

## A LOWER BOUND OF THE POWER EXPONENTIAL FUNCTION

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*Abstract.* In this paper, we consider the lower bound of the power exponential function  $a^{2b} + b^{2a}$  for nonnegative real numbers  $a$  and  $b$ . If  $a + b = 1$ , then it is known that the function has the maximum value 1, but it is no known that the minimum value. In this paper, we show that  $a^{2b} + b^{2a} > 6083/6144 \cong 0.990072$  for nonnegative real numbers  $a$  and  $b$  with  $a + b = 1$ .

### 1. Introduction

The power exponential function is the one of the most classically and important function in the mathematics and sciences. The function has been studied by the inequalities field, especially, Cîrtoaje [1] [2] and Matejíčka [3] showed the power exponential function  $a^{2b} + b^{2a}$  has the maximum value 1 for nonnegative real numbers  $a$  and  $b$  with  $a + b = 1$ . But it is no known that the minimum value of the function. It seems to be that the minimum value is situated between 0.99 and 0.9907 from the computer simulations. In this paper, we show that  $a^{2b} + b^{2a} > 6083/6144 \cong 0.990072$  for nonnegative real numbers  $a$  and  $b$  with  $a + b = 1$ . Our main result is given as follows.

**THEOREM 1.** *We have  $a^{2b} + b^{2a} > 6083/6144 \cong 0.990072$  for nonnegative real numbers  $a$  and  $b$  with  $a + b = 1$ .*

### 2. Proof of Theorem 1

If  $a = 1$  or  $b = 1$  then  $a^{2b} + b^{2a} = 1$ , hence we consider the case of  $a \neq 1$  and  $b \neq 1$ . Without loss of generality, we may assume that  $a = (1-t)/2$  and  $b = (1+t)/2$ , where  $0 < t < 1$ . Here, we have

$$a^{2b} + b^{2a} = 2^{-1+t}(1+t)^{1-t} + 2^{-1-t}(1-t)^{1+t}.$$

**LEMMA 1.** *We have*

$$2^{-1+t}(1+t)^{1-t} > 1 - \frac{1}{2}(t-1)^2 + \frac{1}{8}(t-1)^3 + \frac{1}{12}(t-1)^4 - \frac{3}{64}(t-1)^5 - \frac{1}{192}(t-1)^6,$$

for  $0 < t < 1$ .

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*Proof.* We set

$$f(t) = (-1+t)\ln 2 + (1-t)\ln(1+t) - \ln\left(1 - \frac{1}{2}(t-1)^2 + \frac{1}{8}(t-1)^3 + \frac{1}{12}(t-1)^4 - \frac{3}{64}(t-1)^5 - \frac{1}{192}(t-1)^6\right).$$

Then the derivatives of  $f(t)$  are

$$f'(t) = \frac{(1-t)(65 + 167t + t^2 - 32t^3 - 19t^4 + 9t^5 + t^6)}{(1+t)(-96 - 161t - 3t^2 + 110t^3 - 46t^4 + 3t^5 + t^6)} + \ln 2 - \ln(1+t)$$

and

$$f''(t) = \frac{(1-t)^4 g(t)}{(1+t)^2(-96 - 161t - 3t^2 + 110t^3 - 46t^4 + 3t^5 + t^6)^2},$$

where

$$g(t) = -2303 + 3852t + 19846t^2 + 14219t^3 - 3654t^4 - 2603t^5 + 518t^6 + 85t^7 - 7t^8 - t^9.$$

The derivatives of  $g(t)$  are

$$g'(t) = 3852 + 39692t + 42657t^2 - 14616t^3 - 13015t^4 + 3108t^5 + 595t^6 - 56t^7 - 9t^8$$

and

$$g''(t) = 2(19846 + 42657t - 21924t^2 - 26030t^3 + 7770t^4 + 1785t^5 - 196t^6 - 36t^7),$$

then we have

$$\begin{aligned} 19846 + 42657t - 21924t^2 - 26030t^3 &= 19846 + t(42657 - 21924t - 26030t^2) \\ &> 19846 + t(42657 - 21924 \times 1 - 26030 \times 1^2) \\ &= 19846 - 5297t > 14549 > 0 \end{aligned}$$

and

$$\begin{aligned} 7770t^4 + 1785t^5 - 196t^6 - 36t^7 &= t^4(7734 + 1785t - 196t^2) \\ &> t^4(7734 - 196 \times 1^2) = 7538t^4 > 0. \end{aligned}$$

Therefore, we can get  $g''(t) > 0$  and  $g'(t)$  is strictly increasing for  $0 < t < 1$ . By  $g'(0) = 3852 > 0$ ,  $g'(t) > 0$ , for  $0 < t < 1$ , and  $g(t)$  is strictly increasing for  $0 < t < 1$ . From  $g(0) = -2303 < 0$  and  $g(1) = 29952 > 0$ , there exists a unique real number  $t_0$  such that  $g(t) < 0$ , for  $0 < t < t_0$ , and  $g(t) > 0$ , for  $t_0 < t < 1$ . Hence,  $f'(t)$  is strictly decreasing for  $0 < t < t_0$  and strictly increasing for  $t_0 < t < 1$ . Since  $f'(0) = \ln 2 - 65/96 > 693/1000 - 65/96 = 191/12000 > 0$  and  $f'(1) = 0$ , there exists a unique real number  $t_1$  such that  $f(t)$  is strictly increasing for  $0 < t < t_1$  and strictly decreasing for  $t_1 < t < 1$ . By  $f(0) = f(1) = 0$ , we obtain  $f(t) > 0$ , for  $0 < t < 1$ .  $\square$  We show Lemmas 2 and 3 to prove Lemmas 4 and 5. The following Lemma 2 is given by Cırtoaje [2].

LEMMA 2. *We have*

$$\ln(1-t) \leq -t - \frac{t^2}{2} - \frac{t^3}{3}, \quad \text{for } 0 < t < 1.$$

LEMMA 3. *We have*

$$\ln(1-t) > -t - \frac{t^2}{2} - \frac{t^3}{3} - \frac{t^4}{2}, \quad \text{for } 0 < t < 1/2.$$

*Proof.* We set

$$f(t) = \ln(1-t) - \left( -t - \frac{t^2}{2} - \frac{t^3}{3} - \frac{t^4}{2} \right),$$

then the derivative of  $f(t)$  is

$$f'(t) = \frac{t^3(1-2t)}{1-t} > 0,$$

for  $0 < t < 1/2$ . Thus,  $f(t)$  is strictly increasing for  $0 < t < 1/2$ . From  $f(0) = 0$ , we have  $f(t) > 0$ , for  $0 < t < 1/2$ .  $\square$

We set

$$F(t) = 2^{-1-t}(1-t)^{1+t} + \left( 1 - \frac{1}{2}(t-1)^2 + \frac{1}{8}(t-1)^3 + \frac{1}{12}(t-1)^4 - \frac{3}{64}(t-1)^5 - \frac{1}{192}(t-1)^6 \right),$$

for  $0 < t \leq 1/2$ . By Lemma 1, we have  $2^{-1+t}(1+t)^{1-t} + 2^{-1-t}(1-t)^{1+t} > F(t)$ , for  $0 < t \leq 1/2$ .

LEMMA 4. *The function  $F(t)$  is strictly decreasing for  $0 < t < 1/2$ .*

*Proof.* The derivatives of  $F(t)$  is

$$F'(t) = \frac{1}{192}(1-t)(161 + 167t - 163t^2 + 21t^3 + 6t^4) - 2^{-1-t}(1-t)^{1+t} \ln 2 - 2^{-1-t}(1-t)^{1+t} \left( \frac{1+t}{1-t} - \ln(1-t) \right).$$

Since  $F'(t) < 0$  is equivalent to

$$g(t) = -\ln 192 + \ln(161 + 167t - 163t^2 + 21t^3 + 6t^4) + (1+t)\ln 2 - t\ln(1-t) - \ln \left( \ln 2 + \frac{1+t}{1-t} - \ln(1-t) \right) < 0,$$

we may show  $g(t) < 0$ , for  $0 < t < 1/2$ . From Lemmas 2 and 3 we have

$$-t - \frac{t^2}{2} - \frac{t^3}{3} - \frac{t^4}{2} < \ln(1-t) < -t - \frac{t^2}{2} - \frac{t^3}{3},$$

for  $0 < t < 1/2$  and  $g(t) < h(t)$ , where

$$h(t) = -\ln 192 + \ln(161 + 167t - 163t^2 + 21t^3 + 6t^4) + (1+t)\frac{694}{1000} \\ - t \left( -t - \frac{t^2}{2} - \frac{t^3}{3} - \frac{t^4}{2} \right) - \ln \left( \frac{693}{1000} + \frac{1+t}{1-t} - \left( -t - \frac{t^2}{2} - \frac{t^3}{3} \right) \right).$$

The derivative of  $h(t)$  is

$$h'(t) = \frac{1}{1500(1-t)(161 + 167t - 163t^2 + 21t^3 + 6t^4)} \\ \times \frac{k(t)}{5079 + 3921t - 1500t^2 - 500t^3 - 1000t^4},$$

where

$$k(t) = -49965021 - 1161666045t + 4493699946t^2 - 342761904t^3 - 1329740811t^4 \\ - 666084099t^5 - 5848252316t^6 + 350609750t^7 + 3306897750t^8 \\ - 616264750t^9 + 612152500t^{10} - 578625000t^{11} + 79500000t^{12} + 22500000t^{13}.$$

The derivatives of  $k(t)$  are

$$k'(t) = -1161666045 + 8987399892t - 1028285712t^2 - 5318963244t^3 \\ - 3330420495t^4 - 35089513896t^5 + 2454268250t^6 + 26455182000t^7 \\ - 5546382750t^8 + 6121525000t^9 - 6364875000t^{10} + 954000000t^{11} \\ + 292500000t^{12},$$

$$k''(t) = 12(748949991 - 171380952t - 1329740811t^2 - 1110140165t^3 \\ - 14620630790t^4 + 1227134125t^5 + 15432189500t^6 - 3697588500t^7 \\ + 4591143750t^8 - 5304062500t^9 + 874500000t^{10} + 292500000t^{11})$$

and

$$k'''(t) = 12(-171380952 - 2659481622t - 3330420495t^2 - 58482523160t^3 \\ + 6135670625t^4 + 92593137000t^5 - 25883119500t^6 + 36729150000t^7 \\ - 47736562500t^8 + 8745000000t^9 + 3217500000t^{10}) \\ = 12(-171380952 - 2659481622t - 3330420495t^2 \\ + 5t^3(-11696504632 + 1227134125t) + 92593137000t^5 \\ + 1500t^6(-17255413 + 24486100t - 31824375t^2 + 5830000t^3 + 2145000t^4)).$$

By

$$\begin{aligned}
& 5t^3(-11696504632 + 1227134125t) + 92593137000t^5 \\
& < 5t^3\left(-11696504632 + 1227134125\left(\frac{1}{2}\right)\right) + 92593137000t^5 \\
& = 5t^3\left(-\frac{22165875139}{2}\right) + 92593137000t^5 = \frac{5}{2}t^3(-22165875139 + 37037254800t^2) \\
& < \frac{5}{2}t^3\left(-22165875139 + 37037254800\left(\frac{1}{2}\right)^2\right) = -\frac{5}{2}t^3 \times 12906561439 < 0
\end{aligned}$$

and

$$\begin{aligned}
& 1500t^6(-17255413 + 24486100t - 31824375t^2 + 5830000t^3 + 2145000t^4) \\
& < 1500t^6\left(-17255413 + 24486100\left(\frac{1}{2}\right) - 31824375t^2 + 5830000\left(\frac{1}{2}\right)^3\right. \\
& \quad \left.+ 2145000\left(\frac{1}{2}\right)^4\right) = -1500t^6\left(\frac{8299101}{2} + 31824375t^2\right) < 0,
\end{aligned}$$

we can get  $k'''(t) < 0$ , for  $0 < t < 1/2$ , and  $k''(t)$  is strictly decreasing for  $0 < t < 1/2$ . From  $k''(0) = 8987399892 > 0$  and  $k''(1/2) = -22203487497/4 < 0$ , there exists a unique real number  $t_0$  such that  $k''(t)$  is strictly increasing for  $0 < t < t_0$  and strictly decreasing for  $t_0 < t < 1/2$ . Hence,  $k'(t)$  is strictly increasing for  $0 < t < t_0$  and strictly decreasing for  $t_0 < t < 1/2$ . By  $k'(0) = -1161666045 < 0$  and  $k'(1/2) = 170884429721/128 > 0$ , there exists a unique real number  $t_1$  such that  $k(t)$  is strictly decreasing for  $0 < t < t_1$  and strictly increasing for  $t_1 < t < 1/2$ . From  $k(0) = -499650211 < 0$  and  $k(1/2) = 34466474099/128 > 0$ , there exists a unique real number  $t_2$  such that  $h(t)$  is strictly decreasing for  $0 < t < t_2$  and strictly increasing for  $t_2 < t < 1/2$ . By

$$\begin{aligned}
h(0) &= \frac{347}{500} - \ln \frac{1693}{1000} + \ln 161 - \ln 192 = \frac{347}{500} + \ln \frac{20125}{40632} < \frac{347}{500} + \ln \frac{62}{125} \\
&= \frac{347}{500} + \ln 2 + \ln 31 - 3\ln 5 < \frac{347}{500} + \frac{694}{1000} + \frac{1717}{500} - 3\frac{8047}{5000} = -\frac{31}{5000} < 0
\end{aligned}$$

and

$$\begin{aligned}
h\left(\frac{1}{2}\right) &= \frac{33359}{24000} - \ln \frac{13079}{3000} - \ln 192 + \ln \frac{827}{4} = \frac{33359}{24000} + \ln \frac{103375}{418528} \\
&< \frac{33359}{24000} + \ln \frac{99}{400} = \frac{33359}{24000} + 2\ln 3 + \ln 11 - 4\ln 2 - 2\ln 5 \\
&< \frac{33359}{24000} + 2\frac{54931}{50000} + \frac{23979}{10000} - 4\frac{693}{1000} - 2\frac{8047}{5000} = -\frac{3421}{600000} < 0,
\end{aligned}$$

we can get  $h(t) < 0$  and  $g(t) < 0$  for  $0 < t < 1/2$ .  $\square$

Next, we set

$$G(t) = 2^{-1-t}(1-t)^{1+t} + \left(1 - \frac{1}{2}(t-1)^2 + \frac{1}{8}(t-1)^3 + \frac{1}{12}(t-1)^4 - \frac{3}{64}(t-1)^5 - \frac{1}{16}(t-1)^6\right),$$

for  $1/2 \leq t < 1$ . By Lemma 1, we have  $2^{-1+t}(1+t)^{1-t} + 2^{-1-t}(1-t)^{1+t} > G(t)$ , for  $1/2 \leq t < 1$ .

LEMMA 5. *The function  $G(t)$  is strictly increasing for  $1/2 < t < 1$ .*

*Proof.* The derivative of  $G(t)$  is

$$G'(t) = \frac{1}{192}(1-t)(227 - 97t + 233t^2 - 243t^3 + 72t^4) - 2^{-1-t}(1-t)^{1+t} \ln 2 - 2^{-1-t}(1-t)^{1+t} \left( \frac{1+t}{1-t} - \ln(1-t) \right).$$

From the polynomial

$$\begin{aligned} p(t) &= 227 - 97t + 233t^2 - 243t^3 + 72t^4 = 227 + t(-97 + 233t) + t^3(-243 + 72t) \\ &> 227 + t \left( -97 + 233 \left( \frac{1}{2} \right) \right) + t^3 \left( -243 + 72 \left( \frac{1}{2} \right) \right) > 227 - 107t^3 > 0, \end{aligned}$$

we can take logarithm. Since  $G'(t) > 0$  is equivalent to

$$\begin{aligned} g(t) &= -\ln 192 + \ln(227 - 97t + 233t^2 - 243t^3 + 72t^4) + (1+t)\ln 2 - t\ln(1-t) \\ &\quad - \ln \left( \ln 2 + \frac{1+t}{1-t} - \ln(1-t) \right) > 0, \end{aligned}$$

we may show  $g(t) > 0$ , for  $1/2 < t < 1$ . The derivative of  $g(t)$  is

$$\begin{aligned} g'(t) &= \frac{t}{1-t} + \frac{-97 + 466t - 729t^2 + 288t^3}{227 - 97t + 233t^2 - 243t^3 + 72t^4} + \ln 2 - \frac{\frac{2}{1-t} + \frac{1+t}{(1-t)^2}}{\frac{1+t}{1-t} + \ln 2 - \ln(1-t)} \\ &\quad - \ln(1-t). \end{aligned}$$

From Lemma 2, we have

$$\ln(1-t) < -t - \frac{t^2}{2} - \frac{t^3}{3},$$

for  $1/2 < t < 1$  and  $g'(t) > h(t)$ , where

$$\begin{aligned} h(t) &= \frac{t}{1-t} + \frac{-97 + 466t - 729t^2 + 288t^3}{227 - 97t + 233t^2 - 243t^3 + 72t^4} + \frac{693}{1000} \\ &\quad - \frac{\frac{2}{1-t} + \frac{1+t}{(1-t)^2}}{\frac{1+t}{1-t} + \frac{693}{1000} - \left( -t - \frac{t^2}{2} - \frac{t^3}{3} \right)} - \left( -t - \frac{t^2}{2} - \frac{t^3}{3} \right). \end{aligned}$$

Here, we can get

$$h(t) = \frac{1}{3000(227 - 97t + 233t^2 - 243t^3 + 72t^4)} \\ \times \frac{k(t)}{5079 + 3921t - 1500t^2 - 500t^3 - 1000t^4},$$

where

$$k(t) = -5210041293 + 12236231916t - 5286409470t^2 + 2463991284t^3 \\ + 5236275915t^4 - 5240435352t^5 + 1472819000t^6 - 1671294000t^7 \\ + 985374000t^8 - 125000000t^9 + 99000000t^{10} - 72000000t^{11}.$$

The derivative of  $k(t)$  is

$$k'(t) = 12236231916 - 10572818940t + 7391973852t^2 + 20945103660t^3 \\ - 26202176760t^4 + 8836914000t^5 - 11699058000t^6 + 7882992000t^7 \\ - 1125000000t^8 + 990000000t^9 - 792000000t^{10}.$$

By

$$12236231916 - 10572818940t > 12236231916 - 10572818940 \times 1 = 1663412976, \\ 7391973852t^2 + 20945103660t^3 - 26202176760t^4 \\ = 12t^2(615997821 + 1745425305t - 2183514730t^2) \\ = 12t^2(615997821 + 5t(349085061 - 436702946t)) \\ > 12t^2(615997821 + 5t(349085061 - 436702946 \times 1)) \\ = 12t^2(615997821 - 5t \times 87617885) > 12t^2(615997821 - 5 \times 1 \times 87617885) \\ = 12t^2 \times 177908396 > 12 \left(\frac{1}{2}\right)^2 \times 177908396 = 533725188, \\ 8836914000t^5 - 11699058000t^6 + 7882992000t^7 - 1125000000t^8 \\ = 6000t^5(1472819 - 1949843t + 1313832t^2 - 187500t^3) \\ > 6000t^5(1472819 + t(-1949843 + 1313832t) - 187500 \times 1) \\ > 6000t^5(1472819 + t(-1949843 + 1313832 \left(\frac{1}{2}\right)) - 187500 \times 1) \\ > 6000t^5(1285319 - 1292927t) > -45648000t^5 > -45648000 \times 1 = -45648000$$

and

$$990000000t^9 - 792000000t^{10} = 198000000t^9(5 - 4t) > 198000000 \left(\frac{1}{2}\right)^9 = \frac{1546875}{4},$$

we have

$$k'(t) > 1663412976 + 533725188 - 45648000 + \frac{1546875}{4} = 2151490164 + \frac{1546875}{4} > 0$$

and  $k(t)$  is strictly increasing for  $1/2 < t < 1$ . From  $k(1/2) = 1145554567/16 > 0$ ,  $k(t) > 0$  and  $h(t) > 0$ , for  $1/2 < t < 1$ . Thus,  $g(t)$  is strictly increasing for  $1/2 < t < 1$ . Since we can get

$$\begin{aligned} g\left(\frac{1}{2}\right) &= 2\ln 2 - \ln 192 + \ln \frac{1687}{8} - \ln(3 + 2\ln 2) \\ &> 2\ln 2 - \ln 192 + \ln \frac{1687}{8} - \ln\left(3 + 2\frac{694}{1000}\right) = \ln \frac{210875}{210624} > 0, \end{aligned}$$

we obtain  $g(t) > 0$ , for  $1/2 < t < 1$ .  $\square$

*Proof of Theorem 1.* By Lemma 1 and 4, we have

$$2^{-1+t}(1+t)^{1-t} + 2^{-1-t}(1-t)^{1+t} > F\left(\frac{1}{2}\right) = \frac{4059}{4096} \cong 0.990967,$$

for  $0 < t < 1/2$ . From Lemma 1 and 5, we have

$$2^{-1+t}(1+t)^{1-t} + 2^{-1-t}(1-t)^{1+t} > G\left(\frac{1}{2}\right) = \frac{6083}{6144} \cong 0.990072,$$

for  $1/2 < t < 1$ . Thus, the proof of Theorem 1 is complete.  $\square$

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