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GENERAL TAUBERIAN CONDITIONS FOR WEIGHTED MEAN METHODS OF SUMMABILITY

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Abstract. In this paper we recover convergence of a complex sequence (u_n) out of its summability by weighted means under certain supplementary conditions that control the oscillatory behavior of (u_n). As corollaries, we obtain classical Hardy-type Tauberian conditions for various weighted mean methods.

1. Introduction

Let $p=(p_n)$ be a sequence of nonnegative numbers such that $p_0>0$ and

$$P_n := \sum_{k=0}^{n} p_k \to \infty \text{ as } n \to \infty. \tag{1}$$

The n^{th} weighted mean of a complex sequence $u=(u_n)$ is defined by

$$\sigma_{n,p}^{(1)}(u) := \frac{1}{P_n}\sum_{k=0}^{n} p_k u_k \tag{2}$$

for all $n \in \mathbb{N}_0$. A sequence (u_n) is said to be summable by the weighted mean method determined by the sequence p; in short, summable $(\overline{N},p)$ to a finite number s if

$$\lim_{n\to\infty} \sigma_{n,p}^{(1)}(u) = s. \tag{3}$$

A summability method $(\overline{N},p)$ is said to be regular if $\lim_{n\to\infty} u_n = s$ implies $\lim_{n\to\infty}\sigma_{n,p}^{(1)}(u)=s$. It is well known that $(\overline{N},p)$ is regular if and only if (1) is satisfied.

In this paper, we are interested in converse conclusions. We determine conditions imposed on (p_n) and (u_n) under which convergence of $(\sigma_{n,p}^{(1)}(u))$ implies that of (u_n). Such conditions are known as Tauberian conditions and theorems in this direction are called Tauberian theorems.

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2. Preliminaries

The backward difference of (u_n) is denoted by $\Delta u_n = u_n - u_{n-1}$ with $u_{-1} = 0$ for all $n \in \mathbb{N}_0$. The difference between (u_n) and its n^{th} weighted mean $(\sigma^{(1)}_{n,p}(u))$, which is called the weighted Kronecker identity [4], is given by

$$u_n - \sigma^{(1)}_{n,p}(u) = V^{(0)}_{n,p}(\Delta u), \tag{4}$$

where

$$V^{(0)}_{n,p}(\Delta u) := \frac{1}{P_n}\sum_{k=0}^{n} P_{k-1}\Delta u_k.$$

Now we introduce a device for the control of the oscillatory behavior of a sequence (u_n). Denote the weighted classical control modulo of (u_n) by $\omega^{(0)}_{n,p}(u) = \frac{P_{n-1}}{p_n}\Delta u_n$. For each integer $m \geqslant 1$, define the weighted general control modulo of order m recursively by

$$\omega^{(m)}_{n,p}(u) = \omega^{(m-1)}_{n,p}(u) - \frac{1}{P_n}\sum_{k=0}^{n} p_k \omega^{(m-1)}_{k,p}(u) \tag{5}$$

as in [12]. Classical and general control modulos have been used to generate Tauberian conditions for various summability methods [5, 8, 10, 11, 13].

The following two classes of sequences have an important role in controlling the divergent process [1].

A sequence (u_n) is slowly oscillating in the sense of Schmidt [9] if

$$\lim_{\lambda\to 1^+}\overline{\lim_{n\to\infty}}\max_{n<k\leqslant[\lambda n]}|u_k - u_n| = 0.$$

Here, $[\lambda n]$ denotes the integer part of the product λn and will be used throughout this paper.

A nondecreasing sequence of positive numbers (P_n) is called regularly varying of index $\theta > 0$ in the sense of Karamata [7] if

$$\lim_{n\to\infty}\frac{P_{[\lambda n]}}{P_n} = \lambda^{\theta},\ \lambda > 1.$$

We remind that if (P_n) is regularly varying of index $\theta > 0$, then conditions (1) and

$$\underline{\lim}_{n\to\infty}\frac{P_{[\lambda n]}}{P_n} > 1,\ \lambda > 1$$

are satisfied (see [3]).

The proofs of our results hinge on the following lemmas.

LEMMA 1. ([4]) *Let (u_n) be a sequence of complex numbers. For $\lambda > 1$ and sufficiently large n,*

$$u_n - \sigma^{(1)}_{n,p}(u) = \frac{P_{[\lambda n]}}{P_{[\lambda n]} - P_n}\left(\sigma^{(1)}_{[\lambda n],p}(u) - \sigma^{(1)}_{n,p}(u)\right) - \frac{1}{P_{[\lambda n]} - P_n}\sum_{k=n+1}^{[\lambda n]} p_k \sum_{j=n+1}^{k}\Delta u_j.$$

LEMMA 2. ([4]) *For a sequence* (u_n), $V_{n,p}^{(0)}(\Delta u) = \frac{P_{n-1}}{p_n}\Delta\sigma_{n,p}^{(1)}(u)$.

LEMMA 3. *For a sequence* (u_n), $\omega_{n,p}^{(1)}(u) = \frac{P_{n-1}}{p_n}\Delta V_{n,p}^{(0)}(\Delta u)$.

Proof. By the definition of weighted general control modulo, we have

$$\omega_{n,p}^{(1)}(u) = \omega_{n,p}^{(0)}(u) - \frac{1}{P_n}\sum_{k=0}^{n} p_k \omega_{k,p}^{(0)}(u).$$

It then follows from Lemma 2 and (4) that

$$\begin{aligned}
\omega_{n,p}^{(1)}(u) &= \omega_{n,p}^{(0)}(u) - V_{n,p}^{(0)}(\Delta u) \\
&= \frac{P_{n-1}}{p_n}\Delta u_n - \frac{P_{n-1}}{p_n}\Delta\sigma_{n,p}^{(1)}(u) \\
&= \frac{P_{n-1}}{p_n}\Delta(u_n - \sigma_{n,p}^{(1)}(u)) \\
&= \frac{P_{n-1}}{p_n}\Delta V_{n,p}^{(0)}(\Delta u).
\end{aligned}$$

3. The Main Results

In this section, we establish new general Tauberian theorems for $(\overline{N},p)$ methods of summability.

THEOREM 1. *Let* (P_n) *be regularly varying of index* $\theta > 0$. *If* (u_n) *is summable* $(\overline{N},p)$ *to* s *and if*

$$\lim_{\lambda\to 1^+}\overline{\lim_{n\to\infty}}\sum_{k=n+1}^{[\lambda n]}\frac{p_k}{P_{k-1}}|\omega_{k,p}^{(0)}(u)|^q < \infty,\ q > 1, \tag{6}$$

then (u_n) *converges to* s.

Proof. Considering Lemma 1, we have

$$\left|u_n - \sigma_{n,p}^{(1)}(u)\right| \leqslant \frac{P_{[\lambda n]}}{P_{[\lambda n]} - P_n}\left|\sigma_{[\lambda n],p}^{(1)}(u) - \sigma_{n,p}^{(1)}(u)\right| + \sum_{k=n+1}^{[\lambda n]}|\Delta u_k|.$$

Regular variation of (P_n) implies

$$\overline{\lim_{n\to\infty}}\frac{P_{[\lambda_n]}}{P_{[\lambda_n]} - P_n} = \left\{1 - \frac{1}{\underline{\lim}_{n\to\infty}\frac{P_{[\lambda n]}}{P_n}}\right\}^{-1} < \infty. \tag{7}$$

Taking (7) and the assumed summability $(\overline{N},p)$ of (u_n) into account, we obtain

$$\overline{\lim_{n\to\infty}}\left|u_n-\sigma^{(1)}_{n,p}(u)\right|\leqslant\overline{\lim_{n\to\infty}}\sum_{k=n+1}^{[\lambda n]}|\Delta u_k|\,. \tag{8}$$

Applying the weighted version of Hölder inequality ([2], p.39) to right hand side of (8), we observe

$$\begin{aligned}\sum_{k=n+1}^{[\lambda n]}|\Delta u_k| &= \sum_{k=n+1}^{[\lambda n]}\frac{p_k}{P_{k-1}}|\omega^{(0)}_{k,p}(u)|\\ &\leqslant\left(\sum_{j=n+1}^{[\lambda n]}p_k\right)^{\frac{1}{r}}\left(\sum_{k=n+1}^{[\lambda n]}p_k\frac{|\omega^{(0)}_{k,p}(u)|^q}{P_{k-1}{}^q}\right)^{\frac{1}{q}},\text{where } \tfrac{1}{r}+\tfrac{1}{q}=1\\ &\leqslant\left(P_{[\lambda n]}-P_n\right)^{\frac{1}{r}}\left(\frac{1}{P_n{}^{q-1}}\sum_{k=n+1}^{[\lambda n]}p_k\frac{|\omega^{(0)}_{k,p}(u)|^q}{P_{k-1}}\right)^{\frac{1}{q}}\\ &\leqslant\left(\frac{P_{[\lambda n]}-P_n}{P_n}\right)^{\frac{1}{r}}\left(\sum_{k=n+1}^{[\lambda n]}\frac{p_k}{P_{k-1}}|\omega^{(0)}_{k,p}(u)|^q\right)^{\frac{1}{q}}\end{aligned}$$

and consequently

$$\overline{\lim_{n\to\infty}}\left|u_n-\sigma^{(1)}_{n,p}(u)\right|\leqslant\left(\lambda^{\theta}-1\right)^{\frac{1}{r}}\overline{\lim_{n\to\infty}}\left(\sum_{k=n+1}^{[\lambda n]}\frac{p_k}{P_{k-1}}|\omega^{(0)}_{k,p}(u)|^q\right)^{\frac{1}{q}}. \tag{9}$$

Now, letting $\lambda\to 1^+$ in (9) and taking (6) into account, we conclude

$$\overline{\lim_{n\to\infty}}\left|u_n-\sigma^{(1)}_{n,p}(u)\right|=0.$$

This completes the proof.
As a corollary, we may give the following classical Hardy-type ([6], p.149) Tauberian theorem for $(\overline{N},p)$ summability

COROLLARY 1. *Let (P_n) be regularly varying of index $\theta>0$. If (u_n) is summable $(\overline{N},p)$ to s and if*

$$\frac{P_{n-1}}{p_n}|\Delta u_n|\leqslant M \tag{10}$$

for some $M>0$ and all n, then (u_n) converges to s.

Here, we investigate variations of condition (10) for particular weighted mean methods.

(i) If $p_n = 1$ for all $n \in \mathbb{N}_0$, then $(\overline{N},p)$ summability corresponds to the $(C,1)$ summability, where $P_n = n$. In this case, (10) reduces to condition $n|\Delta u_n| \leqslant C$.

(ii) If $p_n = \frac{1}{n+1}$ for all $n \in \mathbb{N}_0$, then $(\overline{N},p)$ summability corresponds to the $(\ell,1)$ summability, where $P_n \sim \log n$. In this case, (10) reduces to condition $n\log n|\Delta u_n| \leqslant C$.

(iii) If $p_n = \frac{1}{n\log(n+1)}$ for all $n \in \mathbb{N}_0$, then $(\overline{N},p)$ summability corresponds to the $(\ell,2)$ summability, where $P_n \sim \log\log n$. In this case, (10) reduces to condition $n\log\log n|\Delta u_n| \leqslant C$.

Note that, for all three cases, we do not need to assume the regular variation condition on (P_n).

Instead of retrieving convergence of (u_n) from the existence of the limit (3) under additional conditions imposed on the sequences (u_n) and (p_n), we may deduce more general information about the oscillatory behavior of (u_n) by replacing $(\overline{N},p)$ summability of (u_n) by $(\overline{N},p)$ summability of $(V_{n,p}^{(0)}(\Delta u))$.

THEOREM 2. *Let (P_n) be regularly varying of index $\theta > 0$. If $(V_{n,p}^{(0)}(\Delta u))$ is summable $(\overline{N},p)$ to s and if*

$$\lim_{\lambda\to 1^+}\overline{\lim_{n\to\infty}}\sum_{k=n+1}^{[\lambda n]}\frac{p_k}{P_{k-1}}|\omega_{k,p}^{(1)}(u)|^q < \infty,\ q > 1 \tag{11}$$

then (u_n) is slowly oscillating.

Proof. Applying Lemma 1 to $(V_{n,p}^{(0)}(\Delta u))$, we have

$$\left|V_{n,p}^{(0)}(\Delta u) - V_{n,p}^{(1)}(\Delta u)\right| \leqslant \sum_{k=n+1}^{[\lambda n]}|\Delta V_{k,p}^{(0)}(\Delta u)|$$

by (7) and $(\overline{N},p)$ summability of $(V_{n,p}^{(0)}(\Delta u))$ to s. Since

$$\sum_{k=n+1}^{[\lambda n]}|\Delta V_{k,p}^{(0)}(\Delta u)| = \sum_{k=n+1}^{[\lambda n]}\frac{p_k}{P_{k-1}}|\omega_{k,p}^{(1)}(u)|$$

by Lemma 3, we obtain

$$\overline{\lim_{n\to\infty}}\left|V_{n,p}^{(0)}(\Delta u) - V_{n,p}^{(1)}(\Delta u)\right| \leqslant \left(\lambda^{\theta}-1\right)^{\frac{1}{r}}\overline{\lim_{n\to\infty}}\left(\sum_{k=n+1}^{[\lambda n]}\frac{p_k}{P_{k-1}}\left|\omega_{k,p}^{(1)}(u)\right|^q\right)^{\frac{1}{q}} \tag{12}$$

by using the weighted Hölder inequality. Considering the hypothesis (11) and taking the limit of (12) as $\lambda \to 1^+$ yields

$$\overline{\lim_{n\to\infty}} \left| V_{n,p}^{(0)}(\Delta u) - V_{n,p}^{(1)}(\Delta u) \right| = 0. \tag{13}$$

It is clear that (13) implies $\lim_{n\to\infty} V_{n,p}^{(0)}(\Delta u) = s$. By Lemma 2, we have

$$\begin{aligned} |\sigma_{k,p}^{(1)}(u) - \sigma_{n,p}^{(1)}(u)| &= \left| \sum_{j=n+1}^{k} \Delta\sigma_{j,p}^{(1)}(u) \right| \\ &\leqslant \sum_{j=n+1}^{k} \frac{p_j}{P_{j-1}} \left| V_{j,p}^{(0)}(\Delta u) \right|. \end{aligned}$$

Since $(V_{n,p}^{(0)}(\Delta u))$ is bounded, we get

$$\begin{aligned} \max_{n<k\leqslant[\lambda n]} |\sigma_{k,p}^{(1)}(u) - \sigma_{n,p}^{(1)}(u)| &\leqslant \frac{M}{P_n} \sum_{j=n+1}^{[\lambda n]} p_j \\ &= M\left(\frac{P_{[\lambda n]}}{P_n} - 1 \right), \end{aligned}$$

where M is a positive constant. The last inequality above leads us to

$$\lim_{\lambda\to 1^+} \overline{\lim_{n\to\infty}} \max_{n<k\leqslant[\lambda n]} |\sigma_{k,p}^{(1)}(u) - \sigma_{n,p}^{(1)}(u)| = 0,$$

which means that $(\sigma_{n,p}^{(1)}(u))$ is slowly oscillating. It follows from (4) that (u_n) is slowly oscillating. This completes the proof.

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