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RATE OF CONVERGENCE OF GUPTA–SRIVASTAVA OPERATORS BASED ON CERTAIN PARAMETERS

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Abstract. In the present paper, we consider the Bézier variant of the Gupta-Srivastava operators [14] and discuss some direct convergence results by using of Lipschitz type spaces, Ditzian-Totik modulus of smoothness, weighted modulus of continuity and for functions whose derivatives are of bounded variation. In the end some graphical representation for comparison with other variants have been presented.

1. Introduction

In the year 2003, Srivastava et al. [15] proposed a sequence of positive linear operators, containing some well-known operators as special cases and they studied the convergence properties. After a gap of two year Ispir and Yüksel proposed an important Bézier variant of Srivastava-Gupta operators and discussed the rate of convergence for the functions of bunded variation in the interval $[0,\infty)$. Several researchers have proposed different generalizations of the Srivastava-Gupta operators (see for instance [1], [4], [10], [11], [16], [17], etc.). One of them, Yadav [17] studied a modification of Srivastava-Gupta operators which preserve the constant functions as well as linear functions, established the Voronovskaya type theorem and statistical convergence. In 2017, Neer et al. [12] proposed the Bézier variant of the operators, which were introduced by Yadav [17] and discussed several convergence properties.

Recently, Gupta and Srivastava [7] proposed a general family of a positive linear operator, which preserve constant functions as well as linear functions for all $c \in \mathbb{N} \cup \{0\} \cup \{-1\}$. They have considered the general sequence of positive linear operators containing some well-known operators as special cases. For $m \in \mathbb{Z}$, and $c \in \mathbb{N} \cup \{0\} \cup \{-1\}$ operators defined in [7] are given as:

$$L_{n,m}^{c}(f(t);x) = \{n+(m+1)c\} \sum_{k=1}^{\infty} p_{n+mc,k}(x;c) \int_{0}^{\infty} p_{n+(m+2)c,k-1}(t;c) f(t) dt + p_{n+mc,0}(x;c) f(0), \tag{1}$$

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where

$$p_{n,k}(x;c) = \frac{(-x)^k}{k!}\phi_{n,c}^{(k)}(x)$$

and

$$\phi_{n,c}^{(k)}(x) = \begin{cases} e^{-nx}, & c = 0, \\ (1-x)^n, & c = -1, \\ (1+cx)^{\frac{-n}{c}}, & c = 1,2,3,\dots. \end{cases}$$

Very recently, Pratap and Deo[14] studied some approximation properties of these operators and termed these operators as Gupta-Srivastava operators.

Inspired from the recent work depending upon some parameter $\alpha \geqslant 1$, we propose here the following sequence of operators (also called Bézier variant) (1) as follows:

$$\begin{aligned} F_{n,m}^{c,\alpha}(f(t);x) = \{n+(m+1)c\} \sum_{k=1}^{\infty} Q_{n+mc}^{(\alpha)}(x;c) \int_0^{\infty} p_{n+(m+2)c,k-1}(t;c) f(t)dt \\ + Q_{n+mc,0}^{(\alpha)}(x;c) f(0), \end{aligned} \tag{2}$$

where $Q_{n+mc,k}^{(\alpha)}(x;c) = \left(J_{n+mc,k}(x,c)\right)^{\alpha} - \left(J_{n+mc,k+1}(x,c)\right)^{\alpha}$, $\alpha \geqslant 1$, with $J_{n+mc,k}(x,c) = \sum_{j=k}^{\infty} p_{n+mc,k}(x,c)$, where $k < \infty$ and otherwise zero. It is obvious that the operators $F_{n,m}^{c,\alpha}(.;x)$ are the linear positive operator. For $\alpha = 1$ the operators (2) immediately reduce to the form (1).
The special cases of operators (2) are given below:

(i) For $c = 0$, $\alpha = 1$ and $\phi_{n,0}(x) = e^{-nx}$, we get Phillips operators

$$L_{n,m}^{0}(f(t);x) = n\sum_{k=1}^{\infty} p_{n+mc,k}(x;0) \int_0^{\infty} p_{n+mc,k-1}(t;0) f(t)dt + p_{n,0}(x;0) f(0),$$

where

$$p_{n,k}(x,0) = \frac{e^{-nx}(nx)^k}{k!} \text{ and } x \in [0,\infty).$$

(ii) For $c \in \mathbb{N}$, $\alpha = 1$ and $\phi_{n,c}(x) = (1+cx)^{-\frac{n}{c}}$, we get genuine Baskakov-Durrmeyer type operators. These operators are similar to (1) except for $c = \{0,-1\}$, called summation integral type of operators, where

$$p_{n,k}(x;c) = \frac{\left(\frac{n}{c}\right)_k}{k!} \frac{(cx)_k}{(1+cx)^{\frac{n}{c}+k}}$$

and $(n)_i$ denotes the rising factorial given by

$$(n)_i = n(n+1)(n+2)\dots(n+i-1) \;\&\; (n)_0 = 1 (i \in \mathbb{N}).$$

(iii) For $c=-1$, $\alpha=1$ and $\phi_{n,-1}(x)=(1-x)^n$, we have a sequence of Bernstein-Durrmeyer operators

$$L_{n,m}^{-1}(f,x)=(n-m-1)\sum_{k=1}^{n-m-1}p_{n-m,k}(x,-1)\int_0^1 p_{n-m-2,k-1}(t,-1)f(t)dt \\ +p_{n-m,0}(x,-1)f(0)+p_{n-m,n-m}(x,-1), \tag{3}$$

where

$$p_{n,k}(x;-1)=\binom{n}{k}x^k(1-x)^{n-k}.$$

The purpose of this article is to investigate the approximation results by using of Lipchitz type space, Ditzian-Totik modulus of smoothness, weighted modulus of continuity and functions of bounded variation.

2. Basic results

In this section, we give some auxiliary results and with help of these results we study our main results.

Let $C[0,\infty)$ denotes the space of all continuous functions in $[0,\infty)$ and let $C_B[0,\infty)$ be the space of all continuous and bounded functions in $[0,\infty)$.

LEMMA 1. *Let $f(t)=e_i=t^i$, $i=0,1,2,3,4$, $c\in\mathbb{N}\cup\{0\}\cup\{-1\}$ and $m\in\mathbb{Z}$, then we have:*

(i) $L_{n,m}^c(e_0;x)=1$;

(ii) $L_{n,m}^c(e_1;x)=x$;

(iii)

$$L_{n,m}^c(e_2;x)=\frac{(n+(m+1)c)}{(n+(m-1)c)}x^2+\frac{2}{(n+(m-1)c)}x;$$

(iv)

$$L_{n,c}(e_3;x)=\frac{(n+(m+1)c)(n+(m+2)c)}{(n+(m-1)c)(n+(m-2)c)}x^3+\frac{6(n+(m+1)c)}{(n+(m-1)c)(n+(m-2)c)}x^2 \\ +\frac{6}{(n+(m-1)c)(n+(m-2)c)}x;$$

(v)

$$L_{n,c}(e_4;x)=\frac{(n+(m+1)c)(n+(m+2)c)(n+(m+3)c)}{(n+(m-1)c)(n+(m-2)c)(n+(m-3)c)}x^4$$

$$+\frac{12(n+(m+1)c)(n+(m+2)c)}{(n+(m-1)c)(n+(m-2)c)(n+(m-3)c)}x^3$$
$$+\frac{36(n+(m+1)c)}{(n+(m-1)c)(n+(m-2)c)(n+(m-3)c)}x^2$$
$$+\frac{24}{(n+(m-1)c)(n+(m-2)c)(n+(m-3)c)}x.$$

All the moments of operators (1) can be obtained in terms of hyper geometric function of order $r \in \mathbb{N}$. For details see [7].

LEMMA 2. *The central moment of the operators (1) are given as*

$$\mu_{n,s}^c(x) = L_{n,c}((t-x)^s;x),$$

for $s = 2,4$, then we have:

(i)

$$\mu_{n,2}^c(x) = \frac{2x(1+cx)}{(n+(m-1)c)};$$

(ii)

$$\mu_{n,4}^c(x) = \frac{12c^2(n+(m+7)c)}{(n+(m-1)c)(n+(m-2)c)(n+(m-3)c)}x^4$$
$$+\frac{24c^2(13n+(13m+1)c)}{(n+(m-1)c)(n+(m-2)c)(n+(m-3)c)}x^3$$
$$+\frac{12(n+(m+9)c)}{(n+(m-1)c)(n+(m-2)c)(n+(m-3)c)}x^2$$
$$+\frac{24}{(n+(m-1)c)(n+(m-2)c)(n+(m-3)c)}x.$$

REMARK 1. If n is sufficiently large, then the central moment of the operators (1) are

$$\mu_{n,2}^c(x) \leqslant C\frac{x(1+cx)}{n}$$

and

$$\mu_{n,4}^c(x) \leqslant C\frac{(x(1+cx))^2}{n^2},$$

where $C > 0$ is constant.

REMARK 2. We know that $\sum_{j=0}^{\infty} p_{n+mc,j}(x,c) = 1$ and from (2), we have

$$F_{n,m}^{c,\alpha}(1;x) = \{n+(m+1)c\}\sum_{k=1}^{\infty} Q_{n+mc}^{(\alpha)}(x;c)\int_0^{\infty} p_{n+(m+2)c,k-1}(t;c)dt + Q_{n+mc,0}^{(\alpha)}(x;c)$$

$$= \sum_{k=0}^{\infty} Q_{n+mc}^{(\alpha)}(x;c) = (J_{n+mc,0}(x,c))^{(\alpha)} = \left(\sum_{k=0}^{\infty} p_{n+mc,j}(x,c)\right)^{(\alpha)} = 1.$$

LEMMA 3. *For each* $f \in C_B[0,\infty)$ *then we have*

$$\left|F_{n,m}^{c,\alpha}(f(t);x)\right| \leqslant \|f\|.$$

Proof. It is easy to prove the above result by using Remark 2, therefore we skip the proof. □

LEMMA 4. *For every* $f \in C_B[0,\infty)$ *then we have*

$$\left|F_{n,m}^{c,\alpha}(f(t);x)\right| \leqslant \alpha L_{n,m}^{c}(\|f\|;x).$$

Proof. For $0 \leqslant c \leqslant d \leqslant 1$, $\alpha \geqslant 1$, using the inequality

$$|c^{\alpha} - d^{\alpha}| \leqslant \alpha |c-d|,$$

from the definition of $Q_{n+mc,k}^{(\alpha)}(x;c)$, for all $k \in \mathbb{N} \cup \{0\}$, we get

$$0 < \left(J_{n+mc,k}(x,c)\right)^{\alpha} - \left(J_{n+mc,k+1}(x,c)\right)^{\alpha} \leqslant \alpha\left(J_{n+mc,k}(x,c) - J_{n+mc,k+1}(x,c)\right)$$
$$= \alpha p_{n+mc}(x,c).$$

Hence

$$\left|F_{n,m}^{c,\alpha}(f(t);x)\right| \leqslant \alpha L_{n,m}^{c}(\|f\|;x). \quad \square$$

3. Main theorems

For $x > 0$, $t \geqslant 0$ and $0 < \gamma \leqslant 1$, we can see in Özarslan et al. [13], the Lipschitz type space is defined as:

$$Lip_M(\gamma) := \left\{ f \in C[0,\infty) : |f(t) - f(x)| \leqslant M \frac{|t-x|^{\gamma}}{(t+x)^{\gamma/2}} \right\}.$$

Now we estimate the rate of convergence of the function $f \in Lip_M(\gamma)$ by the operators $F_{n,m}^{c,\alpha}(.;x)$.

THEOREM 1. (Main) *For* $f \in Lip_M(\gamma)$ *and* $\gamma \in (0,1]$. *Then for* $x > 0$ *we have*

$$\left|F_{n,m}^{c,\alpha}(f(t);x) - f(x)\right| \leqslant \alpha M \left(\frac{\delta_{n,m}^{c}(x)}{x}\right)^{\gamma/2},$$

where $\delta_{n,m}^{c}(x) = \sqrt{\frac{2x(1+cx)}{(n+(m-1)c)}}$.

Proof. Using Lemma 4, we get

$$\begin{aligned}\left|F_{n,m}^{c,\alpha}(f(t);x)-f(x)\right| &\leqslant F_{n,m}^{c,\alpha}(|f(t)-f(x)|;x)\leqslant \alpha L_{n,m}^{c}(|f(t)-f(x)|;x)\\ &\leqslant \alpha M L_{n,m}^{c}\left(\frac{|t-x|^{\gamma}}{(t+x)^{\gamma/2}};x\right)\leqslant \frac{\alpha M}{x^{\gamma/2}}L_{n,m}^{c}\left(|t-x|^{\gamma};x\right).\end{aligned} \tag{4}$$

Using Hölder's inequality by taking $p=2/\gamma$ and $q=2/(2-\gamma)$, we get

$$\begin{aligned}L_{n,m}^{c}\left(|t-x|^{\gamma};x\right) &\leqslant \left\{L_{n,m}^{c}\left((t-x)^{2};x\right)\right\}^{\frac{\gamma}{2}}.\left\{L_{n,m}^{c}\left(1^{\frac{2}{(2-\gamma)}};x\right)\right\}^{\frac{(2-\gamma)}{2}}\\ &\leqslant \left\{L_{n,m}^{c}\left((t-x)^{2};x\right)\right\}^{\frac{\gamma}{2}}=\left(\delta_{n,m}^{c}(x)\right)^{\frac{\gamma}{2}}.\end{aligned} \tag{5}$$

From (4) and (5), we get

$$\left|F_{n,m}^{c,\alpha}(f(t);x)-f(x)\right|\leqslant \alpha M\left(\frac{\delta_{n,m}^{c}(x)}{x}\right)^{\frac{\gamma}{2}}.$$

Hence the proof. □

In our next theorem, we estimate the rate of convergence by using of Ditzian-Totik modulus of smoothness $\omega_{\phi^{\beta}}(f,\delta)$ and Peetre $K-$functional $K_{\phi^{\beta}}(f,\delta)$, $0\leqslant\beta\leqslant 1$. For $f\in C_B[0,\infty)$ and $\phi(x)=\sqrt{x(1+cx)}$, the Ditzian-Totik modulus of smoothness is defined as

$$\omega_{\phi^{\beta}}(f,\delta)=\sup_{0\leqslant i\leqslant\delta}\ \sup_{x\pm\frac{i\phi^{\beta}(x)}{2}\in[0,\infty)}\left|f\left(x+\frac{i\phi^{\beta}(x)}{2}\right)-f\left(x-\frac{i\phi^{\beta}(x)}{2}\right)\right|$$

and the Peetre $K-$functional is defined as

$$\omega_{\phi^{\beta}}(f,\delta)=\inf_{g\in W_{\beta}}\left\{\|f-g\|-\delta\left\|\phi^{\beta}g'\right\|\right\},$$

where W_{β} is subspace of the space which is locally absolutely continuous functions g on $[0,\infty)$, with the normed $\left\|\phi^{\beta}g'\right\|\leqslant\infty$. In [6, Theorem 2.1.1], there exists a constant $C>0$ such that

$$C^{-1}\omega_{\phi^{\beta}}(f,\delta)\leqslant K_{\phi^{\beta}}(f,\delta)\leqslant C\omega_{\phi^{\beta}}(f,\delta). \tag{6}$$

THEOREM 2. (Main) *For $f\in C_B[0,\infty)$ then we have*

$$\left|F_{n,m}^{c,\alpha}(f(t);x)-f(x)\right|\leqslant C\omega_{\phi^{\beta}}\left(f;\frac{\phi^{1-\beta}(x)}{\sqrt{n}}\right),$$

for sufficiently large n and C is a positive constant independent from f and n.

Proof. For $g \in W_\beta$, we get

$$g(t) = g(x) + \int_x^t g'(u)du. \tag{7}$$

Applying $F_{n,m}^{c,\alpha}$ in (7) and using Hölder's inequality then, we have

$$\begin{aligned} \left|F_{n,m}^{c,\alpha}(g(t);x) - g(x)\right| &\leqslant F_{n,m}^{c,\alpha}\left(\int_x^t |g'|du;x\right) \leqslant \left\|\phi^\beta g'\right\| F_{n,m}^{c,\alpha}\left(\left|\int_x^t \frac{du}{\phi^\beta(u)}\right|;x\right) \\ &\leqslant \left\|\phi^\beta g'\right\| F_{n,m}^{c,\alpha}\left(|t-x|^{1-\beta}\left|\int_x^t \frac{du}{\phi(u)}\right|^\beta;x\right). \end{aligned} \tag{8}$$

Let us take $A = \left|\int_x^t \frac{du}{\phi(u)}\right|$, then we get

$$\begin{aligned} A &\leqslant \left|\int_x^t \frac{du}{\sqrt{u}}\right|\left|\left(\frac{1}{\sqrt{1+cx}} + \frac{1}{\sqrt{1+ct}}\right)\right| \leqslant 2\left|\sqrt{t}-\sqrt{x}\right|\left(\frac{1}{\sqrt{1+cx}} + \frac{1}{\sqrt{1+ct}}\right) \\ &\leqslant 2\frac{|t-x|}{\sqrt{x}+\sqrt{t}}\left(\frac{1}{\sqrt{1+cx}} + \frac{1}{\sqrt{1+ct}}\right) \leqslant 2\frac{|t-x|}{\sqrt{x}}\left(\frac{1}{\sqrt{1+cx}} + \frac{1}{\sqrt{1+ct}}\right). \end{aligned} \tag{9}$$

The inequality $|a+b|^\beta \leqslant |a|^\beta + |b|^\beta$, $0 \leqslant \beta \leqslant 1$, then from (9) we get

$$\left|\int_x^t \frac{du}{\phi(u)}\right|^\beta \leqslant 2^\beta \frac{|t-x|^\beta}{x^{\beta/2}}\left(\frac{1}{(1+cx)^{\beta/2}} + \frac{1}{(1+ct)^{\beta/2}}\right). \tag{10}$$

From (8), (10) and using Cauchy inequality then we get

$$\begin{aligned} \left|F_{n,m}^{c,\alpha}(g(t);x) - g(x)\right| &\leqslant \frac{2^\beta\left\|\phi^\beta g'\right\|}{x^{\beta/2}} F_{n,m}^{c,\alpha}\left(|t-x|\left(\frac{1}{(1+cx)^{\beta/2}} + \frac{1}{(1+ct)^{\beta/2}}\right);x\right) \\ &\leqslant \frac{2^\beta\left\|\phi^\beta g'\right\|}{x^{\beta/2}}\left(\frac{1}{(1+cx)^{\beta/2}}\left(F_{n,c}^{(\alpha)}\left((t-x)^2;x\right)\right)^{1/2}\right. \\ &\quad \left. + \left(F_{n,m}^{c,\alpha}\left((t-x)^2;x\right)\right)^{1/2} . \left(F_{n,m}^{c,\alpha}\left((1+ct)^{-\beta};x\right)\right)^{1/2}\right). \end{aligned} \tag{11}$$

If n is sufficiently large then we get

$$\left(F_{n,m}^{c,\alpha}\left((t-x)^2;x\right)\right)^{1/2} \leqslant \sqrt{\frac{2\alpha}{n}}\phi(x), \tag{12}$$

where $\phi(x)=\sqrt{x(1+cx)}$. For each $x\in[0,\infty)$, $F_{n,m}^{c,\alpha}\left((1+ct)^{-\beta};x\right)\rightarrow(1+cx)^{-\beta}$ as $n\rightarrow\infty$. Thus for $\varepsilon>0$, there exists $n_0\in\mathbb{N}$ such that

$$F_{n,m}^{c,\alpha}\left((1+ct)^{-\beta};x\right)\leqslant(1+cx)^{-\beta}+\varepsilon,\quad\text{for all } n\geqslant n_0.$$

By choosing $\varepsilon=(1+cx)^{-\beta}$ then, we get

$$F_{n,m}^{c,\alpha}\left((1+ct)^{-\beta};x\right)\leqslant 2(1+cx)^{-\beta},\quad\text{for all } n\geqslant n_0. \tag{13}$$

From (11) to (13), we have

$$\begin{aligned}\left|F_{n,m}^{c,\alpha}(g(t);x)-g(x)\right| &\leqslant 2^{\beta}\left\|\phi^{\beta}g'\right\|\sqrt{\frac{2\alpha}{n}}\phi(x)\left(\phi^{-\beta}(x)+\sqrt{2}x^{-\frac{\beta}{2}}(1+cx)^{-\frac{\beta}{2}}\right)\\ &\leqslant 2^{\beta+\frac{1}{2}}(1+\sqrt{2})\left\|\phi^{\beta}g'\right\|\sqrt{\frac{\alpha}{n}}\phi^{1-\beta}(x).\end{aligned} \tag{14}$$

We may write

$$\begin{aligned}\left|F_{n,m}^{c,\alpha}(f(t);x)-f(x)\right| &\leqslant\left|F_{n,m}^{c,\alpha}(f(t)-g(t);x)\right|+\left|F_{n,m}^{c,\alpha}(g(t);x)-g(x)\right|+|g(x)-f(x)|\\ &\leqslant 2\|f-g\|+\left|F_{n,m}^{c,\alpha}(g(t);x)-g(x)\right|.\end{aligned} \tag{15}$$

From (14) to (15) and for sufficiently large n, we get

$$\begin{aligned}\left|F_{n,m}^{c,\alpha}(f(t);x)-f(x)\right| &\leqslant 2\|f-g\|+2^{\beta+\frac{1}{2}}(1+\sqrt{2})\sqrt{\frac{\alpha}{n}}\phi^{1-\beta}(x)\left\|\phi^{\beta}g'\right\|\\ &\leqslant C_1\left\{\|f-g\|+\frac{\phi^{1-\beta}(x)}{\sqrt{n}}\left\|\phi^{\beta}g'\right\|\right\}\leqslant CK_{\phi^{\beta}}\left(f,\frac{\phi^{1-\beta}(x)}{\sqrt{n}}\right),\end{aligned} \tag{16}$$

where $C_1=max(2,2^{\beta+\frac{1}{2}}(1+\sqrt{2})\sqrt{\alpha})$ and $C=2C_1$. From (6) and (16), we get the required result. □

For the estimation of the rate of convergence of the function $f\in C_2[0,\infty)$ by using the weighted modulus of continuity, which was introduced by Ispir and Yüksal [18] as follows:

$$\Omega(f;\delta)=\sup_{x\in[0,\infty),0<\beta<\delta}\frac{f(x+\beta)-f(x)}{1+(x+\beta)^2}. \tag{17}$$

Many authors have discussed weighted modulus of continuity for various linear positive operators. For more information (see [2], [3], [5]).

There are several properties of weighted modulus of continuity $\Omega(.;\delta)$ which are stated in following Lemma.

LEMMA 5. [18] *For* $f \in C_2[0,\infty)$ *the following properties hold:*

(i) $\Omega(f;\delta)$ *is monotonically increasing in* δ;

(ii) $\lim_{\delta\to 0^+}\Omega(f;\delta)=0$;

(iii) for each $r\in\mathbb{N}$, $\Omega(f;r\delta)\leqslant r\Omega(f;\delta)$;

(iv) for each $\lambda\in[0,\infty)$, $\Omega(f;\lambda\delta)\leqslant(\lambda+1)\Omega(f;\delta)$.

THEOREM 3. (Main) *Let* $f\in C_2[0,\infty)$, $\alpha>0$, *for fixed* m *and sufficiently large* n *then we have*

$$\sup_{x\in[0,\infty)}\frac{\left|F_{n,m}^{c,\alpha}(f;x)-f(x)\right|}{(1+x)^{5/2}}\leqslant C\Omega\left(f;\frac{1}{\sqrt{n}}\right),$$

where C *is positive constant depends on* n *and* f.

Proof. By the definition of the weighted modulus of continuity and Lemma 5, we have

$$\begin{aligned}|f(t)-f(x)| &\leqslant \left(1+(x+|t-x|)^2\right)\Omega\left(f;|t-x|\right)\\ &\leqslant 2(1+x^2)\left(1+(t-x)^2\right)\left(1+\frac{|t-x|}{\delta}\right)\Omega(f;\delta).\end{aligned}\tag{18}$$

Applying $F_{n,m}^{c,\alpha}(.;x)$ on both side of (18), we can write

$$\left|F_{n,m}^{c,\alpha}(f;x)-f(x)\right|\leqslant\left[1+F_{n,m}^{c,\alpha}((t-x)^2;x)+F_{n,m}^{c,\alpha}\left((1+(t-x)^2)\frac{|t-x|}{\delta};x\right)\right].\tag{19}$$

From Remark 1 and using Cauchy-Schwarz inequality in the last term of (19), we have

$$F_{n,m}^{c,\alpha}\left((1+(t-x)^2)\frac{|t-x|}{\delta};x\right)\leqslant\frac{1}{\delta}\left(\alpha\mu_{n,2}^{c}(x)\right)^{1/2}+\frac{1}{\delta}\left(\alpha\mu_{n,4}^{c}(x)\right)^{1/2}\left(\alpha\mu_{n,2}^{c}(x)\right)^{1/2}.\tag{20}$$

Combining the estimate from (18) to (20) and taking $C=2(1+\sqrt{\alpha C}+2C)$ and $\delta=\frac{1}{\sqrt{n}}$ then we get the required result. □

In next theorem, we study the rate of convergence of the Bézier variant of Gupta-Srivastava operators (2) in the class $DBV[0,\infty)$, the class of all absolutely continuous functions f defined on $[0,\infty)$ having a derivative coinciding a.e. with a function of bounded variation on $[0,\infty)$. It can be seen that for $f\in DBV[0,\infty)$, we can write

$$f(x)=\int_0^x g(t)\,dt+f(0),$$

where $g(t)$ is a function of bounded variation on each finite subinterval of $[0,\infty)$. The operators (2) can be rewritten in the following form:

$$F_{n,m}^{c,\alpha}(f(t);x)=\int_0^\infty M_{n,m,c}^{(\alpha)}(x,t)f(t)dt, \tag{21}$$

where

$$M_{n,m,c}^{(\alpha)}(x,t)=\{n+(m+1)c\}\sum_{k=1}^\infty Q_{n+mc,k}^{(\alpha)}(x;c)p_{n+(m+2)c,k}(t,c)+Q_{n+mc,0}^{(\alpha)}(x;c)\delta(t),$$

where $\delta(t)$ is Dirac delta function.

LEMMA 6. *For a fixed $x\in[0,\infty)$ and n sufficiently large then we have:*

(i) $\zeta_{n,c}^{(\alpha)}(x;y)=\int_0^y M_{n,m,c}^{(\alpha)}(x;t)dt\leqslant\frac{2\alpha x(1+cx)}{n(x-y)^2},\ \ 0\leqslant y\leqslant x;$

(ii) $1-\zeta_{n,c}^{(\alpha)}(x;z)=\int_z^\infty M_{n,m,c}^{(\alpha)}(x;t)dt\leqslant\frac{2\alpha x(1+cx)}{n(z-x)^2},\ \ x\leqslant z\leqslant\infty.$

Proof. From (21) and using Remark 1 then we have

$$\zeta_{n,c}^{(\alpha)}(x;y)\leqslant\int_0^y M_{n,m,c}^{(\alpha)}(x;t)\left(\frac{x-t}{x-y}\right)^2dt\leqslant\frac{\alpha}{(x-y)^2}L_{n,c}\left((e_1-x)^2;x\right)\leqslant\frac{2\alpha x(1+cx)}{n(x-y)^2}.$$

We can prove the second part of Lemma in same way. □

THEOREM 4. *Let $f\in DBV[0,\infty)$ then for every $x\in[0,\infty)$ and n sufficiently large we have*

$$\begin{aligned}\left|F_{n,m}^{c,\alpha}(f;x)-f(x)\right|\leqslant&\frac{1}{\alpha+1}\left|f'(x+)+\alpha f'(x-)\right|\sqrt{\frac{2\alpha x(1+cx)}{n}}\\&+\frac{\alpha}{\alpha+1}\left|f'(x+)-f'(x-)\right|\sqrt{\frac{2\alpha x(1+cx)}{n}}\\&+\frac{2\alpha(1+cx)}{n}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x-\frac{x}{k}}^{x}f'_x+\frac{x}{\sqrt{n}}\bigvee_{x-\frac{x}{\sqrt{n}}}^{x}f'_x\\&+\frac{2\alpha(1+cx)}{nx}\left|f(2x)-f(x)-xf(x+)\right|\\&+\frac{2\alpha x(1+cx)}{n}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x}^{x+\frac{x}{k}}f'_x+\frac{x}{\sqrt{n}}\bigvee_{x}^{x+\frac{x}{\sqrt{n}}}(f'_x)+M(\gamma,r,x)\\&+\frac{2\alpha(1+cx)}{nx}\left|f(x)\right|+\sqrt{\frac{2\alpha x(1+cx)}{n}}\left|f(x+)\right|,\end{aligned}$$

where $\overset{b}{\underset{a}{V}} f(x)$ *denotes the total variation of* f *on* $[a,b]$, f_x *is an auxiliary operator given by*

$$f_x(t) = \begin{cases} f(t) - f(x-), & 0 \leqslant t < x, \\ 0, & t = x, \\ f(t) - f(x+), & x < t < \infty. \end{cases} \tag{22}$$

Proof. From Remark 2, $F_{n,m}^{c,\alpha}(1;x) = 1$ and using the alternative form of the operators (21) for each $x \in [0,\infty)$ then we have

$$F_{n,m}^{c,\alpha}(f(t);x) - f(x) = \int_0^\infty M_{n,m,c}^{(\alpha)}(x,t)\big(f(t) - f(x)\big)dt = \int_0^\infty M_{n,m,c}^{(\alpha)}(x,t)\left(\int_x^t f'(u)du\right)dt. \tag{23}$$

For each $f \in DBV[0,\infty)$ and from (22), we can write

$$\begin{aligned} f'(u) =& f_x'(u) + \frac{1}{\alpha+1}(f'(x+) + \alpha f'(x-)) \\ &+ \frac{1}{2}(f'(x+) + \alpha f'(x-))\left(\operatorname{sgn}(u-x) + \frac{\alpha-1}{\alpha+1}\right) \\ &\times \delta_x(u)\left[f'(u) - \big(f'(x+) + f'(x-)\big)\right], \end{aligned} \tag{24}$$

where

$$\delta_x(u) = \begin{cases} 1, & u = x, \\ 0, & u \neq x. \end{cases}$$

From (23) and (24) we have

$$\begin{aligned} F_{n,m}^{c,\alpha}(f(t);x) - f(x) = \int_0^\infty M_{n,m,c}^{(\alpha)}(x,t)\int_x^t \Bigg(& f_x'(u) + \frac{1}{\alpha+1}(f'(x+) + \alpha f'(x-)) \\ &+ \frac{1}{2}(f'(x+) + \alpha f'(x-))\left(\operatorname{sgn}(u-x) + \frac{\alpha-1}{\alpha+1}\right) \\ &\times \delta_x(u)[f'(u) - \frac{1}{2}(f'(x+) + f'(x-)]\Bigg)dudt. \end{aligned} \tag{25}$$

It is easy to say that

$$\int_0^\infty M_{n,m,c}^{(\alpha)}(x,t)\int_x^t [f'(u) - \frac{1}{2}(f'(x+) + f'(x-)]\delta_x(u)dudt = 0. \tag{26}$$

Now

$$B_1 = \int_0^\infty M_{n,m,c}^{(\alpha)}(x,t) \int_x^t \frac{1}{\alpha+1}(f'(x+)+\alpha f'(x-))dudt$$
$$= \frac{1}{\alpha+1}(f'(x+)+\alpha f'(x-)) \int_0^\infty M_{n,m,c}^{(\alpha)}(x,t)(t-x)dt \tag{27}$$
$$= \frac{1}{\alpha+1}(f'(x+)+\alpha f'(x-))F_{n,c}^{(\alpha)}((t-x);x)$$

and

$$B_2 = \int_0^\infty M_{n,m,c}^{(\alpha)}(x,t) \int_x^t \frac{1}{2}(f'(x+)+\alpha f'(x-)) \left(\operatorname{sgn}(u-x)+\frac{\alpha-1}{\alpha+1}\right) dudt$$
$$= \frac{1}{2}(f'(x+)+\alpha f'(x-)) \left(-\int_0^x M_{n,m,c}^{(\alpha)}(x,t) \int_x^t \left(\operatorname{sgn}(u-x)+\frac{\alpha-1}{\alpha+1}\right) dudt \right.$$
$$\left. + \int_x^\infty M_{n,m,c}^{(\alpha)}(x,t) \int_x^t \left(\operatorname{sgn}(u-x)+\frac{\alpha-1}{\alpha+1}\right)\right) \tag{28}$$
$$\leqslant \frac{\alpha}{\alpha+1}(f'(x+)+\alpha f'(x-)) \int_0^\infty M_{n,m,c}^{(\alpha)}(x,t)\,|t-x|\,dt$$
$$\leqslant \frac{\alpha}{\alpha+1}(f'(x+)+\alpha f'(x-)) \left(F_{n,c}^{(\alpha)}\left((e_1-x)^2;x\right)\right)^{\frac{1}{2}}.$$

By using Remark 1 and Lemma 4, from (25) – (28) then we have

$$F_{n,m}^{c,\alpha}(f;x) - f(x) \leqslant \left|A_n^{(\alpha)}(f';x)+B_n^{(\alpha)}(f';x)\right| + \frac{2\alpha}{\alpha+1}\left|f'(x+)+\alpha f'(x-)\right| \frac{x(1+cx)}{n}$$
$$+ \frac{\alpha}{\alpha+1}\left|f'(x+)-f'(x-)\right| \sqrt{\frac{2\alpha x(1+cx)}{n}}, \tag{29}$$

where

$$A_n^{(\alpha)}(f';x) = \int_0^x \left(\int_x^t f_x'(u)du\right) M_{n,m,c}^{(\alpha)}(x,t)dt$$

and

$$B_n^{(\alpha)}(f';x) = \int_x^\infty \left(\int_x^t f_x'(u)du\right) M_{n,m,c}^{(\alpha)}(x,t)dt.$$

To estimate $A_n^{(\alpha)}(f';x)$, using integration by parts and applying Lemma 6 with $y = x - \frac{x}{\sqrt{n}}$, we obtain

$$\begin{aligned} A_n^{(\alpha)}(f';x) &= \left| \int_0^x \left(\int_x^t f'_x(u)du \right) d_t \zeta_{n,c}^{(\alpha)}(x;t) \right| = \left| \int_0^x \zeta_{n,c}^{(\alpha)}(x;t) f'_x(t)dt \right| \\ &\leqslant \int_0^y |f'_x(t)| \left| \zeta_{n,c}^{(\alpha)}(x;t) \right| dt + \int_0^y |f'_x(t)| \left| \zeta_{n,c}^{(\alpha)}(x;t) \right| dt \\ &\leqslant \frac{2\alpha x(1+cx)}{n} \int_0^y \bigvee_t^x f'_x (x-t)^2 dt + \int_y^x \bigvee_t^x f'_x dt \\ &\leqslant \frac{2\alpha x(1+cx)}{n} \int_0^{x-\frac{x}{\sqrt{n}}} \bigvee_t^x f'_x (x-t)^2 dt + \frac{x}{\sqrt{n}} \bigvee_{x-\frac{x}{\sqrt{n}}}^x f'_x. \end{aligned} \tag{30}$$

Substituting $u = \frac{x}{x-t}$, we get

$$\begin{aligned} \frac{2\alpha x(1+cx)}{n} \int_0^{x-\frac{x}{\sqrt{n}}} \bigvee_t^x f'_x (x-t)^2 dt &= \frac{2\alpha x(1+cx)}{nx} \int_1^{\sqrt{n}} \bigvee_{x-\frac{x}{u}}^x f'_x du \\ &\leqslant \frac{2\alpha(1+cx)}{n} \sum_{k=1}^{[\sqrt{n}]} \int_k^{k+1} \bigvee_{x-\frac{x}{k}}^x f'_x du \\ &\leqslant \frac{2\alpha(1+cx)}{n} \sum_{k=1}^{[\sqrt{n}]} \bigvee_{x-\frac{x}{k}}^x f'_x. \end{aligned} \tag{31}$$

From (30) and (31) we get

$$A_n^{(\alpha)}(f';x) = \frac{2\alpha(1+cx)}{n} \sum_{k=1}^{[\sqrt{n}]} \bigvee_{x-\frac{x}{k}}^x f'_x + \frac{x}{\sqrt{n}} \bigvee_{x-\frac{x}{\sqrt{n}}}^x f'_x. \tag{32}$$

We can write

$$B_n^{(\alpha)}(f';x) \leqslant \left| \int_x^{2x} \left(\int_x^t f'_x(u)du \right) d_t (1 - \zeta_{n,c}^{(\alpha)}(x;t)) \right| + \left| \int_{2x}^{\infty} \left(\int_x^t f'_x(u)du \right) d_t M_{n,m,c}^{(\alpha)}(x,t) \right|.$$

From the second part of the Lemma 6, we get

$$M_{n,m,c}^{(\alpha)}(x,t) = d_t((1 - \zeta_{n,c}^{(\alpha)}(x;t)), \qquad \text{for } t > x.$$

Hence

$$B_n^{(\alpha)}(f';x) = B_{n,1}^{(\alpha)}(f';x) + B_{n,2}^{(\alpha)}(f';x),$$

where

$$B_{n,1}^{(\alpha)}(f';x) = \left| \int_x^{2x} \left(\int_x^t f'_x(u)du \right) d_t(1-\zeta_{n,c}^{(\alpha)}(x;t)) \right|$$

and

$$B_{n,2}^{(\alpha)}(f';x) = \left| \int_{2x}^{\infty} \left(\int_x^t f'_x(u)du \right) d_t M_{n,m,c}^{(\alpha)}(x,t) \right|.$$

Using integration by parts, applying Lemma 6, $1-\zeta_{n,c}^{(\alpha)}(x;t) \leqslant 1$ and taking $t = x + \frac{x}{u}$ successively,

$$\begin{aligned} B_{n,1}^{(\alpha)}(f';x) =& \left| \int_x^{2x} f'_x(u)du(1-\zeta_{n,c}^{(\alpha)}(x;2x)) - \int_x^{2x} f'_x(t)(1-\zeta_{n,c}^{(\alpha)}(x;t))dt \right| \\ \leqslant& \left| \int_x^{2x} (f'(u)-f'(x+))du \right| \left| 1-\zeta_{n,c}^{(\alpha)}(x;2x) \right| + \left| \int_x^{2x} f'_x(t)(1-\zeta_{n,c}^{(\alpha)}(x;t))dt \right| \\ \leqslant& \frac{2\alpha(1+cx)}{nx} |f(2x)-f(x)-xf(x+)| \\ &+ \frac{2\alpha x(1+cx)}{n} \int_{x+\frac{x}{\sqrt{n}}}^{2x} \frac{\bigvee_x^t f'_x}{(t-x)^2} dt + \int_x^{x+\frac{x}{\sqrt{n}}} \bigvee_x^t f'_x dt \\ \leqslant& \frac{2\alpha(1+cx)}{nx} |f(2x)-f(x)-xf(x+)| \\ &+ \frac{2\alpha x(1+cx)}{n} \sum_{k=1}^{[\sqrt{n}]} \bigvee_x^{x+\frac{x}{k}} f'_x + \frac{x}{\sqrt{n}} \bigvee_x^{x+\frac{x}{\sqrt{n}}} (f'_x). \end{aligned} \tag{33}$$

Using Remark 1 then we have

$$\begin{aligned} B_{n,2}^{(\alpha)}(f';x) =& \left| \int_{2x}^{\infty} \left(\int_x^t (f'(u)-f'(x+))du \right) M_{n,m,c}^{(\alpha)}(x,t)dt \right| \\ \leqslant& \int_{2x}^{\infty} |f(t)-f(x)| M_{n,m,c}^{(\alpha)}(x,t)dt + \int_{2x}^{\infty} |t-x| \, |f(x+)| \, M_{n,m,c}^{(\alpha)}(x,t)dt \\ \leqslant& \left| \int_{2x}^{\infty} f(t) M_{n,m,c}^{(\alpha)}(x,t)dt \right| + |f(x)| \left| \int_{2x}^{\infty} M_{n,m,c}^{(\alpha)}(x,t)dt \right| \\ &+ |f(x+)| \left(\int_{2x}^{\infty} (e_1-x)^2 M_{n,m,c}^{(\alpha)}(x,t)dt \right)^{\frac{1}{2}} \end{aligned} \tag{34}$$

$$\leqslant M\int_{2x}^{\infty} t^{\gamma} M_{n,m,c}^{(\alpha)}(x,t)dt + |f(x)|\left|\int_{2x}^{\infty} M_{n,m,c}^{(\alpha)}(x,t)dt\right| + \sqrt{\frac{2\alpha x(1+cx)}{n}}\,|f(x+)|.$$

For $t \geqslant 2x$, we get $t \leqslant 2(t-x)$ and $x \leqslant t-x$, applying Hölder's inequality, we have

$$\begin{aligned} B_{n,2}^{(\alpha)}(f';x) \leqslant & M2^{\gamma}\left(\int_{2x}^{\infty}(e_1-x)^{2r}M_{n,m,c}^{(\alpha)}(x,t)dt\right)^{\frac{\gamma}{2r}} \\ & + \frac{2\alpha(1+cx)}{nx}|f(x)| + \sqrt{\frac{2\alpha x(1+cx)}{n}}\,|f(x+)| \\ = & M(\gamma,c,r,x) + \frac{2\alpha(1+cx)}{nx}|f(x)| + \sqrt{\frac{2\alpha x(1+cx)}{n}}\,|f(x+)|. \end{aligned} \tag{35}$$

From (33) and (35), we get

$$\begin{aligned} B_{n}^{(\alpha)}(f';x) = & \frac{2\alpha(1+cx)}{nx}|f(2x)-f(x)-xf(x+)| \\ & + \frac{2\alpha x(1+cx)}{n}\sum_{k=1}^{[\sqrt{n}]} \bigvee_{x}^{x+\frac{x}{k}} f'_x + \frac{x}{\sqrt{n}} \bigvee_{x}^{x+\frac{x}{\sqrt{n}}} (f'_x) \\ & + M(\gamma,c,r,x) + \frac{2\alpha(1+cx)}{nx}|f(x)| + \sqrt{\frac{2\alpha x(1+cx)}{n}}\,|f(x+)|. \end{aligned} \tag{36}$$

From (29), (32) and (36), we get our desired result. □

4. Graphical results and discussion

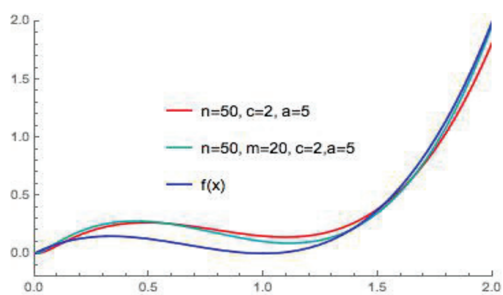

Figure 1: Comparison between Bézier variant of Srivastava-Gupta operators [8] (red) with operators (2)(cyan) along with function $f(x)$.

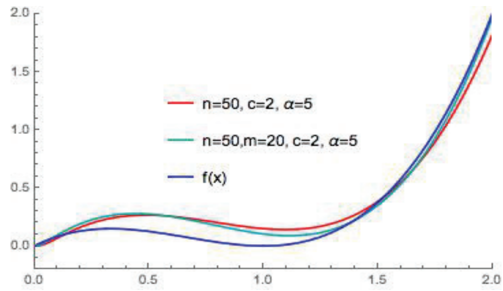

Figure 2: Comparison between Bézier variant of Srivastava-Gupta Operators [9] (red) with operators (2) (cyan) along with function $f(x)$ (blue).

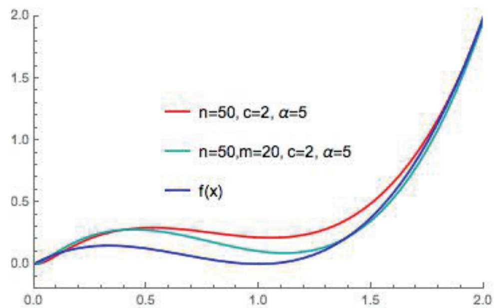

Figure 3: Comparison between Bézier variant of modified Srivastava-Gupta operators [12] (red) with operators (2) (cyan) along with function $f(x)$ (blue).

In operators (2) by taking $m = -1$, we obtain Bézier variant of Srivastava-Gupta operators which were propoesd by Ispir and Yüksel [8] and its modification by Neer et al.[12]. For $m = 1$ in (2), we obtain another Bézier form of Srivastava-Gupta operators considered by Kajla [9]. The proposed operators (2) have generalized form with different values of m. Here, we show graphical comparison between operators (2) for $m = 20$ with above discussed operators ([8], [9], [12]) for the function $f(x) = x^3 - 2x^2 + x$.

From the above graphs we observe here that we have better approximation for the Bézier variant of Gupta-Srivastava operators (2), discussed in the present paper than the other variants of [15], therefore it is justified to study this form of operators.

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