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FUNDAMENTAL THEOREMS OF SUMMABILITY THEORY FOR A NEW TYPE OF SUBSEQUENCES OF DOUBLE SEQUENCES

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Abstract. In 2000, the notion of a subsequence of a double sequence was introduced [3]. Using this definition, a multidimensional analogue to a result from H. Steinhaus, that states that for any regular matrix A there exists a sequence of zeros and ones that is not A-summable, was proved. Additionally, an analogue of a result of R. C. Buck that states that a sequence x is convergent if and only if there exists a regular matrix A that sums every subsequence of x was presented. However, this definition imposes a restrictive condition on the entries of the double sequence that can be considered for the subsequence. In this article, we introduce a less restrictive new definition of a subsequence. We denote them by β-subsequences of a double sequence and show that analogues to these two fundamental theorems of summability still hold for these new subsequences.

1. Introduction

In a seminal article, Patterson introduced the definition of a subsequence of a double sequence [3]. He, then, established two fundamental theorems of summability theory for these subsequences. Namely, the author showed that for any regular 4-dimensional matrix transformation, in the sense of Robison and Hamilton [2, 6], A, there exists a double sequence of 0's and 1's that is not A-summable. Additionally, he showed that the following characterization holds for these subsequences: "A double sequence x is convergent in the Pringsheim sense if and only if there exists a regular 4-dimensional matrix transformation, A, such that A sums every subsequence of x."

However, the construction of these subsequences requires that one imposes a very stringent condition on the subindices eligible to form them. It is the goal of this article to introduce a family of sequences, to be denoted β-subsequences ($\beta > 1$), of double sequences that still satisfy the stated summability theorems but that do not impose such stringent condition.

Therefore, in Section 2, we use an idea similar to that used for β-rearrangements [4], to introduce the concept of a β-subsequence of a double sequence. In Section 3, we start by establishing the following basic notions of analysis of sequences for β-subsequences, that is, we show that if a double sequence is convergent, all of its β-subsequences are convergent and converge to the same limit (see Proposition 1). Following that, we show that for any β-regular 4-dimensional matrix transformation,

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A, there exists a double sequence of 0's and 1's that is not A-summable (see Definition 7 and Theorem 2). We conclude by showing that a double sequence x is convergent in the Pringsheim sense if and only if there exists a β-regular 4-dimensional matrix transformation, A, such that A sums every β-subsequence of x.

2. Definitions and notation

Let $\psi : \mathbb{N}\times\mathbb{N}\to\mathbb{N}$ be given in the following way

$$\begin{aligned} (1,1) &\mapsto 1 & (1,2) &\mapsto 2\\ (2,2) &\mapsto 3 & (2,1) &\mapsto 4\\ (1,3) &\mapsto 5 & (2,3) &\mapsto 6\\ (3,3) &\mapsto 7 & (3,2) &\mapsto 8\ldots. \end{aligned}$$

In matrix form, this can be encoded as

$$\begin{pmatrix} \psi(1,1) & \psi(1,2) & \psi(1,3) & \psi(1,4) & \cdots\\ \psi(2,1) & \psi(2,2) & \psi(2,3) & \psi(2,4) & \cdots\\ \psi(3,1) & \psi(3,2) & \psi(3,3) & \psi(3,4) & \cdots\\ \psi(4,1) & \psi(4,2) & \psi(4,3) & \psi(4,4) & \cdots\\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} 1 & 2 & 5 & 10 & \cdots\\ 4 & 3 & 6 & 11 & \cdots\\ 9 & 8 & 7 & 12 & \cdots\\ 16 & 15 & 14 & 13 & \cdots\\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Clearly, ψ is a bijection between $\mathbb{N}$ and $\mathbb{N}\times\mathbb{N}$. Thus, it is invertible. This map ψ should be though of as a flattening function of the double sequence. We use this flattening function to introduce the definition of a β-subsequence of a double sequence. Before that, we start by defining a β-*section* $S_\beta \subset \mathbb{N}\times\mathbb{N}$ by

$$S_\beta := \left\{ (m,n)\in\mathbb{N}\times\mathbb{N} \;\middle|\; \frac{1}{\beta}\leqslant\frac{m}{n}\leqslant\beta \right\}.$$

DEFINITION 1. (β-subsequence) Let $x=[x_{k,l}]$ be a double sequence and let $\beta>1$ be an extended real. The double sequence $y^{(\pi,\beta)}$ is called a β-*subsequence* of the double sequence x if and only if there exists a strictly increasing function $\pi:\psi(S_\beta)\to\psi(S_\beta)$ such that

$$y^{(\pi,\beta)}_{p,q} = \begin{cases} z_{\psi(p,q)}, & \text{if } \frac{1}{\beta}>\frac{p}{q} \text{ or } \frac{p}{q}>\beta,\\ z_{\pi(\psi(p,q))}, & \text{if } \frac{1}{\beta}\leqslant\frac{p}{q}\leqslant\beta, \end{cases}$$

where $z_i = x_{\psi^{-1}(i)}$. If $\beta=+\infty$, the inequalities are understood in the limit sense.

Some remarks are in order.

REMARK 1. Firstly, it must be noted that a double subsequence in the sense of [3] of x can be realized as a $+\infty$-subsequence of x. However, an arbitrary β-sequence, cannot be realized as a double subsequence in the sense of [3]. Thus, the previous definition provides a generalization of the concept of subsequence of a double sequence.

REMARK 2. Second, a double subsequence in the sense of [3] is not a subsequence of itself. However, every double subsequence is a β-subsequence of itself where the map π is the identity map on S_β.

For convenience, we consider the compatible decomposition of the double sequence x as

$$x = \Pi(x) + B(x) + \Upsilon(x),$$

where

$$B(x)_{m,n} = \begin{cases} x_{m,n}, & \text{if } \frac{1}{\beta} \leqslant \frac{p}{q} \leqslant \beta, \\ 0, & \text{otherwise,} \end{cases}$$

$$\Pi(x)_{m,n} = \begin{cases} x_{m,n}, & \text{if } \frac{p}{q} > \beta, \\ 0, & \text{otherwise,} \end{cases}$$

$$\Upsilon(x)_{m,n} = \begin{cases} x_{m,n}, & \text{if } \frac{p}{q} < \frac{1}{\beta}, \\ 0, & \text{otherwise.} \end{cases}$$

For computational convenience, we assume the convention $\psi^{-1}(i) = (m_i, n_i)$.

DEFINITION 2. (Summability method [6]) Let A be a four dimensional summability method that maps the complex double sequences x into the double sequence Ax where the m,n-th term of Ax is given by

$$(Ax)_{m,n} = \sum_{k,l=1}^{\infty} a_{m,n,k,l} x_{k,l}.$$

DEFINITION 3. (P-convergence [5]) A double sequence $x = [x_{k,l}]$ has a *Pringsheim limit* L if and only if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|x_{k,l} - L| < \varepsilon,$$

whenever $k,l > N$. In this case, we say x is *P-convergent* and we denote it by

$$L = \lim_{k,l\to\infty} x_{k,l}.$$

Unless otherwise specified, the notation $\lim_{k,l\to\infty}$ is reserved in this article to limits in the Pringsheim sense.

For our purposes, we need to give an equivalent definition of a P-limit point as the one given in [3]. This is due to the fact stated in Remark 3. The advantage of the following definition is its independence from the definition of subsequence.

DEFINITION 4. (P-limit points) A double sequence $x = [x_{k,l}]$ has a *Pringsheim limit point* L if and only if for every $\varepsilon > 0$ and $N \in \mathbb{N}$, there exist $k,l \geqslant N$ such that

$$|x_{k,l} - L| < \varepsilon.$$

REMARK 3. Subsequences in the sense of [3] satisfy the following statement: "*If L is a P-limit point of x, then there exists a subsequence of x whose P-limit is L.*" This is not the case for β-subsequences. Indeed, consider the double sequence x such that

$$x_{m,n} = \begin{cases} 1, & \text{if } m = n, \\ 0, & \text{if } m \neq n, \end{cases}$$

and a finite $\beta > 1$. Clearly, 0 and 1 are P-limit points of x. While there are β-subsequences of x converging to 0, there are no β-subsequences converging to 1.

Pringsheim also introduces a stronger notion of divergence.

DEFINITION 5. (Definite divergence [5]) A double sequence $x = [x_{k,l}]$ is said to be *definite divergent* if for every $G > 0$, there exist naturals n, m such that $|x_{k,l}| > G$ for all $k > n, l > m$.

DEFINITION 6. (RH-regular [6]) Let A be a four dimensional matrix. A is said to be *RH-regular* if it maps every bounded P-convergent sequence into a P-convergent sequence with the same P-limit.

THEOREM 1. (Hamilton [2], Robison [6]) *A 4-dimensional matrix A is RH-regular if and only if:*

(RH1) $\lim_{m,n\to\infty} a_{m,n,k,l} = 0$, *for each* $(k,l) \in \mathbb{N}^2$*;*

(RH2) $\lim_{m,n\to\infty} \sum_{k,l=0}^{\infty,\infty} a_{m,n,k,l} = 1$*;*

(RH3) $\lim_{m,n\to\infty} \sum_{k=0}^{\infty} |a_{m,n,k,l}| = 0$, *for each* $l \in \mathbb{N}$*;*

(RH4) $\lim_{m,n\to\infty} \sum_{l=0}^{\infty} |a_{m,n,k,l}| = 0$, *for each* $k \in \mathbb{N}$*;*

(RH5) $\lim_{m,n\to\infty} \sum_{k,l=0}^{\infty,\infty} |a_{m,n,k,l}|$ *is P-convergent;*

(RH6) there exist finite positive integers A and B such that

$$\sum_{\substack{k>B \\ l>B}} |a_{m,n,k,l}| < A,$$

for each $(m,n) \in \mathbb{N}^2$.

DEFINITION 7. (β-regular) A 4-dimensional matrix A is said to be *β-regular* if and only if A is RH-regular and $\lim_{m,n\to\infty} \sum_{(k,l)\in S_\beta} a_{m,n,k,l} = 1$.

3. Results

PROPOSITION 1. *Let $\beta > 1$. Then, the double sequence x is P-convergent to L if and only if every β-subsequence of x is P-convergent to L.*

Proof. Assume x is P-convergent to L and let $\varepsilon > 0$. Assume, $y^{(\pi,\beta)}$ is a subsequence of x. By the P-convergence of x, there exists $N \in \mathbb{N}$ such that

$$|x_{k,l} - L| < \varepsilon,$$

whenever $k,l > N$.

We consider two cases, when $(k,l) \notin S_\beta$ and when $(k,l) \in S_\beta$. In the former case, $x_{k,l} = y^{(\pi,\beta)}_{k,l}$ so

$$|y^{(\pi,\beta)}_{k,l} - L| < \varepsilon,$$

in this case.

The latter case, when $(k,l) \in S_\beta$ is a little more delicate. Consider $y^{(\pi,\beta)}_{k,l}$ where $k > N, l > N$. Then,

$$y^{(\pi,\beta)}_{k,l} = x_{\psi^{-1}(\pi(\psi(k,l)))}.$$

If $\pi(\psi(k,l)) = \psi(p,q)$, there is no guarantee that $p > N$ or $q > N$, thus $|y^{(\pi,\beta)}_{k,l} - L| < \varepsilon$ may not be satisfied. See Figure 1.

To circumvent this situation, define $M \in \mathbb{N}$ by

$$M = \max\{p \in \mathbb{N} \mid 1/\beta \leqslant p/N \leqslant \beta\}$$

and consider $y^{(\pi,\beta)}_{k,l} = x_{p,q}$, where $k > M, l > M$ where $(k,l) \in S_\beta$. Notice that since π is strictly increasing and $\pi(1) \geqslant 1$ we have that $\psi(p,q) = \pi(\psi(k,l)) \geqslant \psi(k,l)$. If equality holds, there is nothing to show, so assume that $\psi(p,q) > \psi(k,l)$. By the construction of ψ, this implies $p > k$ or $q > l$. We claim it is not possible for $p \leqslant N$ or $q \leqslant N$. For a contradiction, assume $q \leqslant N$. By the definition of N, it is clear that $N \leqslant M$. Thus, $q \leqslant N \leqslant M < k < p$ as it is not possible for $q > l$ to hold.

This implies that $1 \leqslant \frac{p}{N} \leqslant \frac{p}{q}$. Since $(p,q) \in S_\beta$, we have

$$\frac{1}{\beta} < 1 \leqslant \frac{p}{N} \leqslant \frac{p}{q} < \beta.$$

However, $p > M$ which contradicts the maximality of M. The other case is handled similarly.

Thus, for all $k,l > M$

$$|y^{(\pi,\beta)}_{k,l} - L| < \varepsilon$$

and the β-subsequence is convergent.

For the converse, just note that x is the β-subsequence where $\pi(x) = x$ on S_β. Therefore, x is P-convergent by hypothesis. □

In the following, we modify the proof of [3, Theorem 3.1] to obtain a “Steinhaus-type” theorem. This will prove a powerful tool in establishing a characterization of P-convergence in terms of β-subsequences below.

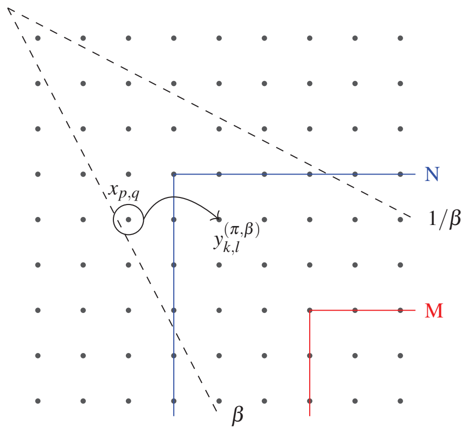

Figure 1: Pictorial representation of how an element in the subsequence (circled) may fail to belong to $\{(k,l) \mid k,l > N\}$.

THEOREM 2. *Let $\beta > 1$ and let A be a β-regular four dimensional matrix. Then, there exists a sequence x with support on S_β whose entries are only equal to 1 or 0 such that x is not A-summable.*

Proof. As in [3, Theorem 3.1] for each $i \in \mathbb{N}$, we pick coefficients

$$m_0 < \cdots < m_i, \qquad k_0 < \cdots < k_i,$$
$$n_0 < \cdots < n_i, \qquad l_0 < \cdots < l_i,$$

inductively with such that by (RH1),

$$\sum_{\substack{k \leqslant k_i \\ l \leqslant l_i \\ (k,l) \in S_\beta}} |a_{m_i,n_i,k,l}| \leqslant \sum_{\substack{k \leqslant k_i \\ l \leqslant l_i}} |a_{m_i,n_i,k,l}| < \frac{1}{(i+2)^2},$$

and by (RH3), (RH4)

$$\sum_{\substack{k \leqslant k_i \\ l > l_i \\ (k,l) \in S_\beta}} |a_{m_i,n_i,k,l}| \leqslant \sum_{\substack{k \leqslant k_i \\ l > l_i}} |a_{m_i,n_i,k,l}| < \frac{1}{(i+2)^2},$$

$$\sum_{\substack{k > k_i \\ l \leqslant l_i \\ (k,l) \in S_\beta}} |a_{m_i,n_i,k,l}| \leqslant \sum_{\substack{k > k_i \\ l \leqslant l_i}} |a_{m_i,n_i,k,l}| < \frac{1}{(i+2)^2}. \tag{1}$$

In addition, by the β-regularity of A, pick m_i and n_i so that

$$\left|\sum_{(k,l)\in S_\beta} a_{m_i,n_i,k,l}\right| > 1-\frac{1}{(i+2)^2}.$$

So that

$$\sum_{\substack{k>k_i\\ l>l_i\\ (k,l)\in S_\beta}} |a_{m_i,n_i,k,l}| \geqslant \left|\sum_{(k,l)\in S_\beta} a_{m_i,n_i,k,l}\right| - \sum_{\substack{k\leqslant k_i\\ l\leqslant l_i\\ (k,l)\in S_\beta}} |a_{m_i,n_i,k,l}| - \sum_{\substack{k\leqslant k_i\\ l>l_i\\ (k,l)\in S_\beta}} |a_{m_i,n_i,k,l}|$$
$$- \sum_{\substack{k>k_i\\ l\leqslant l_i\\ (k,l)\in S_\beta}} |a_{m_i,n_i,k,l}| > 1-\frac{4}{(i+2)^2}.$$

With these coefficients chosen, we proceed to choose $k_{i+1}>k_i$ and $l_{i+1}>l_i$ such that

$$\left|\sum_{\substack{k_i<k<k_{i+1}\\ l_i<l<l_{i+1}\\ (k,l)\in S_\beta}} a_{m_i,n_i,k,l}\right| > 1-\frac{4}{(i+2)^2}, \quad \sum_{\substack{k\geqslant k_{i+1}\\ l\geqslant l_{i+1}}} |a_{m_i,n_i,k,l}| < \frac{1}{(i+2)^2},$$
$$\sum_{\substack{k_i<k<k_{i+1}\\ l\geqslant l_{i+1}}} |a_{m_i,n_i,k,l}| < \frac{1}{(i+2)^2}, \quad \sum_{\substack{k\geqslant k_{i+1}\\ l_i<l<l_{i+1}}} |a_{m_i,n_i,k,l}| < \frac{1}{(i+2)^2}. \tag{2}$$

Now, we define the double sequence x by

$$x_{k,l} = \begin{cases} 1, & \text{if } (k,l)\in S_\beta, k_{2p}<k<k_{2p+1} \text{ and } l_{2p}<l<l_{2p+1}, \text{ for } p\in\mathbb{N},\\ 0, & \text{otherwise.}\end{cases} \tag{3}$$

Noting that $a_{m,n,k,l}x_{k,l}=0$ whenever $(k,l)\notin S_\beta$, we have that the m_i,n_i'th term of the double sequence Ax is given by

$$\begin{aligned}(Ax)_{m_i,n_i} &= \sum_{(k,l)\in S_\beta} a_{m_i,n_i,k,l}\\ &= \sum_{\substack{k\leqslant k_i\\ l\leqslant l_i\\ (k,l)\in S_\beta}} a_{m_i,n_i,k,l}x_{k,l} + \sum_{\substack{k\leqslant k_i\\ l>l_i\\ (k,l)\in S_\beta}} a_{m_i,n_i,k,l}x_{k,l} + \sum_{\substack{k>k_i\\ l\leqslant l_i\\ (k,l)\in S_\beta}} a_{m_i,n_i,k,l}x_{k,l}\\ &\quad + \sum_{\substack{k_i<k<k_{i+1}\\ l_i<l<l_{i+1}\\ (k,l)\in S_\beta}} a_{m_i,n_i,k,l}x_{k,l} + \sum_{\substack{k_i<k<k_{i+1}\\ l\geqslant l_{i+1}\\ (k,l)\in S_\beta}} a_{m_i,n_i,k,l}x_{k,l} + \sum_{\substack{k\geqslant k_{i+1}\\ l_i<l<l_{i+1}\\ (k,l)\in S_\beta}} a_{m_i,n_i,k,l}x_{k,l}\end{aligned} \tag{4}$$

$$+\sum_{\substack{k\geqslant k_{i+1}\\ l\geqslant l_{i+1}\\ (k,l)\in S_\beta}} a_{m_i,n_i,k,l}x_{k,l} =: \sum_{j=1}^{7} I_j.$$

We index each of the sums by I_j for $j=1,\dots,7$. Now based on (3), we note that $I_4=0$ or $I_4>1-\frac{4}{(i+2)^2}$ depending on whether i is odd or even, respectively. So, whenever i is odd, we have that

$$|(Ax)_{m_i,n_i}| \leqslant \sum_{j\neq 4}|I_j| \leqslant \frac{6}{(i+2)^2},$$

which has P-limit equal to zero. On the other hand, however, when i is even, by the reverse triangle inequality and (4) we obtain

$$|(Ax)_{m_i,n_i}| \geqslant |I_4| - \sum_{j\neq 4}|I_j|$$

and by (1) and (2), we have

$$|(Ax)_{m_i,n_i}| > 1-\frac{4}{(i+2)^2}-\sum_{j\neq 4}\frac{1}{(i+2)^2}.$$

The latter expression has P-limit equal to 1. Thus, A cannot sum x. $\square$

LEMMA 1. *Suppose that y and z are two convergent β-subsequences of x. If $\lim_{m,n\to\infty} y_{m,n} = \lim_{m,n\to\infty} z_{m,n}$, then $\lim_{m,n\to\infty}(B(y)_{m,n}-B(z)_{m,n})=0$. In particular, if $B(x)$ is not summable, then x is not A summable.*

Proof. Notice that $\Pi(y)=\Pi(z)$ and $\Upsilon(y)=\Upsilon(z)$ as y,z are subsequences of x. Therefore,

$$(y-z)_{m,n} = (B(y)-B(z))_{m,n},$$

for all $m,n\in\mathbb{N}$. Therefore, by assumption

$$\lim_{m,n\to\infty}(B(y)_{m,n}-B(z)_{m,n})=0.$$

For the second statement, suppose that for some bounded subsequences y,z of x,

$$\lim_{m,n\to\infty}(B(Ay)_{m,n}-B(Az)_{m,n})\neq 0.$$

Then, by what we just showed we have that

$$\lim_{m,n\to\infty}(Ay)_{m,n} \neq \lim_{m,n\to\infty}(Az)_{m,n},$$

thus implying that x is not summable. $\square$

REMARK 4. The converse of the lemma is not true. Consider the double sequence z such that $z_{m,n} = 1$ if $\frac{m}{n} < 1/\beta$ and $z_{m,n} = 0$ otherwise. Further, let y be the null-double-sequence. In that case,

$$\lim_{m,n\to\infty} (B(y)_{m,n} - B(z)_{m,n}) = 0,$$

but clearly $\lim\limits_{m,n\to\infty} z_{m,n}$ is undefined, while $\lim\limits_{m,n\to\infty} y_{m,n} = 0$.

In [3], Patterson showed that for a special type of β-subsequence a "Buck-type" result (see [1, 3]) holds for this special case of double subsequence. As it turns out, this happens to be true for the more general β-subsequences.

THEOREM 3. *Let $\beta > 1$. A bounded double sequence x is P-convergent if and only if there exists a β-regular matrix A such that A sums every β-subsequence of x.*

Proof. The implication follows from Proposition 1, as any β-subsequence of a bounded convergent double sequence is bounded and convergent. Thus, any RH-regular matrix A sums it. In particular, any β-regular matrix sums it.

For the converse, we shall show that for a bounded but not P-convergent x and any β-regular matrix A there exists a β-subsequence of x that is not summed by A. By Lemma 1, it suffices to consider subsequences of $B(x)$. Therefore, assume x is supported on S_β.

If x is bounded but not P-convergent, it must have more than one limit point. Consider the flattened sequence corresponding to x, namely the sequence defined by $(x_{\psi^{-1}(i)})_{i=1}^{\infty} = (x_{m_i,n_i})_{i=1}^{\infty}$ and define

$$\alpha = \limsup_{i\to\infty} x_{m_i,n_i} \qquad \text{and} \qquad \beta = \liminf_{i\to\infty} x_{m_i,n_i}.$$

Since, the P-limit is not unique, we necessarily have that $\alpha \neq \beta$.

As in [3], we define the double sequence $[y_{m,n}]$ by

$$y_{m,n} = \frac{x_{m,n} - \beta}{\alpha - \beta}, \qquad \text{for all } n, m \in \mathbb{N}.$$

Note that $[y_{m,n}]$ is supported on S_β, as is x. It is also clear that

$$\limsup_{i\to\infty} y_{m_i,n_i} = 1 \qquad \text{and} \qquad \liminf_{i\to\infty} y_{m_i,n_i} = 0.$$

Then, there exists a subsequences $(y_{m_{i_j},n_{i_j}})_{j=1}^{\infty}$ and $(y_{m_{i_k},n_{i_k}})_{k=1}^{\infty}$ of the flattened sequence $(y_{m_i,n_i})_{i=1}^{\infty}$ such that

$$\frac{1}{\beta} < \frac{m_{i_k}}{n_{i_k}} < \beta \quad \text{and} \quad \frac{1}{\beta} < \frac{m_{i_j}}{n_{i_j}} < \beta$$

and

$$\lim_{j\to\infty} y_{m_{i_j},n_{i_j}} = 0 \quad \text{and} \quad \lim_{k\to\infty} y_{m_{i_k},n_{i_k}} = 1.$$

Notice that by Remark 3, there are no such β-subsequences. Indeed, the Pringsheim limit of the β-subsequence corresponding to $(y_{m_{i_k},n_{i_k}})_{k=1}^{\infty}$ is undefined. However, this flattened subsequence shall suffice for our purposes.

Define

$$y^*_{m,n} = \begin{cases} 1, & \text{if } (m,n) = (m_{i_k}, n_{i_k}) \text{ for some } k \in \mathbb{N}, \\ 0, & \text{if } (m,n) = (m_{i_j}, n_{i_j}) \text{ for some } j \in \mathbb{N}, \\ y_{m,n}, & \text{otherwise.} \end{cases}$$

Since y^* has infinitely many 0's and 1's in its S_β component, by Theorem 2, there exists a β-subsequence $z^{(\pi,\beta)}$ of y^* that is not A-summable. Let $y^{(\pi,\beta)}$ denote the β-subsequence of y induced by the same injection π that defines $z^{(\pi,\beta)}$. It is easy to see that

$$\lim_{m,n\to\infty} (y^{(\pi,\beta)}_{m,n} - z^{(\pi,\beta)}_{m,n}) = 0.$$

Thus by the linearity and regularity of A, we have

$$\lim_{m,n\to\infty} (Ay^{(\pi,\beta)}_{m,n} - Az^{(\pi,\beta)}_{m,n}) = 0.$$

This, in turn, implies that the β-subsequence $y^{(\pi,\beta)}$ is not A-summable. By the definition of $y_{m,n}$, this implies that the corresponding subsequence $x^{(\pi,\beta)}$ of x is not A-summable. □

REMARK 5. In the particular case when $\beta = +\infty$, Theorem 3 implies Theorem 3.2 in [3], as the set of all subsequences in their sense is contained in the set of all $+\infty$-subsequences. Thus, this theorem presents a generalization of the results therein.

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