Journal of
Classical
Analysis
Volume 15, Number 2 (2019), 99–117 doi:10.7153/jca-2019-15-10

# 2D–SHEFFER–MITTAG–LEFFLER POLYNOMIALS: PROPERTIES AND EXAMPLES

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*Abstract.* In this work, the 2D-Sheffer polynomials and the Mittag-Leffler polynomials are combined to introduce the family of the 2D-Sheffer-Mittag-Leffler polynomials. The generating function, quasi-monomial properties and series definition of these polynomials are established. Examples of some members belonging to this family are considered. The graphs of some hybrid special polynomials are also drawn for suitable values of the indices.

## 1. Introduction and preliminaries

In 1939, I. M. Sheffer [13] studied in detail the polynomials sets of type zero $s_n(x)$; $n \in \mathbb{N}_0$. Since then, these polynomial sets are called the Sheffer polynomials. These polynomials have been extensively studied not only due to the fact that they arise in various branches of mathematics but also because of their importance in applied sciences and engineering. The Sheffer polynomials are defined by the following generating function:

$$A(t)\exp(xH(t)) = \sum_{n=0}^{\infty} s_n(x)\frac{t^n}{n!},$$

where $A(t)$ and $H(t)$ are power series such that

$$A(t) = \sum_{n=0}^{\infty} a_n \frac{t^n}{n!}, \quad a_0 \neq 0 \tag{1}$$

and

$$H(t) = \sum_{n=1}^{\infty} h_n \frac{t^n}{n!}, \quad h_1 \neq 0. \tag{2}$$

REMARK 1. It should be noted that for $H(t) = t$, the Sheffer polynomials become the Appell polynomials [1] defined by generating function:

$$A(t)\exp(xt) = \sum_{n=0}^{\infty} A_n(x)\frac{t^n}{n!}.$$

*Mathematics subject classification* (2010): 33C45, 33C99, 33E20.
*Keywords and phrases*: 2D-Sheffer polynomials, Mittag-Leffler polynomials, generating function.
This work has been sponsored by Dr. D. S. Kothari Post Doctoral Fellowship (Award letter No. F.4-2/2006(BSR)/MA/17-18/0025) awarded to **Dr. Mahvish Ali** by the University Grants Commission, Government of India, New Delhi.
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The special polynomials of two variables provided new means of analysis for the solution of large classes of partial differential equations often encountered in physical problems. Most of the special functions of mathematical physics and their generalizations have been suggested by physical problems. The importance of multi-variable Hermite polynomials has been recognized in [6] and these polynomials have been exploited to deal with quantum mechanical and optical beam transport problems. The 2-variable Laguerre polynomials are the natural solutions of heat diffusion equation of Fokker-Plank type and are used to study the beam life-time due to quantum fluctuation in storage rings [18].

We consider the family of 2D-Sheffer polynomials $f_n(x,y)$ defined by the following generating function:

$$A(t)\phi(y,H(t))\exp(xH(t)) = \sum_{n=0}^{\infty} f_n(x,y)\frac{t^n}{n!}, \tag{3}$$

where $A(t)$ and $H(t)$ are defined by equations (1) and (2), respectively and $\phi$ is a function of $y$ and $H(t)$ such that $\phi(0,H(t)) = 1$.

REMARK 2. For $\phi(y,H(t)) = \exp(y(H(t))^m)$, equation (3) reduces to the following generating function of the Gould-Hopper-Sheffer polynomials ${}_{H^{(m)}}s_n(x,y)$ [8]:

$$A(t)\exp(xH(t)+y(H(t))^m) = \sum_{n=0}^{\infty} {}_{H^{(m)}}s_n(x,y)\frac{t^n}{n!}. \tag{4}$$

REMARK 3. Taking $\phi(y,H(t)) = \frac{1}{1-y(H(t))^r}$ in equation (3), it yields the following generating function of the 2-variable truncated exponential-Sheffer polynomials ${}_{e^{(r)}}s_n(x,y)$ [11]:

$$A(t)\frac{1}{(1-y(H(t))^r)}\exp(xH(t)) = \sum_{n=0}^{\infty} {}_{e^{(r)}}s_n(x,y)\frac{t^n}{n!}. \tag{5}$$

In view of equations (3)-(5), it follows that the Gould-Hopper-Sheffer polynomials ${}_{H^{(m)}}s_n(x,y)$ and 2-variable truncated exponential-Sheffer polynomials ${}_{e^{(r)}}s_n(x,y)$ belong to the 2D-Sheffer family.

The combination of monomiality principle and operational methods provide a fairly unique tool to treat various polynomials from unified point of view, see for example [3]. According to the monomiality principle [5, 4, 17] a polynomial set $r_n(x)$ ($n \in \mathbb{N},\ x \in \mathbb{C}$), is quasi-monomial, if there exist two operators $\hat{\mathscr{M}}$ and $\hat{\mathscr{P}}$, called multiplicative and derivative operators respectively, which when acting on the polynomials $r_n(x)$ yield:

$$\hat{\mathscr{M}}\{r_n(x)\} = r_{n+1}(x)$$

and

$$\hat{\mathscr{P}}\{r_n(x)\} = n\, r_{n-1}(x),$$

respectively.

The operators $\hat{\mathscr{M}}$ and $\hat{\mathscr{P}}$ satisfy the commutation relation

$$[\hat{\mathscr{P}},\hat{\mathscr{M}}]=\hat{1}$$

and thus display the Weyl group structure.

If $\hat{\mathscr{M}}$ and $\hat{\mathscr{P}}$ have differential realizations, then it can be easily shown that the differential equation satisfied by $r_n(x)$ is

$$\hat{\mathscr{M}}\hat{\mathscr{P}}\{r_n(x)\}=n\,r_n(x). \tag{6}$$

Assuming here and in the sequel $r_0(x)=1$, then $r_n(x)$ can be explicitly constructed as:

$$r_n(x)=\hat{\mathscr{M}}^n\{1\} \tag{7}$$

and consequently the generating function of $r_n(x)$ can be cast in the form

$$G(x,t)=\exp(t\hat{\mathscr{M}})\{1\}=\sum_{n=0}^{\infty} r_n(x)\frac{t^n}{n!}. \tag{8}$$

The Gould-Hopper-Sheffer polynomials ${}_{H^{(m)}}s_n(x,y)$ are quasi-monomial with respect to the following multiplicative and derivative operators:

$$\hat{\mathscr{M}}_{H^{(m)}s}=\Big(x+my\frac{\partial^{m-1}}{\partial x^{m-1}}\Big)H'(H^{-1}(D_x))+\frac{A'(H^{-1}(D_x))}{A(H^{-1}(D_x))}$$

and

$$\hat{\mathscr{P}}_{H^{(m)}s}=H^{-1}(D_x),$$

respectively, where $D_x:=\frac{\partial}{\partial x}$.

The 2-variable truncated exponential-Sheffer polynomials ${}_{e^{(r)}}s_n(x,y)$ are quasi-monomial with respect to the following multiplicative and derivative operators:

$$\hat{\mathscr{M}}_{e^{(r)}s}=\Big(x+ryD_yyD_x^{r-1}\Big)H'(H^{-1}(D_x))+\frac{A'(H^{-1}(D_x))}{A(H^{-1}(D_x))}$$

and

$$\hat{\mathscr{P}}_{e^{(r)}s}=H^{-1}(D_x),$$

respectively.

Consequently, it is observed that the 2D-Sheffer polynomials $f_n(x,y)$ are quasi-monomial with respect to the following multiplicative and derivative operators:

$$\hat{\mathscr{M}}_f=xH'(H^{-1}(D_x))+\frac{\phi'(y,D_x)}{\phi(y,D_x)}H'(H^{-1}(D_x))+\frac{A'(H^{-1}(D_x))}{A(H^{-1}(D_x))}$$

and

$$\hat{\mathscr{P}}_f = H^{-1}(D_x),$$

respectively.

In [5], Dattoli introduced and studied Hermite-Bessel and Laguerre-Bessel functions providing the usefulness of the point of view based on the concept of quasi monomiality. Several other hybrid special polynomials related to the Laguerre and Hermite polynomials are studied. The hybrid special polynomial families related to the Appell and Sheffer polynomial sequences are first introduced and studied by Khan and her co-authors, see for example [9, 10]. These hybrid special polynomials associated with Appell and Sheffer sequences are studied in several contexts. Many works have been devoted to the study of hybrid and mixed families of special functions. Recently, a novel approach has been used to study some new types of mixed special polynomial families related to the Appell and Sheffer sequences, see for example [14, 15, 16, 19].

The Mittag-Leffler polynomials $M_n(x)$ [2, 12] are defined by the following generating function:

$$\left(\frac{1+t}{1-t}\right)^x = \sum_{n=0}^{\infty} M_n(x)\frac{t^n}{n!}$$

or

$$\exp\left(x\ln\left(\frac{1+t}{1-t}\right)\right) = \sum_{n=0}^{\infty} M_n(x)\frac{t^n}{n!}. \tag{9}$$

These polynomials satisfy the following orthogonality property:

$$\int_{-\infty}^{\infty} M_n(ix)M_m(-ix)\frac{dx}{x\sinh \pi x} = \frac{2}{n}\delta_{m,n}, \quad m,n \in \mathbb{N}.$$

The first few Mittag-Leffler polynomials are

$$M_0(x) = 1, \quad M_1(x) = 2x, \quad M_2(x) = 4x^2,$$

$$M_3(x) = 8x^3 + 4x, \tag{10a}$$

$$M_4(x) = 16x^4 + 32x^2. \tag{10b}$$

In this paper, the 2D-Sheffer-Mittag-Leffler polynomials are introduced and their properties are derived. In Section 2, the generating function, quasi-monomial properties and series definition of these polynomials are established. In Section 3, examples of some members belonging to this family are considered. The graphs of some hybrid special polynomials are also drawn for suitable values of the indices.

## 2. 2D-Sheffer-Mittag-Leffler polynomials

The 2D-Sheffer-Mittag-Leffler polynomials are introduced by means of generating function by proving the following result:

THEOREM 1. *For the 2D-Sheffer-Mittag-Leffler polynomials* ${}_fM_n(x,y)$, *the following generating function holds true:*

$$A\left(\ln\left(\frac{1+t}{1-t}\right)\right)\phi\left(y,H\left(\ln\left(\frac{1+t}{1-t}\right)\right)\right)\exp\left(xH\left(\ln\left(\frac{1+t}{1-t}\right)\right)\right)$$
$$=\sum_{n=0}^{\infty}{}_fM_n(x,y)\frac{t^n}{n!}. \tag{11}$$

*Proof.* In order to derive the generating function for the 2D-Sheffer-Mittag-Leffler polynomials, the 2D-Sheffer polynomials $f_n(x,y)$ are taken as base in generating function (9) of the Mittag-Leffler polynomials. Thus, replacing $x$ by the multiplicative operator $\hat{\mathscr{M}}_f$ of the 2D-Sheffer polynomials $f_n(x,y)$ in the l.h.s. of equation (9) and denoting the resultant 2D-Sheffer-Mittag-Leffler polynomials in the r.h.s. by ${}_fM_n(x,y)$, it follows that

$$\exp\left(\hat{\mathscr{M}}_f\ln\left(\frac{1+t}{1-t}\right)\right)=\sum_{n=0}^{\infty}{}_fM_n(x,y)\frac{t^n}{n!}, \tag{12}$$

which on using equation (8) with $t$ replaced by $\ln\left(\frac{1+t}{1-t}\right)$ and in view of generating function (3) in the resultant equation proves assertion (11).

REMARK 4. From equations (9) and (12), the following operational correspondence between the 2D-Sheffer-Mittag-Leffler polynomials ${}_fM_n(x,y)$ and Mittag-Leffler polynomials $M_n(x)$ is obtained:

$$M_n(\hat{\mathscr{M}}_f)={}_fM_n(x,y). \tag{13}$$

REMARK 5. Since for $y=0$, the 2D-Sheffer polynomials $f_n(x,y)$ reduce to the Sheffer polynomials $s_n(x)$. Therefore, taking $y=0$ in the left hand side of equation (11) and denoting the resultant Sheffer-Mittag-Leffler polynomials ${}_sM_n(x)$ in the right hand side, the following consequence of Theorem 1 is deduced:

COROLLARY 1. *For the Sheffer-Mittag-Leffler polynomials* ${}_sM_n(x)$, *the following generating function holds true:*

$$A\left(\ln\left(\frac{1+t}{1-t}\right)\right)\exp\left(xH\left(\ln\left(\frac{1+t}{1-t}\right)\right)\right)=\sum_{n=0}^{\infty}{}_sM_n(x)\frac{t^n}{n!}.$$

REMARK 6. Since for $y=0$ and $H(t)=t$, the 2D-Sheffer polynomials $f_n(x,y)$ reduce to the Appell polynomials $A_n(x)$. Therefore taking $y=0$ and $H\left(\ln\left(\frac{1+t}{1-t}\right)\right)=\ln\left(\frac{1+t}{1-t}\right)$ in the l.h.s. of equation (11) and denoting the resultant Appell-Mittag-Leffler polynomials ${}_AM_n(x)$ in the r.h.s., the following consequence of Theorem 1 is deduced:

COROLLARY 2. *For the Appell-Mittag-Leffler polynomials ${}_AM_n(x)$, the following generating function holds true:*

$$A\left(\ln\left(\frac{1+t}{1-t}\right)\right)\left(\frac{1+t}{1-t}\right)^x = \sum_{n=0}^{\infty} {}_AM_n(x)\frac{t^n}{n!}.$$

To give an application of the operational correspondence between the 2D-Sheffer-Mittag-Leffler polynomials ${}_fM_n(x,y)$ and Mittag-Leffler polynomials $M_n(x)$, the following result is taken:

$$M_n(x) = \sum_{k=0}^{n}\binom{n}{k}(n-1)_{n-k}2^k(x)_k, \tag{14}$$

where $(x)_k$ are the lower factorial polynomials defined by

$$(x)_k = \sum_{l=0}^{n} S(n,l)x^l,$$

in terms of the Stirling numbers of the first kind $S(n,l)$.

The lower factorial polynomials $(x)_n$ are also given explicitly as:

$$(x)_n = x(x-1)(x-2)\cdots(x-n+1).$$

Consequently, definition (14) takes the form

$$M_n(x) = \sum_{k=0}^{n}\sum_{l=0}^{n}\binom{n}{k}(n-1)_{n-k}2^kS(n,l)x^l. \tag{15}$$

Now, replacing $x$ by the multiplicative operator $\hat{\mathscr{M}}_f$ of the 2D-Sheffer polynomials $f_n(x,y)$ in equation (15) and then using equations (13) and (7) in l.h.s. and r.h.s. respectively of the resultant equation, the following expansion for the 2D-Sheffer-Mittag-Leffler polynomials in terms of the Mittag-Leffler polynomials is obtained:

$${}_fM_n(x,y) = \sum_{k=0}^{n}\sum_{l=0}^{n}\binom{n}{k}(n-1)_{n-k}2^kS(n,l)M_l(x).$$

Next, the multiplicative and derivative operators associated with the 2D-Sheffer-Mittag-Leffler polynomials are obtained:

THEOREM 2. *For the 2D-Sheffer-Mittag-Leffler polynomials ${}_fM_n(x,y)$, the following multiplicative and derivative operators exist:*

$$\hat{\mathscr{M}}_{fM} = \left(xH'(H^{-1}(D_x)) + \frac{\phi'(y,D_x)}{\phi(y,D_x)}H'(H^{-1}(D_x)) + \frac{A'(H^{-1}(D_x))}{A(H^{-1}(D_x))}\right)$$
$$\times\left(2(\cosh(H^{-1}(D_x)+1)\right) \tag{16}$$

*and*

$$\hat{\mathscr{P}}_{fM} = \tanh\left(\frac{H^{-1}(D_x)}{2}\right), \tag{17}$$

*respectively.*

*Proof.* Consider the identity

$$\begin{aligned}
&D_x \left\{A\left(\ln\left(\frac{1+t}{1-t}\right)\right)\phi\left(y,H\left(\ln\left(\frac{1+t}{1-t}\right)\right)\right)\exp\left(xH\left(\ln\left(\frac{1+t}{1-t}\right)\right)\right)\right\}\\
&= H\left(\ln\left(\frac{1+t}{1-t}\right)\right) \left\{A\left(\ln\left(\frac{1+t}{1-t}\right)\right)\phi\left(y,H\left(\ln\left(\frac{1+t}{1-t}\right)\right)\right)\right.\\
&\qquad \left.\times \exp\left(xH\left(\ln\left(\frac{1+t}{1-t}\right)\right)\right)\right\},
\end{aligned}$$

which on simplification gives

$$\begin{aligned}
&\tanh\left(\frac{H^{-1}(D_x)}{2}\right) \left\{A\left(\ln\left(\frac{1+t}{1-t}\right)\right)\phi\left(y,H\left(\ln\left(\frac{1+t}{1-t}\right)\right)\right)\exp\left(xH\left(\ln\left(\frac{1+t}{1-t}\right)\right)\right)\right\}\\
&\quad = t\left\{A\left(\ln\left(\frac{1+t}{1-t}\right)\right)\phi\left(y,H\left(\ln\left(\frac{1+t}{1-t}\right)\right)\right)\exp\left(xH\left(\ln\left(\frac{1+t}{1-t}\right)\right)\right)\right\}.
\end{aligned} \tag{18}$$

Differentiating equation (11) partially with respect to $t$, it follows that

$$\begin{aligned}
&\left(xH'\left(\ln\left(\frac{1+t}{1-t}\right)\right) + \frac{\phi'\left(y,H\left(\ln\left(\frac{1+t}{1-t}\right)\right)\right)}{\phi\left(y,H\left(\ln\left(\frac{1+t}{1-t}\right)\right)\right)}H'\left(\ln\left(\frac{1+t}{1-t}\right)\right) + \frac{A'\left(\ln\left(\frac{1+t}{1-t}\right)\right)}{A\left(\ln\left(\frac{1+t}{1-t}\right)\right)}\right)\\
&\times\left(\frac{2}{1-t^2}\right)\left\{A\left(\ln\left(\frac{1+t}{1-t}\right)\right)\phi\left(y,H\left(\ln\left(\frac{1+t}{1-t}\right)\right)\right)\exp\left(xH\left(\ln\left(\frac{1+t}{1-t}\right)\right)\right)\right\}\\
&= \sum_{n=0}^{\infty} {}_fM_{n+1}(x,y)\frac{t^n}{n!}.
\end{aligned}$$

In view of identity (18), the above equation takes the form

$$\begin{aligned}
&\left(xH'(H^{-1}(D_x)) + \frac{\phi'(y,D_x)}{\phi(y,D_x)}H'(H^{-1}(D_x)) + \frac{A'(H^{-1}(D_x))}{A(H^{-1}(D_x))}\right)\left(2(\cosh(H^{-1}(D_x)+1)\right))\\
&\times\left\{A\left(\ln\left(\frac{1+t}{1-t}\right)\right)\phi\left(y,H\left(\ln\left(\frac{1+t}{1-t}\right)\right)\right)\exp\left(xH\left(\ln\left(\frac{1+t}{1-t}\right)\right)\right)\right\}\\
&= \sum_{n=0}^{\infty} {}_fM_{n+1}(x,y)\frac{t^n}{n!}.
\end{aligned}$$

Now making use of generating function (11) in the l.h.s. of the above equation and then equating the coefficients of like powers of $t$ in both sides of the resultant equation, it follows that

$$\left(xH'(H^{-1}(D_x)) + \frac{\phi'(y,D_x)}{\phi(y,D_x)}H'(H^{-1}(D_x)) + \frac{A'(H^{-1}(D_x))}{A(H^{-1}(D_x))}\right)\left(2(\cosh H^{-1}(D_x)+1)\right)$$

$$\{{}_fM_n(x,y)\} = {}_fM_{n+1}(x,y),$$

which proves assertion (16).

Further, making use of generating function (11) in both sides of identity (18) and then equating the coefficients of like powers of $t$ in both sides of the resultant equation, the following equation is obtained:

$$\tanh\left(\frac{H^{-1}(D_x)}{2}\right)\{{}_fM_n(x,y)\} = n\ {}_fM_{n-1}(x,y),$$

which proves assertion (17).

In view of Remarks 5 and 6, the following consequences of Theorem 2 are deduced:

COROLLARY 3. *The Sheffer-Mittag-Leffler polynomials ${}_sM_n(x)$ are quasi-monomial with respect to the following multiplicative and derivative operators:*

$$\hat{\mathscr{M}}_{sM} = \left(xH'(H^{-1}(D_x)) + \frac{A'(H^{-1}(D_x))}{A(H^{-1}(D_x))}\right)\left(2(\cosh H^{-1}(D_x)+1)\right)$$

*and*

$$\hat{\mathscr{P}}_{sM} = \tanh\left(\frac{H^{-1}(D_x)}{2}\right),$$

*respectively.*

COROLLARY 4. *The Appell-Mittag-Leffler polynomials ${}_AM_n(x)$ are quasi-monomial with respect to the following multiplicative and derivative operators:*

$$\hat{\mathscr{M}}_{AM} = \left(x + \frac{A'(D_x)}{A(D_x)}\right)(2(\cosh D_x + 1))$$

*and*

$$\hat{\mathscr{P}}_{AM} = \tanh\left(\frac{D_x}{2}\right),$$

*respectively.*

THEOREM 3. *For the 2D-Sheffer-Mittag-Leffler polynomials ${}_fM_n(x,y)$, the following differential equation is satisfied:*

$$\left(\left(xH'(H^{-1}(D_x)) + \frac{\phi'(y,D_x)}{\phi(y,D_x)}H'(H^{-1}(D_x)) + \frac{A'(H^{-1}(D_x))}{A(H^{-1}(D_x))}\right)\right.$$

$$\left.\left(2(\cosh H^{-1}(D_x)+1)\tanh\left(\frac{H^{-1}(D_x)}{2}\right)\right) - n\right){}_fM_n(x,y) = 0. \qquad (19)$$

*Proof.* Using expressions (16) and (17) of the multiplicative and derivative operators $\hat{\mathscr{M}}_{fM}$ and $\hat{\mathscr{P}}_{fM}$ in monomiality equation (6) for the 2D-Sheffer-Mittag-Leffler polynomials ${}_fM_n(x,y)$, assertion (19) follows.

In view of Remarks 5 and 6, the following corollaries are deduced as consequences of Theorem 3:

COROLLARY 5. *For the Sheffer-Mittag-Leffler polynomials ${}_sM_n(x)$, the following differential equation is satisfied:*

$$\left(\left(xH'(H^{-1}(D_x))+\frac{A'(H^{-1}(D_x))}{A(H^{-1}(D_x))}\right)\left(2(\cosh H^{-1}(D_x)+1)\tanh\left(\frac{H^{-1}(D_x)}{2}\right)\right)-n\right){}_sM_n(x)$$
$$=0.$$

COROLLARY 6. *For the Appell-Mittag-Leffler polynomials ${}_AM_n(x)$, the following differential equation is satisfied:*

$$\left(\left(x+\frac{A'(D_x)}{A(D_x)}\right)\left(2(\cosh D_x+1)\tanh\left(\frac{D_x}{2}\right)\right)-n\right){}_AM_n(x)=0.$$

In the next section, certain members of the 2D-Sheffer-Mitttag-Leffler family are considered and corresponding results for these hybrid special polynomials are derived.

## 3. Examples

Taking $\phi(y,H(t))$ of particular members belonging to the 2D-Sheffer family, the corresponding members of the 2D-Sheffer family is obtained. The results for these hybrid special polynomials related to the Mittag-Leffler polynomials are obtained by considering the following examples:

EXAMPLE 1. Since, for $\phi(y,H(t))=\exp(y(H(t))^m)$, the 2D-Sheffer polynomials $f_n(x,y)$ reduce to the Gould-Hopper-Sheffer polynomials ${}_{H^{(m)}}s_n(x,y)$. Therefore, from equation (11), it follows that the Gould-Hopper-Sheffer-Mittag-Leffler polynomials ${}_{H^{(m)}s}M_n(x,y)$ are defined by the following generating function:

$$A\left(\ln\left(\frac{1+t}{1-t}\right)\right)\exp\left(xH\left(\ln\left(\frac{1+t}{1-t}\right)\right)+y\left(H\left(\ln\left(\frac{1+t}{1-t}\right)\right)\right)^m\right)$$
$$=\sum_{n=0}^{\infty}{}_{H^{(m)}s}M_n(x,y)\frac{t^n}{n!}. \tag{20}$$

In view of equation (13), the following operational correspondence between the Gould-Hopper-Sheffer-Mittag-Leffler polynomials ${}_{H^{(m)}s}M_n(x,y)$ and Mittag-Leffler polynomials $M_n(x)$ is obtained:

$$M_n(\hat{\mathscr{M}}_{H^{(m)}s})={}_{H^{(m)}s}M_n(x,y), \tag{21}$$

where $\hat{\mathscr{M}}_{H^{(m)}s}$ is the multiplicative operator of the Gould-Hopper-Sheffer polynomials ${}_{H^{(m)}}s_n(x,y)$.

Taking $\phi(y,D_x) = \exp(yD_x^m)$ in equations (16) and (17), the following expression for the multiplicative and derivative operators for the Gould-Hopper-Sheffer-Mittag-Leffler polynomials ${}_{H^{(m)}s}M_n(x,y)$ are obtained:

$$\hat{\mathscr{M}}_{H^{(m)}{}_sM} = \left( xH'(H^{-1}(D_x)) + myD_x^{m-1}H'(H^{-1}(D_x)) + \frac{A'(H^{-1}(D_x))}{A(H^{-1}(D_x))} \right) \times \left(2\left(\cosh(H^{-1}(D_x)+1)\right)\right) \tag{22}$$

and

$$\hat{\mathscr{P}}_{H^{(m)}{}_sM} = \tanh\left(\frac{H^{-1}(D_x)}{2}\right), \tag{23}$$

respectively.

Consequently, the following differential equation for the Gould-Hopper-Sheffer-Mittag-Leffler polynomials ${}_{H^{(m)}s}M_n(x,y)$ is obtained:

$$\left( \left( xH'(H^{-1}(D_x)) + myD_x^{m-1}H'(H^{-1}(D_x)) + \frac{A'(H^{-1}(D_x))}{A(H^{-1}(D_x))} \right) \right.$$

$$\left. \left( 2(\cosh H^{-1}(D_x)+1)\tanh\left(\frac{H^{-1}(D_x)}{2}\right)\right) - n \right) {}_{H^{(m)}s}M_n(x,y) = 0. \tag{24}$$

EXAMPLE 2. Since, for $\phi(y,H(t)) = \frac{1}{1-y(H(t))^r}$, the 2D-Sheffer polynomials $f_n(x,y)$ reduce to the 2-variable truncated exponential-Sheffer-Mittag-Leffler polynomials ${}_{e^{(r)}s}M_n(x,y)$. Therefore, from equation (11), it follows that the 2-variable truncated exponential-Sheffer-Mittag-Leffler polynomials ${}_{e^{(r)}s}M_n(x,y)$ are defined by the following generating function:

$$A\left(\ln\left(\frac{1+t}{1-t}\right)\right) \frac{1}{1-y\left(H\left(\ln\left(\frac{1+t}{1-t}\right)\right)\right)^r} \exp\left(xH\left(\ln\left(\frac{1+t}{1-t}\right)\right)\right) = \sum_{n=0}^{\infty} {}_{e^{(r)}s}M_n(x,y)\frac{t^n}{n!}. \tag{25}$$

In view of equation (13), the following operational correspondence between the 2-variable truncated exponential-Sheffer-Mittag-Leffler polynomials ${}_{e^{(r)}s}M_n(x,y)$ and Mittag-Leffler polynomials $M_n(x)$ is obtained:

$$M_n(\hat{\mathscr{M}}_{e^{(r)}s}) = {}_{e^{(r)}s}M_n(x,y), \tag{26}$$

where $\hat{\mathscr{M}}_{e^{(r)}s}$ is the multiplicative operator of the 2-variable truncated exponential-Sheffer ${}_{e^{(r)}}s_n(x,y)$.

Taking $\phi(y,D_x) = \frac{1}{1-yD_x^r}$ in equations (16) and (17), the following expressions for the multiplicative and derivative operators for the 2-variable truncated exponential-Sheffer-Mittag-Leffler polynomials ${}_{e^{(r)}s}M_n(x,y)$ are obtained:

$$\hat{\mathscr{M}}_{e^{(r)}{}_sM} = \left(xH'(H^{-1}(D_x)) + ryD_yyD_x^{r-1}H'(H^{-1}(D_x)) + \frac{A'(H^{-1}(D_x))}{A(H^{-1}(D_x))}\right) \times \left(2\left(\cosh(H^{-1}(D_x)+1)\right)\right) \tag{27}$$

and

$$\hat{\mathscr{P}}_{e^{(r)}{}_sM} = \tanh\left(\frac{H^{-1}(D_x)}{2}\right), \tag{28}$$

respectively.

Also, from equation (19), the following differential equation for the 2-variable truncated exponential-Sheffer-Mittag-Leffler polynomials ${}_{e^{(r)}s}M_n(x,y)$ is obtained:

$$\left(\left(xH'(H^{-1}(D_x)) + ryD_yyD_x^{r-1}H'(H^{-1}(D_x)) + \frac{A'(H^{-1}(D_x))}{A(H^{-1}(D_x))}\right)\right.$$
$$\left.\left(2(\cosh H^{-1}(D_x)+1)\tanh\left(\frac{H^{-1}(D_x)}{2}\right)\right) - n\right) {}_{e^{(r)}s}M_n(x,y) = 0. \tag{29}$$

Since the functions $A(t)$ and $H(t)$ are the determining functions for the Sheffer polynomials $s_n(x)$. Therefore, for suitable selections of $A(t)$ and $H(t)$, several classical polynomials can be obtained. In particular, for $A(t) = e^{-t^2}$; $H(t) = 2t$, the Hermite polynomials $H_n(x)$ are obtained. Also, for $A(t) = \frac{1}{1-t}$; $H(t) = \frac{-t}{1-t}$, the Laguerre polynomials $L_n(x)$ are obtained.

In the next example, the results for two specific members of the Gould-Hopper-Sheffer-Mittag-Leffler family considered in Example 1 are derived.

EXAMPLE 3. Since for $A(t) = e^{-t^2}$; $H(t) = 2t$, the Gould-Hopper-Sheffer polynomials become the Gould-Hopper-Hermite polynomials ${}_{H^{(m)}}H_n(x,y)$ [8]. Therefore, taking $A\left(\ln\left(\frac{1+t}{1-t}\right)\right) = \exp\left(-\left(\ln\left(\frac{1+t}{1-t}\right)\right)^2\right)$ and $H\left(\ln\left(\frac{1+t}{1-t}\right)\right) = 2\ln\left(\frac{1+t}{1-t}\right)$ in equations (20)-(24), the corresponding results for the Gould-Hopper-Hermite-Mittag-Leffler polynomials ${}_{H^{(m)}H}M_n(x,y)$ are obtained. These results are given in Table 3.1

**Table 3.1. Results for ${}_{H^{(m)}H}M_n(x,y)$**

| S.No. | Results | Mathematical Expressions |
|---|---|---|
| 1. | **Generating function** | $\exp\left(2x\ln\left(\frac{1+t}{1-t}\right) + y\left(2\ln\left(\frac{1+t}{1-t}\right)\right)^m - \left(\ln\left(\frac{1+t}{1-t}\right)\right)^2\right) = \sum_{n=0}^{\infty} {}_{H^{(m)}H}M_n(x,y)\frac{t^n}{n!}$ |
| 2. | **Operational rule** | $M_n(\hat{\mathscr{M}}_{H^{(m)}H}) = {}_{H^{(m)}H}M_n(x,y)$ |
| 3. | **Multiplicative and derivative operators** | $\hat{\mathscr{M}} := \left(2x + 2myD_x^{m-1} + D_x\right)(2(\cosh(D_x/2+1))$ <br> $\hat{\mathscr{P}} := \tanh\left(\frac{D_x}{4}\right)$ |
| 4. | **Differential equation** | $\left(\left(2x + 2myD_x^{m-1} + D_x\right)\left(2(\cosh D_x/2+1)\tanh\left(\frac{D_x}{4}\right)\right) - n\right) {}_{H^{(m)}H}M_n(x,y) = 0$ |

Again, taking $A\left(\ln\left(\frac{1+t}{1-t}\right)\right)=\frac{1}{1-\ln\left(\frac{1+t}{1-t}\right)}$ and $H\left(\ln\left(\frac{1+t}{1-t}\right)\right)=\frac{-\ln\left(\frac{1+t}{1-t}\right)}{1-\ln\left(\frac{1+t}{1-t}\right)}$ in equations (20)-(24), the corresponding results for the Gould-Hopper-Laguerre-Mittag-Leffler polynomials ${}_{H^{(m)}L}M_n(x,y)$ are obtained. These results are given in Table 3.2.

**Table 3.2. Results for ${}_{H^{(m)}L}M_n(x,y)$**

| S.No. | Results | Mathematical Expressions |
|---|---|---|
| 1. | Generating function | $\frac{1}{1-\ln\left(\frac{1+t}{1-t}\right)}\exp\left(x\left(\frac{-\ln\left(\frac{1+t}{1-t}\right)}{1-\ln\left(\frac{1+t}{1-t}\right)}\right)+y\left(\frac{-\ln\left(\frac{1+t}{1-t}\right)}{1-\ln\left(\frac{1+t}{1-t}\right)}\right)^m\right)=\sum_{n=0}^{\infty}{}_{H^{(m)}L}M_n(x,y)\frac{t^n}{n!}$ |
| 2. | Operational rule | $M_n(\hat{\mathscr{M}}_{H^{(m)}L})={}_{H^{(m)}L}M_n(x,y)$ |
| 3. | Multiplicative and derivative operators | $\hat{\mathscr{M}}:=\left((x+myD_x^{m-1})(-D_x^2+2D_x-1)+D_x-1\right)\left(2(\cosh\frac{D_x}{D_x-1}+1)\right)$<br>$\hat{\mathscr{P}}:=\tanh\left(\frac{D_x}{2(D_x-1)}\right)$ |
| 4. | Differential equation | $\left(\left((x+myD_x^{m-1})(-D_x^2+2D_x-1)+D_x-1\right)\left(2(\cosh\frac{D_x}{D_x-1}+1)\tanh\left(\frac{D_x}{2(D_x-1)}\right)\right)-n\right){}_{H^{(m)}L}M_n(x,y)=0$ |

REMARK 7. Since for $m=2$, the Gould-Hopper polynomials $H_n^{(m)}(x,y)$ reduce to the 2-variable Hermite polynomials $H_n(x,y)$. Therefore taking $m=2$ in the results mentioned in Tables 3.1 and 3.2, the corresponding results for the Hermite-Hermite-Mittag-Leffler polynomials ${}_{H}HM_n(x,y)$ and Hermite-Laguerre-Mittag-Leffler polynomials ${}_{H}LM_n(x,y)$ can be obtained.

In the next example, the results for two specific members of the 2-variable truncated exponential-Sheffer-Mittag-Leffler family considered in Example 2 are derived.

EXAMPLE 4. Taking $A\left(\ln\left(\frac{1+t}{1-t}\right)\right)=\exp\left(-\left(\ln\left(\frac{1+t}{1-t}\right)\right)^2\right)$ and $H\left(\ln\left(\frac{1+t}{1-t}\right)\right)=$ $=2\ln\left(\frac{1+t}{1-t}\right)$ in equations (25)-(29), the corresponding results for the 2-variable truncated exponential-Hermite-Mittag-Leffler polynomials ${}_{e^{(r)}H}M_n(x,y)$ are obtained. These results are given in Table 3.3.

**Table 3.3 Results for ${}_{e^{(r)}H}M_n(x,y)$**

| S.No. | Results | Mathematical Expressions |
|---|---|---|
| 1. | Generating function | $\frac{1}{1-y\left(2\ln\left(\frac{1+t}{1-t}\right)\right)^r}\exp\left(2x\ln\left(\frac{1+t}{1-t}\right)-\left(\ln\left(\frac{1+t}{1-t}\right)\right)^2\right)=\sum_{n=0}^{\infty}{}_{e^{(r)}H}M_n(x,y)\frac{t^n}{n!}$ |
| 2. | Operational rule | $M_n(\hat{\mathscr{M}}_{e^{(r)}H})={}_{e^{(r)}H}M_n(x,y)$ |
| 3. | Multiplicative and derivative operators | $\hat{\mathscr{M}}:=\left(2x+2ryD_y yD_x^{r-1}+D_x\right)(2(\cosh D_x/2+1))$<br>$\hat{\mathscr{P}}:=\tanh\left(\frac{D_x}{4}\right)$ |
| 4. | Differential equation | $\left(\left(2x+2ryD_y yD_x^{r-1}+D_x\right)\left(2(\cosh D_x/2+1)\tanh\left(\frac{D_x}{4}\right)\right)-n\right){}_{e^{(r)}H}M_n(x,y)=0$ |

Again, taking $A\left(\ln\left(\frac{1+t}{1-t}\right)\right)=\frac{1}{1-\ln\left(\frac{1+t}{1-t}\right)}$ and $H\left(\ln\left(\frac{1+t}{1-t}\right)\right)=\frac{-\ln\left(\frac{1+t}{1-t}\right)}{1-\ln\left(\frac{1+t}{1-t}\right)}$ in equations (25)-(29), the corresponding results for the 2-variable truncated exponential-Laguerre-Mittag-Leffler polynomials ${}_{e^{(r)}L}M_n(x,y)$ are obtained. These results are given in Table 3.4.

**Table 3.4. Results for ${}_{e^{(r)}L}M_n(x,y)$**

| S.No. | Results | Mathematical Expressions |
|---|---|---|
| 1. | Generating function | $\frac{1}{1-\ln\left(\frac{1+t}{1-t}\right)} \frac{1}{\left(1-y\left(\frac{-\ln\left(\frac{1+t}{1-t}\right)}{1-\ln\left(\frac{1+t}{1-t}\right)}\right)\right)^r} \exp\left(x\left(\frac{-\ln\left(\frac{1+t}{1-t}\right)}{1-\ln\left(\frac{1+t}{1-t}\right)}\right)\right) = \sum_{n=0}^{\infty} {}_{e^{(r)}L}M_n(x,y)\frac{t^n}{n!}$ |
| 2. | Operational rule | $M_n(\hat{\mathscr{M}}_{e^{(r)}L}) = {}_{e^{(r)}L}M_n(x,y)$ |
| 3. | Multiplicative and derivative operators | $\hat{\mathscr{M}} := \left((x+ryD_y yD_x^{r-1})(-D_x^2+2D_x-1)+D_x-1\right)\left(2(\cosh\frac{D_x}{D_x-1}+1)\right)$<br>$\hat{\mathscr{P}} := \tanh\left(\frac{D_x}{2(D_x-1)}\right)$ |
| 4. | Differential equation | $\left(\left((x+ryD_y yD_x^{r-1})(-D_x^2+2D_x-1)+D_x-1\right)\left(2(\cosh\frac{D_x}{D_x-1}+1)\tanh\left(\frac{D_x}{2(D_x-1)}\right)\right)-n\right){}_{e^{(r)}L}M_n(x,y)=0$ |

Since, for $y=0$, the 2D-Sheffer-Mittag-Leffler polynomials ${}_fM_n(x,y)$ reduces to the Sheffer-Mittag-Leffler polynomials ${}_sM_n(x)$. Therefore, taking $y=0$ in the results given in Table 3.1 or Table 3.3, the following results for the Hermite-Mittag-Leffler polynomials ${}_HM_n(x)$ are obtained.

**Table 3.5. Results for ${}_HM_n(x)$**

| S.No. | Results | Mathematical Expressions |
|---|---|---|
| 1. | Generating function | $\exp\left(2x\ln\left(\frac{1+t}{1-t}\right)-\left(\ln\left(\frac{1+t}{1-t}\right)\right)^2\right) = \sum_{n=0}^{\infty} {}_HM_n(x)\frac{t^n}{n!}$ |
| 2. | Operational rule | $M_n(\hat{\mathscr{M}}_H) = {}_HM_n(x)$ |
| 3. | Multiplicative and derivative operators | $\hat{\mathscr{M}} := (2x+D_x)(2(\cosh D_x/2+1))$<br>$\hat{\mathscr{P}} := \tanh\left(\frac{D_x}{4}\right)$ |
| 4. | Differential equation | $\left((2x+D_x)\left(2(\cosh D_x/2+1)\tanh\left(\frac{D_x}{4}\right)\right)-n\right){}_HM_n(x)=0$ |

Similarly, taking $y=0$ in the results given in Table 3.2 or Table 3.4, the following results for the Laguerre-Mittag-Leffler polynomials ${}_LM_n(x)$ are obtained:

**Table 3.6. Results for ${}_LM_n(x)$**

| S.No. | Results | Mathematical Expressions |
|---|---|---|
| 1. | Generating function | $\frac{1}{1-\ln\left(\frac{1+t}{1-t}\right)}\exp\left(x\left(\frac{-\ln\left(\frac{1+t}{1-t}\right)}{1-\ln\left(\frac{1+t}{1-t}\right)}\right)\right) = \sum_{n=0}^{\infty} {}_LM_n(x)\frac{t^n}{n!}$ |
| 2. | Operational rule | $M_n(\hat{\mathscr{M}}_L) = {}_LM_n(x)$ |
| 3. | Multiplicative and derivative operators | $\hat{\mathscr{M}} := \left(-xD_x^2+(2x+1)D_x-(x+1)\right)\left(2(\cosh\frac{D_x}{D_x-1}+1)\right)$<br>$\hat{\mathscr{P}} := \tanh\left(\frac{D_x}{2(D_x-1)}\right)$ |
| 4. | Differential equation | $\left(\left(-xD_x^2+(2x+1)D_x-(x+1)\right)\left(2(\cosh\frac{D_x}{D_x-1}+1)\tanh\left(\frac{D_x}{2(D_x-1)}\right)\right)-n\right){}_LM_n(x)=0$ |

The graphs of the Hermite-Mittag-Leffler polynomials ${}_HM_n(x)$ and Laguerre-Mittag-Leffler polynomials ${}_LM_n(x)$ are drawn. For this, the first few values of the Hermite and Laguerre polynomials are required.

The expressions of the Hermite polynomials $H_n(x)$ and the Laguerre polynomials $L_n(x)$ for $n=0,1,2,3,4,5$ are given in the Table 3.7:

**Table 3.7. Expressions of first six $H_n(x)$ and $L_n(x)$**

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $H_n(x)$ | 1 | $2x$ | $4x^2-2$ | $8x^3-12x$ | $16x^4-48x^2+12$ | $32x^5-160x^3+120x$ |
| $L_n(x)$ | 1 | $-x+1$ | $\frac{1}{2}x^2-2x+1$ | $-\frac{1}{6}x^3+\frac{3}{2}x^2-3x+1$ | $-\frac{1}{24}x^4-\frac{2}{3}x^3+3x^2-4x+1$ | $-\frac{1}{120}x^5+\frac{5}{24}x^4-\frac{5}{3}x^3+5x^2-5x+1$ |

Next, the expressions of ${}_HM_n(x)$ and ${}_LM_n(x)$ are obtained for $n=3$ and $n=4$. Thus, replacing $x$ by the multiplicative operators $\mathscr{M}_H$ and $\mathscr{M}_L$ of the Hermite and Laguerre polynomials respectively in both sides of equations (10a) and (10b) and then using appropriate operational rules, it follows that

$$ {}_HM_3(x) = 64x^3 - 88x, \tag{30}$$

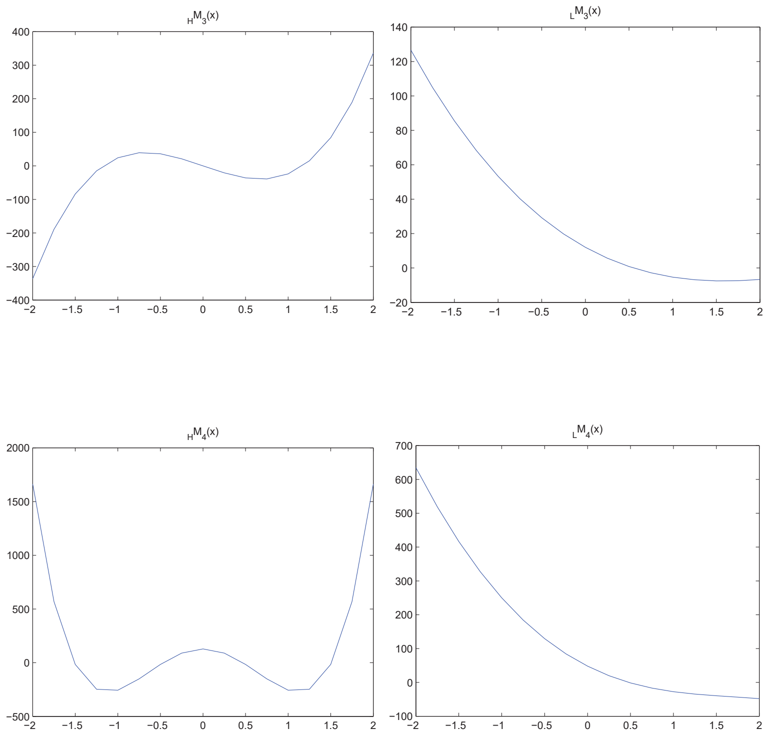
$$ {}_LM_3(x) = -\frac{4}{3}x^3 + 12x^2 - 28x + 12, \tag{31}$$

$$ {}_HM_4(x) = 256x^4 - 640x^2 + 128 \tag{32}$$

and

$$ {}_LM_4(x) = -\frac{2}{3}x^4 - \frac{32}{3}x^3 + 64x^2 - 128x + 48. \tag{33}$$

In view of equations (30)-(33) and using MATLAB the following graphs are drawn:




The above graphs indicate the behavior of the polynomials for odd and even indices.

## 4. Concluding remarks

The hybrid special polynomials of two variables are important from the point of view of applications. These polynomials allow the derivation of a number of useful identities in a fairly straight forward way and help in introducing new families of special polynomials. Most of the multi-variable special polynomials and their generalizations have been suggested by physical problems. In Sections, 2 and 3 the hybrid relatives of the Mittag-Leffler polynomials are considered. Here a general class of polynomials $h_n(x)$ is considered.

Starting from the generating function

$$\exp(xY(t)) = \sum_{n=0}^{\infty} h_n(x)\frac{t^n}{n!}, \tag{34}$$

where $Y(t)$ is the formal power series of the form:

$$Y(t) = \sum_{n=1}^{\infty} y_n \frac{t^n}{n!}, \quad y_1 \neq 0.$$

Now, replacing $x$ by the multiplicative operator $\hat{\mathscr{M}}_f$ of the 2D-Sheffer polynomials $f_n(x,y)$ in the l.h.s. of equation (34) and denoting the resultant 2D-Sheffer-general polynomials in the r.h.s. by ${}_fh_n(x,y)$, it follows that

$$\exp\left(\hat{\mathscr{M}}_f Y(t)\right) = \sum_{n=0}^{\infty} {}_fh_n(x,y)\frac{t^n}{n!},$$

which by virtue of equation (8) with $t$ replaced by $Y(t)$ and then using equation (3) in the resultant equation gives the following generating function for the 2D-Sheffer-general polynomials ${}_fh_n(x,y)$:

$$A(Y(t))\phi(y,H(Y(t)))\exp(xH(Y(t))) = \sum_{n=0}^{\infty} {}_fh_n(x,y)\frac{t^n}{n!}. \tag{35}$$

Next, consider the identity

$$\begin{aligned} &D_x \ \{A(Y(t))\phi(y,H(Y(t)))\exp(xH(Y(t)))\} \\ &= H(Y(t)) \ \ (A(Y(t))\phi(y,H(Y(t)))\exp(xH(Y(t)))), \end{aligned} \tag{36}$$

which can be expressed as:

$$\begin{aligned} &Y^{-1}\left(H^{-1}(D_x)\right) \ \{A(Y(t))\phi(y,H(Y(t)))\exp(xH(Y(t)))\} \\ &= t \ \{A(Y(t))\phi(y,H(Y(t)))\exp(xH(Y(t)))\}, \end{aligned} \tag{37}$$

where $Y^{-1}$ and $H^{-1}$ denote the compositional inverses of the functions $Y$ and $H$, respectively.

Differentiating equation (35) partially with respect to $t$, it follows that

$$\left(xH'(Y(t))+\frac{\phi'(y,H(Y(t)))}{\phi(y,H(Y(t)))}H'(Y(t))+\frac{A'(Y(t))}{A(Y(t))}\right)Y'(t)$$

$$\{A(Y(t))\phi(y,H(Y(t)))\exp(xH(Y(t)))\}=\sum_{n=0}^{\infty}{}_fh_{n+1}(x,y)\frac{t^n}{n!}.$$

In view of identity (36), the above equation takes the form

$$\left(xH'(H^{-1}(D_x))+\frac{\phi'(y,D_x)}{\phi(y,D_x)}H'(H^{-1}(D_x))+\frac{A'(H^{-1}(D_x))}{A(H^{-1}(D_x))}\right)\left(Y'(Y^{-1}(H^{-1}(D_x)))\right)$$
$$\{A(Y(t))\phi(y,H(Y(t)))\exp(xH(Y(t)))\}=\sum_{n=0}^{\infty}{}_fh_{n+1}(x,y)\frac{t^n}{n!}.$$

Now making use of generating function (35) in the l.h.s. of the above equation and then equating the coefficients of like powers of $t$ in both sides of the resultant equation, it follows that

$$\left(xH'(H^{-1}(D_x))+\frac{\phi'(y,D_x)}{\phi(y,D_x)}H'(H^{-1}(D_x))+\frac{A'(H^{-1}(D_x))}{A(H^{-1}(D_x))}\right)$$

$$\left(Y'(Y^{-1}(H^{-1}(D_x)))\right)\{{}_fh_n(x,y)\}={}_fh_{n+1}(x,y).$$

In view of the above identity, the multiplicative operator of the 2D-Sheffer-general polynomials ${}_fh_n(x,y)$ is given as:

$$\hat{\mathscr{M}}_{fh}=\left(xH'(H^{-1}(D_x))+\frac{\phi'(y,D_x)}{\phi(y,D_x)}H'(H^{-1}(D_x))+\frac{A'(H^{-1}(D_x))}{A(H^{-1}(D_x))}\right)\left(Y'(Y^{-1}(H^{-1}(D_x)))\right).$$

Further, making use of generating function (35) in both sides of identity (37) and then equating the coefficients of like powers of $t$ in both sides of resultant equation, the derivative operator of the 2D-Sheffer-general polynomials ${}_fh_n(x,y)$ is obtained as:

$$\hat{\mathscr{P}}_{fh}=Y^{-1}\left(H^{-1}(D_x)\right).$$

In view of equation (6), the following differential equation for the 2D-Sheffer-general polynomials ${}_fh_n(x,y)$ is obtained:

$$\left(\left(xH'(H^{-1}(D_x))+\frac{\phi'(y,D_x)}{\phi(y,D_x)}H'(H^{-1}(D_x))+\frac{A'(H^{-1}(D_x))}{A(H^{-1}(D_x))}\right)\right.$$

$$\left.\left(Y'(Y^{-1}(H^{-1}(D_x)))\right)Y^{-1}\left(H^{-1}(D_x)\right)-n\right){}_fh_n(x,y)=0.$$

In view of Remarks 5 and 6, the following generating functions for the Sheffer-general polynomials ${}_sh_n(x)$ and Appell-general polynomials ${}_Ah_n(x)$ are obtained:

$$A(Y(t))\exp(xH(Y(t))) = \sum_{n=0}^{\infty} {}_sh_n(x)\frac{t^n}{n!}$$

and

$$A(Y(t))\exp(xY(t)) = \sum_{n=0}^{\infty} {}_Ah_n(x)\frac{t^n}{n!},$$

respectively.

Consequently, it follows that the Sheffer-general polynomials ${}_sh_n(x)$ and Appell-general polynomials ${}_Ah_n(x)$ are quasi-monomial w.r.t. the following multiplicative and derivative operators:

$$\hat{\mathscr{M}}_{sh} = \left(xH'(H^{-1}(D_x)) + \frac{A'(H^{-1}(D_x))}{A(H^{-1}(D_x))}\right)\left(Y'(Y^{-1}(H^{-1}(D_x)))\right);$$

$$\hat{\mathscr{P}}_{sh} = Y^{-1}\left(H^{-1}\left(D_x\right)\right)$$

and

$$\hat{\mathscr{M}}_{Ah} = \left(x + \frac{A'(D_x)}{A(D_x)}\right)\left(Y'(Y^{-1}(D_x))\right);$$

$$\hat{\mathscr{P}}_{Ah} = Y^{-1}\left(D_x\right),$$

respectively.

Similarly, the following differential equations for the Sheffer-general polynomials ${}_sh_n(x)$ and Appell-general polynomials ${}_Ah_n(x)$ are obtained:

$$\left(\left(xH'(H^{-1}(D_x)) + \frac{A'(H^{-1}(D_x))}{A(H^{-1}(D_x))}\right)\left(Y'(Y^{-1}(H^{-1}(D_x)))\right)Y^{-1}\left(H^{-1}\left(D_x\right)\right) - n\right){}_sh_n(x,y) = 0$$

and

$$\left(\left(x + \frac{A'(D_x)}{A(D_x)}\right)\left(Y'(Y^{-1}(D_x))\right)Y^{-1}\left(D_x\right) - n\right){}_Ah_n(x,y) = 0.$$

The hybrid polynomials are introduced by combining two polynomial families using their quasi-monomial properties and certain operational rules. These polynomials can also be framed within the context of monomiality principle. The recurrence relations, differential equations and other results for the hybrid special polynomials can be used to solve the existing as well as new emerging problems in certain branches of science. To establish the determinant forms for the hybrid special polynomials is a recent investigation [7] which can be helpful for computation purposes. These polynomials can be studied from different points of view, for example, to establish orthogonality of these polynomials may be taken in future investigations. The study of hybrid numbers associated with the hybrid polynomials from combinatorial aspect can also be taken as future research possibility. To find generalizations of the positive linear operators involving hybrid special polynomials for applications in approximation theory may also be explored.

*Acknowledgements.* The authors are thankful to the Reviewer(s) for several useful comments and suggestions towards the improvement of this paper.

*Conflict of interest statement.* On behalf of all authors, the corresponding author states that there is no conflict of interest.

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(Received May 21, 2019)

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