xi\left(\left(\frac{m+3}{4}\right)x\right)-\mu\left(\left(\frac{m+3}{4}\right)x\right)\right\|_*,\\
&\qquad\left\|\xi\left(\left(\frac{m-1}{2}\right)x\right)-\mu\left(\left(\frac{m-1}{2}\right)x\right)\right\|_*,\\
&\qquad\left\|\xi\left(\left(\frac{m-1}{4}\right)x\right)-\mu\left(\left(\frac{m-1}{4}\right)x\right)\right\|_*,\\
&\qquad\left\|\xi\left(\left(\frac{m+1}{2}\right)x\right)-\mu\left(\left(\frac{m+1}{2}\right)x\right)\right\|_*,\ \|\xi(mx)-\mu(mx)\|_*\Bigg\}.
\end{aligned}$$

So, (**H2**) is valid.

By using mathematical induction, we will show that for each $x\in X_0$ we have

$$\Lambda_m^n\varepsilon_m(x)=\ c\ \left(\frac{m-1}{4}\right)^q\left(\frac{m+1}{2}\right)^p\|x\|^{p+q}\alpha_m^n, \tag{5}$$

where $\alpha_m=\left(\frac{m-1}{4}\right)^{p+q}$. From (4), we obtain that (5) holds for $n=0$. Next, we will assume that (5) holds for $n=k$, where $k\in\mathbb{N}$. Then we have,

$$\Lambda_m^{k+1}\varepsilon_m(x)=\Lambda_m\left(\Lambda_m^k\varepsilon_m(x)\right)$$

$$
\begin{aligned}
&= \max\Big\{ \Lambda_m^k \varepsilon_m\left(\left(\frac{3m+1}{4}\right)x\right), \Lambda_m^k \varepsilon_m\left(\left(\frac{m+3}{4}\right)x\right), \Lambda_m^k \varepsilon_m\left(\left(\frac{m-1}{2}\right)x\right), \\
&\qquad \Lambda_m^k \varepsilon_m\left(\left(\frac{m-1}{4}\right)x\right) \Lambda_m^k \varepsilon_m\left(\left(\frac{m+1}{2}\right)x\right), \Lambda_m^k \varepsilon_m(mx)\Big\} \\
&= c\left(\frac{m-1}{4}\right)^q \left(\frac{m+1}{2}\right)^p \|x\|^{p+q} \alpha_m^k \max\Big\{\left(\frac{3m+1}{4}\right)^{p+q}, \left(\frac{m+3}{4}\right)^{p+q}, \left(\frac{m-1}{2}\right)^{p+q}, \\
&\qquad \left(\frac{m-1}{4}\right)^{p+q}, \left(\frac{m+1}{2}\right)^{p+q}, m^{p+q}\Big\} \\
&= c\left(\frac{m-1}{4}\right)^q \left(\frac{m+1}{2}\right)^p \|x\|^{p+q} \alpha_m^{k+1}, \ x \in X_0.
\end{aligned}
$$

This shows that (5) holds for $n = k+1$. Now we can conclude that the inequality (5) holds for all $n \in \mathbb{N}_0$. From (5), we obtain

$$
\lim_{n \longrightarrow \infty} \Lambda^n \varepsilon_m(x) = 0,
$$

for all $x \in X_0$. Hence, according to Theorem 1, there exists a unique solution $C_m : X_0 \to Y$ of the equation

$$
\begin{aligned}
C_m(x) =& 4C_m\left(\left(\frac{3m+1}{4}\right)x\right) + 4C_m\left(\left(\frac{m+3}{4}\right)x\right) + 3C_m\left(\left(\frac{m-1}{2}\right)x\right) \\
& - 24C_m\left(\left(\frac{m-1}{4}\right)x\right) - 6C_m\left(\left(\frac{m+1}{2}\right)x\right) - C_m(mx),
\end{aligned}
$$

for all $x \in X_0$ such that

$$
\|f(x) - C_m(x)\|_* \leqslant \sup_{n \in \mathbb{N}_0} \left\{ c\left(\frac{m-1}{4}\right)^p \left(\frac{m+1}{2}\right)^p \|x\|^{p+q} \alpha_m^n \right\}, \quad x \in X_0. \tag{6}
$$

Moreover,

$$
C_m(x) := \lim_{n \to \infty} \mathscr{T}_m^n f(x),
$$

for all $x \in X_0$. Now we show that

$$
\begin{aligned}
&\|\mathscr{T}_m^n f(x+2y) + \mathscr{T}_m^n f(x-2y) + 24\mathscr{T}_m^n f(y) + 6\mathscr{T}_m^n f(x) - 4\mathscr{T}_m^n f(x \pm y) \\
&-3\mathscr{T}_m^n f(2y)\|_* \leqslant c\, \alpha_m^n \|x\|^p \|y\|^q,
\end{aligned} \tag{7}
$$

for every $x, y \in X_0$ such that $x+y \neq 0$ and $x-y \neq 0$. Since, the case $n = 0$ is just (2), take $k \in \mathbb{N}$ and assume that (7) holds for $n = k$ and every $x, y \in X_0$ such that $x+y \neq 0$ and $x-y \neq 0$. Then,

$$
\begin{aligned}
&\Big\|\mathscr{T}_m^{n+1} f(x+2y) + \mathscr{T}_m^{n+1} f(x-2y) + 24\mathscr{T}_m^{n+1} f(y) + 6\mathscr{T}_m^{n+1} f(x) - 4\mathscr{T}_m^{n+1} f(x+y) \\
&-4\mathscr{T}_m^{n+1} f(x-y) - 3\mathscr{T}_m^{n+1} f(2y)\Big\|_* \\
&= \Big\| 4\mathscr{T}_m^k f\left(\left(\frac{3m+1}{4}\right)(x+2y)\right) + 4\mathscr{T}_m^k f\left(\left(\frac{m+3}{4}\right)(x+2y)\right)
\end{aligned}
$$

$$
\begin{aligned}
&+3\mathscr{T}_m^k f\left(\left(\frac{m-1}{2}\right)(x+2y)\right)-24\mathscr{T}_m^k f\left(\left(\frac{m-1}{4}\right)(x+2y)\right)\\
&-6\mathscr{T}_m^k f\left(\left(\frac{m+1}{2}\right)(x+2y)\right)-\mathscr{T}_m^k f(m(x+2y))+4\mathscr{T}_m^k f\left(\left(\frac{3m+1}{4}\right)(x-2y)\right)\\
&+4\mathscr{T}_m^k f\left(\left(\frac{m+3}{4}\right)(x-2y)\right)+3\mathscr{T}_m^k f\left(\left(\frac{m-1}{2}\right)(x-2y)\right)\\
&-24\mathscr{T}_m^k f\left(\left(\frac{m-1}{4}\right)(x-2y)\right)-6\mathscr{T}_m^k f\left(\left(\frac{m+1}{2}\right)(x-2y)\right)-\mathscr{T}_m^k f(m(x-2y))\\
&+96\mathscr{T}_m^k f\left(\left(\frac{3m+1}{4}\right)y\right)+96\mathscr{T}_m^k f\left(\left(\frac{m+3}{4}\right)y\right)+72\mathscr{T}_m^k f\left(\left(\frac{m-1}{2}\right)y\right)\\
&-576\mathscr{T}_m^k f\left(\left(\frac{m-1}{4}\right)y\right)-144\mathscr{T}_m^k f\left(\left(\frac{m+1}{2}\right)y\right)-24\mathscr{T}_m^k f(my)\\
&+24\mathscr{T}_m^k f\left(\left(\frac{3m+1}{4}\right)x\right)+24\mathscr{T}_m^k f\left(\left(\frac{m+3}{4}\right)x\right)+18\mathscr{T}_m^k f\left(\left(\frac{m-1}{2}\right)x\right)\\
&-144\mathscr{T}_m^k f\left(\left(\frac{m-1}{4}\right)x\right)-36\mathscr{T}_m^k f\left(\left(\frac{m+1}{2}\right)x\right)-6\mathscr{T}_m^k f(mx)\\
&-16\mathscr{T}_m^k f\left(\left(\frac{3m+1}{4}\right)(x+y)\right)-16\mathscr{T}_m^k f\left(\left(\frac{m+3}{4}\right)(x+y)\right)\\
&-18\mathscr{T}_m^k f\left(\left(\frac{m-1}{2}\right)(x+y)\right)+96\mathscr{T}_m^k f\left(\left(\frac{m-1}{4}\right)(x+y)\right)\\
&+24\mathscr{T}_m^k f\left(\left(\frac{m+1}{2}\right)(x+y)\right)+4\mathscr{T}_m^k f(m(x+y))-16\mathscr{T}_m^k f\left(\left(\frac{3m+1}{4}\right)(x-y)\right)\\
&-16\mathscr{T}_m^k f\left(\left(\frac{m+3}{4}\right)(x-y)\right)-18\mathscr{T}_m^k f\left(\left(\frac{m-1}{2}\right)(x-y)\right)\\
&+96\mathscr{T}_m^k f\left(\left(\frac{m-1}{4}\right)(x-y)\right)+24\mathscr{T}_m^k f\left(\left(\frac{m+1}{2}\right)(x-y)\right)+4\mathscr{T}_m^k f(m(x-y))\\
&-12\mathscr{T}_m^k f\left(\left(\frac{3m+1}{2}\right)y\right)-12\mathscr{T}_m^k f\left(\left(\frac{m+3}{2}\right)y\right)-9\mathscr{T}_m^k f((m-1)y)\\
&+72\mathscr{T}_m^k f\left(\left(\frac{m-1}{2}\right)y\right)+18\mathscr{T}_m^k f((m+1)y)+3\mathscr{T}_m^k f(2my)\Big\|_*\\
\leqslant&\max\left\{\mathscr{T}_{m5},\mathscr{T}_{m6},\mathscr{T}_{m7},\mathscr{T}_{m8},\mathscr{T}_{m9},\mathscr{T}_{m10}\right\}\\
\leqslant&\max\left\{c\,\alpha_m^k\|x\|^p\,\|y\|^q\left(\frac{3m+1}{4}\right)^{p+q},\ c\,\alpha_m^k\|x\|^p\|y\|^q\left(\frac{m+3}{2}\right)^{p+q},\right.\\
&\quad c\,\alpha_m^k\|x\|^p\|y\|^q\left(\frac{m-1}{2}\right)^{p+q},c\,\alpha_m^k\|x\|^p\|y\|^q\left(\frac{m-1}{4}\right)^{p+q},\\
&\quad\left.c\,\alpha_m^k\|x\|^p\|y\|^q\left(\frac{m+1}{2}\right)^{p+q},\ c\,\alpha_m^k\|x\|^p\|y\|^q\,m^{p+q}\right\}
\end{aligned}
$$

$$= c\alpha_m^k \|x\|^p \|y\|^q \max\left\{\left(\frac{3m+1}{4}\right)^{p+q}, \left(\frac{m+3}{4}\right)^{p+q}, \left(\frac{m-1}{2}\right)^{p+q}, \left(\frac{m-1}{4}\right)^{p+q}, \left(\frac{m+1}{2}\right)^{p+q}, m^{p+q}\right\} \leqslant c\alpha_m^{k+1} \|x\|^p \|y\|^q.$$

With

$$\begin{aligned}
\mathscr{T}_{m5} = \Bigg\| & \mathscr{T}_m^n f\left(\left(\frac{3m+1}{4}\right)(x+2y)\right) + \mathscr{T}_m^n f\left(\left(\frac{3m+1}{4}\right)(x-2y)\right) \\
& +24\mathscr{T}_m^n f\left(\left(\frac{3m+1}{4}\right)y\right) + 6\mathscr{T}_m^n f\left(\left(\frac{3m+1}{4}\right)x\right) \\
& -4\mathscr{T}_m^n f\left(\left(\frac{3m+1}{4}\right)(x+y)\right) - 4\mathscr{T}_m^n f\left(\left(\frac{3m+1}{4}\right)(x-y)\right) \\
& -3\mathscr{T}_m^n f\left(\left(\frac{3m+1}{2}\right)y\right)\Bigg\|_*,
\end{aligned}$$

$$\begin{aligned}
\mathscr{T}_{m6} = \Bigg\| & \mathscr{T}_m^n f\left(\left(\frac{m+3}{4}\right)(x+2y)\right) + \mathscr{T}_m^n f\left(\left(\frac{m+3}{4}\right)(x-2y)\right) \\
& +24\mathscr{T}_m^n f\left(\left(\frac{m+3}{4}\right)y\right) + 6\mathscr{T}_m^n f\left(\left(\frac{m+3}{4}\right)x\right) - 4\mathscr{T}_m^n f\left(\left(\frac{m+3}{4}\right)(x+y)\right) \\
& -4\mathscr{T}_m^n f\left(\left(\frac{m+3}{4}\right)(x-y)\right) - 3\mathscr{T}_m^n f\left(\left(\frac{m+3}{2}\right)y\right)\Bigg\|_*,
\end{aligned}$$

$$\begin{aligned}
\mathscr{T}_{m7} = \Bigg\| & \mathscr{T}_m^n f\left(\left(\frac{m-1}{2}\right)(x+2y)\right) + \mathscr{T}_m^n f\left(\left(\frac{m-1}{2}\right)(x-2y)\right) \\
& +24\mathscr{T}_m^n f\left(\left(\frac{m-1}{2}\right)y\right) + 6\mathscr{T}_m^n f\left(\left(\frac{m-1}{2}\right)x\right) - 4\mathscr{T}_m^n f\left(\left(\frac{m-1}{2}\right)(x+y)\right) \\
& -4\mathscr{T}_m^n f\left(\left(\frac{m-1}{2}\right)(x-y)\right) - 3\mathscr{T}_m^n f((m-1)y)\Bigg\|_*,
\end{aligned}$$

$$\begin{aligned}
\mathscr{T}_{m8} = \Bigg\| & \mathscr{T}_m^n f\left(\left(\frac{m-1}{4}\right)(x+2y)\right) + \mathscr{T}_m^n f\left(\left(\frac{m-1}{4}\right)(x-2y)\right) \\
& +24\mathscr{T}_m^n f\left(\left(\frac{m-1}{4}\right)y\right) + 6\mathscr{T}_m^n f\left(\left(\frac{m-1}{4}\right)x\right) - 4\mathscr{T}_m^n f\left(\left(\frac{m-1}{4}\right)(x+y)\right) \\
& -4\mathscr{T}_m^n f\left(\left(\frac{m-1}{4}\right)(x-y)\right) - 3\mathscr{T}_m^n f\left(\left(\frac{m-1}{2}\right)y\right)\Bigg\|_*,
\end{aligned}$$

$$\begin{aligned}
\mathscr{T}_{m9} = \Bigg\| & \mathscr{T}_m^n f\left(\left(\frac{m+1}{2}\right)(x+2y)\right) + \mathscr{T}_m^n f\left(\left(\frac{m+1}{2}\right)(x-2y)\right) \\
& +24\mathscr{T}_m^n f\left(\left(\frac{m+1}{2}\right)y\right) + 6\mathscr{T}_m^n f\left(\left(\frac{m+1}{2}\right)x\right) - 4\mathscr{T}_m^n f\left(\left(\frac{m+1}{2}\right)(x+y)\right) \\
& -4\mathscr{T}_m^n f\left(\left(\frac{m+1}{2}\right)(x-y)\right) - 3\mathscr{T}_m^n f((m+1)y)\Bigg\|_*
\end{aligned}$$

and

$$\begin{aligned}\mathscr{T}_{m10} = \| & \mathscr{T}_m^n f(m(x+2y)) + \mathscr{T}_m^n f(m(x-2y)) + 24\mathscr{T}_m^n f(my) \\ & +6\mathscr{T}_m^n f(mx) - 4\mathscr{T}_m^n f(m(x+y)) - 4\mathscr{T}_m^n f(m(x-y)) \\ & -3\mathscr{T}_m^n f(2my)\|_*,\end{aligned}$$

for all $x, y \in X_0$ such that $x+y \neq 0$ and $x-y \neq 0$. Thus, by induction we have shown that (7) holds for every $n \in \mathbb{N}_0$. Letting $n \longrightarrow \infty$ in (7), we obtain that

$$C_m(x+2y) + C_m(x-2y) = 4C_m(x+y) + 4C_m(x-y) - 24C_m(y) - 6C_m(x) + 3C_m(2y),$$

for all $x, y \in X_0$ such that $x+y \neq 0$ and $x-y \neq 0$. In this way we obtain a sequence $\{C_m\}_{m \geqslant m_0}$ of a mixed type cubic-quartic functions on X_0 such that

$$\|f(x) - C_m(x)\|_* \leqslant \sup_{n \in \mathbb{N}_0} \left\{ c \left(\frac{m-1}{4}\right)^q \left(\frac{m+1}{2}\right)^p \|x\|^{p+q} \alpha_m^n \right\}, \ x \in X_0.$$

This implies that

$$\|f(x) - C_m(x)\|_* \leqslant c \left(\frac{m-1}{4}\right)^q \left(\frac{m+1}{2}\right)^p \|x\|^{p+q}, \ x \in X_0.$$

It follows, with $m \longrightarrow \infty$, that f is a mixed type cubic-quartic on X_0.

Second case: $p+q > 0$.

In a similar way we can prove the second case. Take $m \in \mathbb{N}$ such that

$$\alpha_m := \left(\frac{m-2}{m}\right)^{p+q} < 1 \text{ and } m \geqslant m_0.$$

Since $p+q > 0$, one of p, q must be positive; let $p > 0$ and replace x by $\left(\frac{2}{m}\right)x$ and y by $\left(\frac{m-2}{2m}\right)x$ in (2). Thus,

$$\begin{aligned}&\left\| 4f\left(\left(\frac{m+2}{2m}\right)x\right) + 4f\left(\left(\frac{6-m}{2m}\right)x\right) + 3f\left(\left(\frac{m-2}{m}\right)x\right) - 24f\left(\left(\frac{m-2}{2m}\right)x\right)\right. \\ &\left. -6f\left(\left(\frac{2}{m}\right)x\right) - f\left(\left(\frac{4-m}{m}\right)x\right) - f(x)\right\|_* \leqslant c\left(\frac{2}{m}\right)^p \left(\frac{m-2}{2m}\right)^q \|x\|^{p+q},\end{aligned} \tag{8}$$

Write

$$\begin{aligned}\mathscr{T}_m\xi(x) :=& 4\xi\left(\left(\frac{m+2}{2m}\right)x\right) + 4\xi\left(\left(\frac{6-m}{2m}\right)x\right) + 3\xi\left(\left(\frac{m-2}{m}\right)x\right) \\ & - 24\xi\left(\left(\frac{m-2}{2m}\right)x\right) - 6\xi\left(\left(\frac{2}{m}\right)x\right) - \xi\left(\left(\frac{4-m}{m}\right)x\right), \ \xi \in Y^{X_0}, x \in X_0,\end{aligned} \tag{9}$$

and

$$\varepsilon_m(x) := c\left(\frac{2}{m}\right)^q\left(\frac{m-2}{2m}\right)^p \|x\|^{p+q},\ x\in X_0, \tag{10}$$

then, (8) takes form

$$\|\mathscr{T}_m f(x) - f(x)\|_* \leqslant \varepsilon_m(x),\ x\in X_0.$$

Define

$$\Lambda_m\delta(x) = \max\left\{\delta\left(\left(\frac{m+2}{2m}\right)x\right), \delta\left(\left(\frac{6-m}{2m}\right)x\right), \delta\left(\left(\frac{m-2}{m}\right)x\right),\right.$$
$$\left.\delta\left(\left(\frac{m-2}{2m}\right)x\right), \delta\left(\left(\frac{2}{m}\right)x\right), \delta\left(\left(\frac{4-m}{m}\right)x\right)\right\},\ \delta\in\mathbb{R}_+^{X_0}, x\in X_0.$$

Then, it is easily seen that Λ_m has the form described in (**H3**) with $k=6$,

$$f_1(x) = \left(\frac{m+2}{2m}\right)x,\ f_2(x) = \left(\frac{6-m}{2m}\right)x,\ f_3(x) = \left(\frac{m-2}{m}\right)x,$$
$$f_4(x) = \left(\frac{m-2}{2m}\right)x,\ f_5(x) = \left(\frac{2}{m}\right)x,\ f_6(x) = \left(\frac{4-m}{m}\right)x$$

and

$$L_1(x) = L_2(x) = L_3(x) = L_4(x) = L_5(x) = L_6(x) = 1.$$

The rest of the proof is similar to the proof of Theorem 2. □

It easy to show the hyperstability of a mixed type cubic-quartic equation on the set containing 0. We present the following theorem and we refer to see [4, Theorem 5].

COROLLARY 1. *Let $(X,\|\cdot\|)$ and $(Y,\|\cdot\|_*)$ be normed space and ultrametric Banach space, respectively, $c\geqslant 0$, $p,q\in\mathbb{R}$, $p,q>0$ and let $f: X\longrightarrow Y$ satisfy*

$$\begin{aligned}&\|4f(x+y)+4f(x-y)+3f(2y)-24f(y)-6f(x)-f(x+2y)-f(x-2y)\|_*\\ &\leqslant c\,\|x\|^p\,\|y\|^q,\end{aligned} \tag{11}$$

for all $x,y\in X_0$.

First case: If f is odd, then f is cubic on X_0. Second case: If f is pair, then f is quartic on X_0.

Proof. First case: putting $x=0$ in (11), we get that

$$f(2x) = 8f(x),\ x\in X.$$

Second case: putting $y=0$ in (11), we get that

$$f(2x) = 16f(x),\ x\in X. \quad \square$$

The above theorems imply in particular the following corollary, which shows their simple application.

COROLLARY 2. *Let $(X,\|\cdot\|)$ and $(Y,\|\cdot\|_*)$ be normed space and ultrametric Banach space, respectively, $G : X^2 \longrightarrow Y$ and $G(x_0,y_0) \neq 0$, for some $x_0,y_0 \in X$ and*

$$\left\|G(x,y)\right\|_* \leqslant c\,\|x\|^p\,\|y\|^q,\ x,y \in X, \tag{12}$$

where $c \geqslant 0$, $p,q \in \mathbb{R}$. Assume that the numbers p,q satisfy one of the following conditions:

1. *$p+q<0$ and (2) holds, for all $x,y \in X_0$;*
2. *$p+q>0$ and (2) holds, for all $x,y \in X_0$.*

Then the functional equation

$$\begin{aligned} f(x+2y)+f(x-2y) =& 4(f(x+y)+f(x-y))-24f(y)-6f(x)\\ &+3f(2y)+G(x,y),\ x,y \in X, \end{aligned} \tag{13}$$

has no solution in the class of functions $g : X \longrightarrow Y$.

In the following theorem, we present a general hyperstability for the mixed type cubic-quartic equation where the control function is $\varphi(x)+\varphi(y)$.

THEOREM 3. *Let $(X,\|\cdot\|)$ be a normed space, $(Y,\|\cdot\|_*)$ be an ultrametric Banach space over a field $\mathbb{K}$ and $\varphi : X \longrightarrow \mathbb{R}_+$ be a function such that*

$$\begin{aligned} U := \{n \in \mathbb{N} : \alpha_n := \max\{&\lambda(n)\,,\,\lambda(3n+1)\,,\,\lambda(n+1)\,,\,\lambda(4n+1)\,,\\ &\lambda(2n+1)\,,\,\lambda(2n)\} < 1\} \end{aligned} \tag{14}$$

is a non-empty set, where $\lambda(a) := \inf\{t \in \mathbb{R}_+ : \varphi(ax) \leqslant t\varphi(x)$, for all $x \in X\}$, for all $a \in \mathbb{K}_0$ such that

$$\lim_{a\to\infty} \lambda(a) = 0.$$

Suppose that $f : X \longrightarrow Y$ satisfies the inequality

$$\begin{aligned} &\|4f(x+y)+4f(x-y)+3f(2y)-24f(y)-6f(x)-f(x+2y)\\ &-f(x-2y)\|_* \leqslant \varphi(x)+\varphi(y), \end{aligned} \tag{15}$$

for all $x,y \in X_0$. Then, f is a mixed type cubic-quartic on X_0.

Proof. Replacing y by mx and x by $(2m+1)x$, for $m \in \mathbb{N}$, in (15) we get

$$\begin{aligned} &\|4f((3m+1)x)+4f((m+1)x)+3f(2mx)-24f(mx)-6f((2m+1)x)\\ &-f((4m+1)x)\ -f(x)\|_* \leqslant \varphi\big((2m+1)x\big)+\varphi(mx), \end{aligned} \tag{16}$$

for all $x \in X_0$. For each $m \in U$ we define the operator $\mathscr{T}_m : Y^{X_0} \longrightarrow Y^{X_0}$ by

$$\begin{aligned} \mathscr{T}_m\xi(x) :=& 4\xi((3m+1)x)+4\xi((m+1)x)+3\xi(2mx)-24\xi(mx)-6\xi((2m+1)x)\\ &-\xi((4m+1)mx),\ \xi \in Y^{X_0},\ x \in X_0. \end{aligned}$$

Further put

$$\varepsilon_m(x) := \varphi((2m+1)x) + \varphi(mx) \leqslant (\lambda(2m+1) + \lambda(m))\varphi(x),\ x \in X_0. \tag{17}$$

Then the inequality (16) takes the form

$$\|\mathscr{T}_m f(x) - f(x)\|_* \leqslant \varepsilon_m(x),\ x \in X_0.$$

For each $m \in U$ the operator $\Lambda_m : \mathbb{R}_+^{X_0} \longrightarrow \mathbb{R}_+^{X_0}$ which is defined by

$$\Lambda_m \delta(x) := \max\{\delta(mx),\ \delta(2mx),\ \delta((3m+1)x),\ \delta((m+1)x)\},\ \delta((2m+1)x),\ \delta((4m+1)x)\},$$

where $\delta \in \mathbb{R}_+^{X_0}$, $x \in X_0$, which has the form described in (**H3**) with $k = 6$,

$$f_1(x) = mx,\ f_2(x) = 2mx, f_3(x) = (3m+1)x,$$
$$f_4(x) = (m+1)x,\ f_5(x) = (2m+1)x,\ f_6(x) = (4m+1)x$$

and

$$L_1(x) = L_2(x) = L_2(x) = L_4(x) = L_5(x) = L_6(x) = 1,$$

for all $x \in X_0$. Moreover, for every $\xi, \mu \in Y^{X_0}, x \in X_0$

$$\begin{aligned}
&\left\|\mathscr{T}_m\xi(x) - \mathscr{T}_m\mu(x)\right\|_* \\
&= \|4\xi((3m+1)x) + 4\xi\left((m+1)x\right) + 3\xi(2mx) - 24\xi(mx) - 6\xi((2m+1)x) \\
&\quad -\xi((4m+1)x) - 4\mu((3m+1)x) - 4\mu\left((m+1)x\right) - 3\mu(2mx) + 24\mu(mx) \\
&\quad +6\mu((2m+1)x) + \xi((4m+1)x)\|_* \\
&\leqslant \max\{\|\xi((3m+1)x) - \mu((3m+1)x)\|_*,\ \left\|\xi((m+1)x) - \mu((m+1)x)\right\|_*, \\
&\qquad \left\|\xi(2mx) - \mu(2mx)\right\|_*, \left\|\xi(mx) - \mu(mx)\right\|_*,\ \left\|\xi((2m+1)x) - \mu((2m+1)x)\right\|_*, \\
&\qquad \left\|\xi((4m+1)x) - \mu((4m+1)x)\right\|_*\}.
\end{aligned}$$

So, (**H2**) is valid. By using mathematical induction, we will show that for each $x \in X_0$ we have

$$\Lambda_m^n \varepsilon_m(x) \leqslant (\lambda(2m+1) + \lambda(m))\alpha_m^n \varphi(x). \tag{18}$$

From (17), we obtain that the inequality (18) holds for $n = 0$. Next, we will assume that (18) holds for $n = k$, where $k \in \mathbb{N}$. Then, we have

$$\begin{aligned}
&\Lambda_m^{k+1}\varepsilon_m(x) = \Lambda_m\left(\Lambda_m^k \varepsilon_m(x)\right) \\
&= \max\Big\{\Lambda_m^k\varepsilon_m(mx),\ \Lambda_m^k\varepsilon_m\left((3m+1)x\right),\ \Lambda_m^k\varepsilon_m\left((m+1)x\right),\ \Lambda_m^k\varepsilon_m\left((4m+1)x\right), \\
&\qquad \Lambda_m^k\varepsilon_m\left((2m+1)x\right),\ \Lambda_m^k\varepsilon_m\left(2mx\right)\Big\} \\
&\leqslant (\lambda(m) + \lambda(2m+1))\,\alpha_m^k\, \max\{\varphi(mx),\ \varphi((3m+1)x),\ \varphi((m+1)x),\ \varphi((4m+1)x),
\end{aligned}$$

$$\varphi((2m+1)x),\ \varphi(2mx)\}$$
$$\leqslant (\lambda(m)+\lambda(-2m+1))\,\alpha_m^{k+1}\varphi(x),\ x\in X_0.$$

This shows that (18) holds for $n=k+1$. Now we can conclude that the inequality (18) holds for all $n\in\mathbb{N}$. From (18), we obtain

$$\lim_{n\longrightarrow\infty}\Lambda^n\varepsilon_m(x)=0,$$

for all $x\in X_0$ and all $m\in U$. Hence, according to Theorem 1, there exists for each $m\in U$ a unique solution $C_m: X_0\longrightarrow Y$ of the equation

$$\begin{aligned}C_m(x)=&4C_m((3m+1)x)+4C_m((m+1)x)+3C_m(2mx)-24C_m(mx)\\&-6C_m((2m+1)x)-C_m((4m+1)x),\end{aligned}\tag{19}$$

for all $x\in X_0$, such that

$$\|f(x)-C_m(x)\|_*\leqslant\sup_{n\in\mathbb{N}_0}\left\{\Big(\lambda(m)+\lambda(2m+1)\Big)\alpha_m^n\varphi(x)\right\},\ x\in X_0.\tag{20}$$

Moreover,

$$C_m(x):=\lim_{n\longrightarrow\infty}\left(\mathscr{T}_m^n f\right)(x),$$

for all $x\in X_0$. Now we show that

$$\begin{aligned}&\|\mathscr{T}_m^n f(x+2y)+\mathscr{T}_m^n f(x-2y)+24\mathscr{T}_m^n f(y)+6\mathscr{T}_m^n f(x)-4\mathscr{T}_m^n f(x+y)\\&-4\mathscr{T}_m^n f(x-y)-3\mathscr{T}_m^n f(2y)\|_*\leqslant\alpha_m^n\ (\varphi(x)+\varphi(y))\end{aligned}\tag{21}$$

for every $x,y\in X_0$ such that $x+y\neq 0$ and $n\in\mathbb{N}$. Since, the case $n=0$ is just (15), take $k\in\mathbb{N}$ and assume that (21) holds for $n=k$ where $k\in\mathbb{N}$ and every $x,y\in X_0$ such that $x+y\neq 0$. Then,

$$\begin{aligned}&\left\|\mathscr{T}_m^{n+1}f(x+2y)+\mathscr{T}_m^{n+1}f(x-2y)+24\mathscr{T}_m^{n+1}f(y)+6\mathscr{T}_m^{n+1}f(x)-4\mathscr{T}_m^{n+1}f(x+y)\right.\\&\left.-4\mathscr{T}_m^{n+1}f(x-y)-3\mathscr{T}_m^{n+1}f(2y)\right\|_*\\=&\Big\|4\mathscr{T}_m^k f((3m+1)(x+2y))+4\mathscr{T}_m^k f((m+1)(x+2y))+3\mathscr{T}_m^k f(2m(x+2y))\\&-24\mathscr{T}_m^k f(m(x+2y))-6\mathscr{T}_m^k f((2m+1)(x+2y))-\mathscr{T}_m^k f((4m+1)(x+2y))\\&+4\mathscr{T}_m^k f((3m+1)(x-2y))+4\mathscr{T}_m^k f((m+1)(x-2y)))+3\mathscr{T}_m^k f(2m(x-2y))\\&-24\mathscr{T}_m^k f(m(x-2y))-6\mathscr{T}_m^k f((2m+1)(x-2y))-\mathscr{T}_m^k f((4m+1)(x-2y))\\&+96\mathscr{T}_m^k f((3m+1)y)+96\mathscr{T}_m^k f((m+1)y))+72\mathscr{T}_m^k f(2my)\\&-576\mathscr{T}_m^k f(my)-144\mathscr{T}_m^k f((2m+1)y)-24\mathscr{T}_m^k f((4m+1)y)\\&+24\mathscr{T}_m^k f((3m+1)x)+24\mathscr{T}_m^k f((m+1)x)+18\mathscr{T}_m^k f(2mx)\\&-144\mathscr{T}_m^k f(mx)-36\mathscr{T}_m^k f((2m+1)x)-6\mathscr{T}_m^k f((4m+1)x)\\&-16\mathscr{T}_m^k f((3m+1)(x+y))-16\mathscr{T}_m^k f((m+1)(x+y))-18\mathscr{T}_m^k f(2m(x+y))\end{aligned}$$

$$
\begin{aligned}
&+96\mathscr{T}_m^k f(m(x+y))+24\mathscr{T}_m^k f((2m+1)(x+y))+4\mathscr{T}_m^k f((4m+1)(x+y))\\
&-16\mathscr{T}_m^k f((3m+1)(x-y))-16\mathscr{T}_m^k f((m+1)(x-y))-18\mathscr{T}_m^k f(2m(x-y))\\
&+96\mathscr{T}_m^k f(m(x-y))+24\mathscr{T}_m^k f((2m+1)(x-y))+4\mathscr{T}_m^k f((4m+1)(x-y))\\
&-12\mathscr{T}_m^k f(2(3m+1)y)-12\mathscr{T}_m^k f(2(m+1)y)-9\mathscr{T}_m^k f(4my)\\
&+72\mathscr{T}_m^k f(2my)+18\mathscr{T}_m^k f(2(2m+1)y)+3\mathscr{T}_m^k f(2(4m+1)y)\Big\|_*.
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant \max\left\{\mathscr{T}_{ma},\mathscr{T}_{mb},\mathscr{T}_{mc},\mathscr{T}_{md},\mathscr{T}_{me},\mathscr{T}_{mf}\right\}\\
&\leqslant \max\Big\{\alpha_m^k(\varphi((3m+1)x)+\varphi((3m+1)y))\ ,\ \alpha_m^k(\varphi((m+1)x)+\varphi((m+1)y)),\\
&\qquad \alpha_m^k(\varphi(2mx)+\varphi(2my))\ ,\ \alpha_m^k(\varphi(mx)+\varphi(my)),\\
&\qquad \alpha_m^k(\varphi((2m+1)x)+\varphi((2m+1)y))\ ,\ \alpha_m^k(\varphi((4m+1)x)+\varphi((4m+1)y))\Big\}\\
&\leqslant \alpha_m^k \max\{\lambda(3m+1),\ \lambda(m+1),\ \lambda(2m),\ \lambda(m),\ \lambda(2m+1),\ \lambda(4m+1)\}(\varphi(x)+\varphi(y))\\
&=\alpha_m^{k+1}(\varphi(x)+\varphi(y)).
\end{aligned}
$$

With

$$
\begin{aligned}
\mathscr{T}_{ma}=\|&\mathscr{T}_m^n f((3m+1)(x+2y))+\mathscr{T}_m^n f((3m+1)(x-2y))+24\mathscr{T}_m^n f((3m+1)y)\\
&+6\mathscr{T}_m^n f((3m+1)x)-4\mathscr{T}_m^n f((3m+1)(x+y))-4\mathscr{T}_m^n f((3m+1)(x-y))\\
&-3\mathscr{T}_m^n f(2(3m+1)y)\|_*,\\
\mathscr{T}_{mb}=\|&\mathscr{T}_m^n f((m+1)(x+2y))+\mathscr{T}_m^n f((m+1)(x-2y))+24\mathscr{T}_m^n f((m+1)y)\\
&+6\mathscr{T}_m^n f((m+1)x)-4\mathscr{T}_m^n f((m+1)(x+y))-4\mathscr{T}_m^n f((m+1)(x-y))\\
&-3\mathscr{T}_m^n f(2(m+1)y)\|_*,\\
\mathscr{T}_{mc}=\|&\mathscr{T}_m^n f(2m(x+2y))+\mathscr{T}_m^n f(2m(x-2y))+24\mathscr{T}_m^n f(2my)+6\mathscr{T}_m^n f(2mx)\\
&-4\mathscr{T}_m^n f(2m(x+y))-4\mathscr{T}_m^n f(2m(x-y))-3\mathscr{T}_m^n f(4my)\|_*,\\
\mathscr{T}_{md}=\|&\mathscr{T}_m^n f(m(x+2y))+\mathscr{T}_m^n f(m(x-2y))+24\mathscr{T}_m^n f(my)+6\mathscr{T}_m^n f(mx)\\
&-4\mathscr{T}_m^n f(m(x+y))-4\mathscr{T}_m^n f(m(x-y))-3\mathscr{T}_m^n f(2my)\|_*,\\
\mathscr{T}_{me}=\|&\mathscr{T}_m^n f((2m+1)(x+2y))+\mathscr{T}_m^n f((2m+1)(x-2y))+24\mathscr{T}_m^n f((2m+1)y)\\
&+6\mathscr{T}_m^n f((2m+1)x)-4\mathscr{T}_m^n f((2m+1)(x+y))-4\mathscr{T}_m^n f((2m+1)(x-y))\\
&-3\mathscr{T}_m^n f(4my)\|_*
\end{aligned}
$$

and

$$
\begin{aligned}
\mathscr{T}_{mf}=\|&\mathscr{T}_m^n f((4m+1)(x+2y))+\mathscr{T}_m^n f((4m+1)(x-2y))+24\mathscr{T}_m^n f((4m+1)y)\\
&+6\mathscr{T}_m^n f((4m+1)x)-4\mathscr{T}_m^n f((4m+1)(x+y))-4\mathscr{T}_m^n f((4m+1)(x-y))\\
&-3\mathscr{T}_m^n f(2(4m+1)my)\|_*.
\end{aligned}
$$

Thus, by induction we have shown that (21) holds for every $n \in \mathbb{N}$. Letting $n \longrightarrow \infty$ in (21), we obtain that

$$
C_m(x+2y)+C_m(x-2y)=4C_m(x+y)+4C_m(x-y)-24C_m(y)-6C_m(x)+3C_m(2y),
$$

for all $x, y \in X_0$ such that $x+y \neq 0$. In this way we obtain a sequence $\{C_m\}_{m\in U}$ of mixed type cubic-quartic functions on X_0 such that

$$\|f(x) - C_m(x)\|_* \leqslant \sup_{n\in\mathbb{N}_0} \{(\lambda(2m+1) + \lambda(m))\, \alpha_m^n \varphi(x)\},\ x \in X_0.$$

This implies that

$$\|f(x) - C_m(x)\|_* \leqslant (\lambda(m) + \lambda(2m+1))\, \varphi(x),\ x \in X_0.$$

It follows, with $m \longrightarrow \infty$, that f is mixed type cubic-quartic on X_0. □

The following corollary is a particular case of Theorem 3 where $\varphi(x) := c\, \|x\|^p$, with $c \geqslant 0$ and $p < 0$.

COROLLARY 3. *Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|_*)$ be normed space and ultrametric Banach space, respectively, $c \geqslant 0$, $p < 0$ and let $f : X \longrightarrow Y$ satisfy*

$$\begin{aligned} &\|4f(x+y) + 4f(x-y) + 3f(2y) - 24f(y) - 6f(x) - f(x+2y) - f(x-2y)\|_* \\ \leqslant & c\ (\|x\|^p + \|y\|^p), \end{aligned} \tag{22}$$

for all $x, y \in X_0$. Then f is mixed type cubic-quartic on X_0.

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