Journal of Classical Analysis
Volume 15, Number 2 (2019), 123–130 doi:10.7153/jca-2019-15-12

ON THEOREMS CONNECTING MELLIN AND HANKEL TRANSFORMS

VIRENDRA KUMAR

Abstract. In the present paper four theorems connecting Mellin and Hankel transforms are established. The theorems are general in nature. As application, four integrals involving special functions are obtained. It is obvious from the examples that we can evaluate integrals involving special functions with the help of the theorems established in this paper. Otherwise it is difficult to evaluate such type of integrals.

1. Introduction

DEFINITION 1. The Mellin transform [3] of $f(x)$ is defined by the equation

$$M(f;\ s) = \int_0^\infty x^{s-1} f(x)dx,$$

where s is a complex variable.

DEFINITION 2. The Laplace transform [3] of $f(x)$ is defined by the equation

$$L(f;\ p) = \int_0^\infty e^{-px} f(x)dx,$$

where p is a complex variable.

DEFINITION 3. The Hankel transform [4] of order v of $f(x)$ is defined by the equation

$$H_v(f;\ \zeta) = \int_0^\infty (\zeta x)^{1/2} J_v(\zeta x) f(x)dx, \zeta > 0,$$

where $J_v(z)$ stands for the Bessel function of the first kind [2, p. 4, Eq. (2)].

DEFINITION 4. Another form of the Hankel transform [4] of order v of $f(x)$ is defined by the equation

$$H_v(f;\ \zeta) = \int_0^\infty x J_v(\zeta x) f(x)dx, \zeta > 0, \tag{1}$$

Mathematics subject classification (2010): 44A05.
Keywords and phrases: Mellin transform, Hankel transform, Laplace transform, Bessel function, Struve's function.

Paper JCA-15-12

2. Main theorems

THEOREM 1. *If* $\zeta > 0$, $Re(s+v) > 0$ *and* $Re(\rho) > 0$, *then*

$$M\{e^{-\rho x}f(x);\ s\} = \int_0^\infty K(s,\ \zeta)H_v(f;\ \zeta)d\zeta, \tag{2}$$

where

$$K(s,\ \zeta) = \frac{\zeta^{v+1}\Gamma(s+v)}{2^v\rho^{v+s}\Gamma(1+v)}\ {}_2F_1\left(\frac{s+v}{2}, \frac{s+v+1}{2}; v+1;\ -\frac{\zeta^2}{\rho^2}\right).$$

Proof. We have by the Hankel inversion theorem [10] that

$$f(x) = \int_0^\infty \zeta\ H_v(f;\ \zeta)J_v(\zeta x)d\zeta.$$

Hence

$$M\{e^{-\rho x}f(x);\ s\} = \int_0^\infty \zeta H_v(f;\ \zeta)M\{e^{-\rho x}J_v(\zeta x);\ s\}d\zeta. \tag{3}$$

The change of order of integration is justified because $\mathrm{Re}(\rho) > 0$ and $J_v(\zeta x)$ is a bounded function for both the variables for Landau's bounds [5] (see also [6]) i.e

$$|J_v(x)| \leqslant b_L v^{-1/3}, \qquad b_L := 2^{1/3}\sup_{x\in R_+}(\mathrm{Ai}(x)) \tag{4}$$

and

$$|J_v(x)| \leqslant c_L|x|^{-1/3}, \qquad c_L := \sup_{x\in R_+} x^{1/3}(J_0(x)), \tag{5}$$

where $\mathrm{Ai}(x)$ stands for the familiar Airy function.
Now, using the following result [3, p. 327, Eq. (6)] in (3)

$$M\{e^{-ax}J_v(\beta x);\ s\} = \frac{\beta^v\Gamma(s+v)}{2^v a^{v+s}\Gamma(1+v)}\ {}_2F_1\left(\frac{s+v}{2}, \frac{s+v+1}{2}; v+1;\ -\frac{\beta^2}{a^2}\right),$$

provided that $\mathrm{Re}(a) > |Im\ \beta|$ and $\mathrm{Re}(s+v) > 0$ we arrive at the desired result (2), where $\zeta > 0$, $\mathrm{Re}(s+v) > 0$ and $\mathrm{Re}(\rho) > 0$.

THEOREM 2. *If* $0 < Re(s) < Re(v) + 3/2$ *and* $\zeta > 0$, *then*

$$M[(x^2+\beta^2)^{-v/2-1/4}f\{(x^2+\beta^2)^{1/2}\};\ s] = \int_0^\infty K(s,\ \zeta)H_v(f;\ \zeta)d\zeta, \tag{6}$$

where

$$K(s,\ \zeta) = \zeta^{1/2-s/2}\ 2^{s/2-1}\ \beta^{s/2-v}\ \Gamma(s/2)\ J_{v-s/2}(\zeta\beta).$$

Proof. We have by the Hankel inversion theorem [10] that

$$f(x) = \int_0^\infty \sqrt{\zeta x}\ H_v(f;\ \zeta)J_v(\zeta x)d\zeta. \tag{7}$$

Hence

$$M[(x^2+\beta^2)^{-\nu/2-1/4}f\{(x^2+\beta^2)^{1/2}\};\ s] = \int_0^\infty \sqrt{\zeta}H_\nu(f;\ \zeta)M[(x^2+\beta^2)^{-\nu/2} \times J_\nu\{\zeta(x^2+\beta^2)^{1/2}\};\ s]d\zeta. \tag{8}$$

The change of order of integration is justified because $\zeta > 0, 0 < \mathrm{Re}\,(s) < \mathrm{Re}\,(\nu) + 3/2$ and $J_\nu\{\zeta(x^2+\beta^2)^{1/2}\}$ is a bounded function for both the variables for Landau's bounds [5, 6], (see (4) and (5)).
Now, using the following result [3, p. 328, Eq. (12] in (8)

$$M[(x^2+\beta^2)^{-\nu/2}J_\nu\{a(x^2+\beta^2)^{1/2};\ s] = 2^{s/2-1}a^{-s/2}\beta^{s/2-\nu}\Gamma(s/2)J_{\nu-s/2}(a\beta), \tag{9}$$

provided that $a > 0$ and $0 < \mathrm{Re}(s) < \mathrm{Re}(\nu) + 3/2$ we arrive at the desired result (6), where $\zeta > 0$ and $0 < \mathrm{Re}(s) < \mathrm{Re}(\nu) + 3/2$.

THEOREM 3. *If Re* $(\nu) > -1, 0 < x < a$ *and Re* $(s) > 0$, *then*

$$M[(a^2-x^2)^{\nu/2-1/4}f\{(a^2-x^2)^{1/2}\};\ s] = \int_0^\infty K(s,\ \zeta)H_\nu(f;\ \zeta)d\zeta, \tag{10}$$

where

$$K(s,\ \zeta) = \zeta^{1/2-s/2}2^{s/2-1}a^{\nu+s/2}\,\Gamma(s/2)\,J_{\nu+s/2}(a\zeta).$$

Proof. Again, by (7) we have that

$$M[(a^2-x^2)^{\nu/2-1/4}f\{(a^2-x^2)^{1/2}\};\ s] = \int_0^\infty \sqrt{\zeta}H_\nu(f;\ \zeta)M[(a^2-x^2)^{\nu/2}J_\nu\{\zeta(a^2-x^2)^{1/2}\};\ s]d\zeta. \tag{11}$$

The change of order of integration is justified because $\mathrm{Re}\,(\nu) > -1, 0 < x < a$, $\mathrm{Re}\,(s) > 0$ and $J_\nu\{\zeta(a^2-x^2)^{1/2}\}$ is a bounded function for both the variables for Landau's bounds [5, 6], (see (4) and (5)).
Now, using the following result [3, p. 329, Eq. (13)] in (11)

$$M[(a^2-x^2)^{\nu/2}J_\nu\{\beta(a^2-x^2)^{1/2};\ s] = 2^{s/2-1}a^{\nu+s/2}\beta^{-s/2}\Gamma(s/2)J_{\nu+s/2}(a\beta), \tag{12}$$

provided that $\mathrm{Re}\,(\nu) > -1, 0 < x < a$ and $\mathrm{Re}\,(s) > 0$ we arrive at the desired result (10), where $\mathrm{Re}\,(\nu) > -1, 0 < x < a$ and $\mathrm{Re}\,(s) > 0$.

THEOREM 4. *If* $0 < x < a$, *Re* $(s) > 0$ *and* $\zeta > 0$, *then*

$$M[(a^2-x^2)^{-\nu/2-1/4}f\{(a^2-x^2)^{1/2}\};\ s] = \int_0^\infty K(s,\ \zeta)H_\nu(f;\ \zeta)d\zeta, \tag{13}$$

where

$$K(s,\ \zeta) = \zeta^{1/2-s/2}2^{1-\nu}a^{s/2-\nu}\,[\Gamma(\nu)]^{-1}\,S_{\nu+s/2-1,\ s/2-\nu}(a\zeta).$$

Proof. Again, by (7) we have that

$$\begin{aligned} &M[(a^2-x^2)^{-\nu/2-1/4}f\{(a^2-x^2)^{1/2}\};\, s] \\ &= \int_0^\infty \sqrt{\zeta}H_\nu(f;\,\zeta)M[(a^2-x^2)^{-\nu/2}J_\nu\{\zeta(a^2-x^2)^{1/2}\};\, s]d\zeta. \end{aligned} \tag{14}$$

The change of order of integration is justified because $0<x<a$, $\mathrm{Re}\,(s)>0, \zeta>0$ and $J_\nu\{\zeta(a^2-x^2)^{1/2}\}$ is a bounded function for both the variables for Landau's bounds [5, 6], (see (4) and (5)).
Now, using the following result [3, p. 329, Eq. (14)] in (14)

$$M[(a^2-x^2)^{-\nu/2}J_\nu\{\beta(a^2-x^2)^{1/2};\, s] = 2^{1-\nu}a^{s/2-\nu}\beta^{-s/2}[\Gamma(\nu)]^{-1}S_{\nu+s/2-1,\, s/2-\nu}(a\beta), \tag{15}$$

where $S_{\mu,\,\nu}(z)$ stands for the Lommel's function [2, p. 40, Eq. (71)], $0<x<a$ and $\mathrm{Re}\,(s)>0$ we arrive at the desired result (13), where $0<x<a$, $\mathrm{Re}\,(s)>0$ and $\zeta>0$.

3. Applications

EXAMPLE 1. Let $f(x)=x^\eta e^{-\xi x}$.
Then

$$M\{e^{-\rho x}f(x);\, s\} = \int_0^\infty e^{-(\rho+\xi)x}x^{(s+\eta)-1}dx. \tag{16}$$

Using the following result [8, p. 322, Eq, 2.3.3.1] in (16)

$$\int_0^\infty e^{-cx}x^{\lambda-1}dx = \frac{\Gamma(\lambda)}{c^\lambda},\ \mathrm{Re}(\lambda)>0,\ \mathrm{Re}(c)>0,$$

we get

$$M\{e^{-\rho x}f(x);\, s\} = \frac{\Gamma(s+\eta)}{(\rho+\xi)^{s+\eta}},\ \mathrm{Re}(s+\eta)>0,\ \mathrm{Re}(\rho+\xi)>0. \tag{17}$$

Now, from (1) we have

$$H_\nu(f;\,\zeta) = L\{x^{\eta+1}J_\nu(\zeta x);\, \xi\}, \tag{18}$$

where L stands for the Laplace transform.
Using the following result [3, p. 182, Eq. (9)] in (18)

$$L\{t^\mu J_\nu(at);\, p\} = \Gamma(\mu+\nu+1)(p^2+a^2)^{(-\mu-1)/2}P_\mu^{-\nu}\left(\frac{p}{\sqrt{p^2+a^2}}\right),$$

where $P_\mu^{-\nu}(z)$ stands for the Legendre function of the first kind [1, p. 143, Eq. (6)], $\mathrm{Re}\,(p)>|Im\ a|$ and $\mathrm{Re}\,(\mu+\nu)>-1$ we get

$$H_\nu(f;\,\zeta) = \Gamma(\eta+\nu+2)(\xi^2+\zeta^2)^{-(\eta+2)/2}P_{\eta+1}^{-\nu}\left(\frac{\xi}{\sqrt{\xi^2+\zeta^2}}\right), \tag{19}$$

where $\zeta > 0,\ \mathrm{Re}(\xi) > 0$ and $\mathrm{Re}(\eta + \nu) > -2$.
Now, using the results (17) and (19) in (2), we get

$$\begin{aligned}\int_0^\infty \zeta^{\nu+1}{}_2F_1\left(\frac{s+\nu}{2}, \frac{s+\nu+1}{2}; \nu+1; -\frac{\zeta^2}{\rho^2}\right)(\xi^2+\zeta^2)^{-(\eta+2)/2} \\ \times P_{\eta+1}^{-\nu}\left(\frac{\xi}{\sqrt{\xi^2+\zeta^2}}\right)d\zeta \\ = \frac{2^\nu \rho^{\nu+s}\Gamma(\nu+1)\Gamma(s+\eta)}{(\rho+\xi)^{s+\eta}\Gamma(s+\nu)\Gamma(\eta+\nu+2)},\end{aligned} \tag{20}$$

where $\zeta > 0,\ \mathrm{Re}(\xi) > 0,\ \mathrm{Re}(s+\eta) > 0$ and $\mathrm{Re}(\nu+1) > 0$.

EXAMPLE 2. Let $f(x) = x^{\nu-\mu+1/2}J_\mu(ax),\ a > 0,\ \ -1 < \mathrm{Re}(\nu) < \mathrm{Re}(\mu)$.
Then

$$f\{(x^2+\beta^2)^{1/2}\} = (x^2+\beta^2)^{\nu/2-\mu/2+1/4}J_\mu\{a(x^2+\beta^2)^{1/2}\}$$

and

$$M[(x^2+\beta^2)^{-\nu/2-1/4}f\{(x^2+\beta^2)^{1/2}\};\ s] = M[(x^2+\beta^2)^{-\mu/2}J_\mu\{a(x^2+\beta^2)^{1/2}\};\ s] \tag{21}$$

Using the result (9) in (21), we get

$$M[(x^2+\beta^2)^{-\nu/2-1/4}f\{(x^2+\beta^2)^{1/2}\};\ s] = 2^{s/2-1}a^{-s/2}\beta^{s/2-\mu}\Gamma(s/2)J_{\mu-s/2}(a\beta), \tag{22}$$

where $a > 0$ and $0 < \mathrm{Re}(s) < \mathrm{Re}(\mu) + 3/2$.

$$H_\nu(f;\ \zeta) = H_\nu\{x^{\nu-\mu+1/2}J_\mu(ax);\ \zeta\}. \tag{23}$$

Using the following result [4, p. 48, Eq. (7)] in (23)

$$H_\nu\{x^{\nu-\mu+1/2}J_\mu(ax);\ y\} = \frac{2^{\nu-\mu+1}y^{\nu+1/2}}{\Gamma(\mu-\nu)a^\mu}(a^2-y^2)^{\mu-\nu-1}, \tag{24}$$

provided that $a > 0,\ -1 < \mathrm{Re}(\nu) < \mathrm{Re}(\mu)$ and $0 < y < a$ we get

$$H_\nu(f;\ \zeta) = \frac{2^{\nu-\mu+1}\zeta^{\nu+1/2}}{\Gamma(\mu-\nu)a^\mu}(a^2-\zeta^2)^{\mu-\nu-1}, \tag{25}$$

where $a > 0,\ -1 < \mathrm{Re}(\nu) < \mathrm{Re}(\mu)$ and $0 < \zeta < a$.
Now, using the results (22) and (25) in (6), we get

$$\begin{aligned}&\int_0^\infty \zeta^{\nu-s/2+1}(a^2-\zeta^2)^{\mu-\nu-1}J_{\nu-s/2}(\zeta\beta)d\zeta \\ &= 2^{\mu-\nu-1}a^{\mu-s/2}\beta^{\nu-\mu}\Gamma(\mu-\nu)J_{\mu-s/2}(a\beta),\end{aligned} \tag{26}$$

where $a > 0,\ -1 < \mathrm{Re}(\nu) < \mathrm{Re}(\mu),\ 0 < \zeta < a$ and $\mathrm{Re}(\mu-\nu) > 0$.

The formula (26) extends the formula collection given in [9], because there [9, p. 178, Eq. (17)] exists an integral representation for

$$\int_a^{\infty} x^{1\pm\nu}(x^2-a^2)^{\beta-1}J_\nu(cx)dx,$$

valid when $a>0$.

EXAMPLE 3. Let $f(x)=x^{\mu-\nu+1/2}J_\mu(bx), b>0, \mathrm{Re}(\nu)>\mathrm{Re}(\mu)>-1$. Then

$$f\{(a^2-x^2)^{1/2}\}=(a^2-x^2)^{\mu/2-\nu/2+1/4}J_\mu\{b(a^2-x^2)^{1/2}\}$$

and

$$M[(a^2-x^2)^{\nu/2-1/4}f\{(a^2-x^2)^{1/2}\};\, s]=M[(a^2-x^2)^{\mu/2}J_\mu\{b(a^2-x^2)^{1/2}\};\, s]. \quad (27)$$

Using the result (12) in (27), we get

$$M[(a^2-x^2)^{\nu/2-1/4}f\{(a^2-x^2)^{1/2}\};\, s]=2^{s/2-1}a^{\mu+s/2}b^{-s/2}\Gamma(s/2)J_{\mu+s/2}(ab), \quad (28)$$

where $0<x<a,\ b>0,\ \mathrm{Re}(\mu)>-1$ and $\mathrm{Re}(s)>0$.

$$H_\nu(f;\,\zeta)=H_\nu\{x^{\mu-\nu+1/2}J_\mu(bx);\,\zeta\}. \quad (29)$$

Using the following result [4, p. 48, Eq. (8)] in (29)

$$H_\nu\{x^{\mu-\nu+1/2}J_\mu(ax);\,y\}=\frac{2^{\mu-\nu+1}a^\mu}{\Gamma(\nu-\mu)y^{\nu-1/2}}(y^2-a^2)^{\nu-\mu-1},$$

provided that $a>0,\ \mathrm{Re}(\nu)>\mathrm{Re}(\mu)>-1$ and $a<y<\infty$ we get

$$H_\nu(f;\,\zeta)=\frac{2^{\mu-\nu+1}b^\mu}{\Gamma(\nu-\mu)\zeta^{\nu-1/2}}(\zeta^2-b^2)^{\nu-\mu-1}, \quad (30)$$

where $\mathrm{Re}(\nu)>\mathrm{Re}(\mu)>-1$ and $0<b<\zeta<\infty$.
Now, using the results (28) and (30) in (10), we get

$$\begin{aligned}&\int_0^{\infty}\zeta^{1-\nu-s/2}(\zeta^2-b^2)^{\nu-\mu-1}J_{\nu+s/2}(a\zeta)d\zeta\\ &=2^{\nu-\mu-1}a^{\mu-\nu}b^{-\mu-s/2}\Gamma(\nu-\mu)J_{\mu+s/2}(ab),\end{aligned} \quad (31)$$

where $a>0,\ 0<b<\zeta<\infty$ and $\mathrm{Re}(\nu-\mu)>0$.

EXAMPLE 4. Let $f(x)=x^{\nu-\mu+1/2}J_\mu(bx),\ b>0, -1<\mathrm{Re}(\nu)<\mathrm{Re}(\mu)$. Then

$$f\{(a^2-x^2)^{1/2}\}=(a^2-x^2)^{\nu/2-\mu/2+1/4}J_\mu\{b(a^2-x^2)^{1/2}\}$$

and

$$M[(a^2-x^2)^{-\nu/2-1/4}f\{(a^2-x^2)^{1/2}\};\, s]=M[(a^2-x^2)^{-\mu/2}J_\mu\{b(a^2-x^2)^{1/2}\};\, s]. \quad (32)$$

Using the result (15) in (32), we get

$$
\begin{aligned}
&M[(a^2-x^2)^{-\nu/2-1/4} f\{(a^2-x^2)^{1/2}\};\, s] \\
&= 2^{1-\mu} a^{s/2-\mu} b^{-s/2} [\Gamma(\mu)]^{-1} S_{\mu+s/2-1,\, s/2-\mu}(ab),
\end{aligned}
\tag{33}
$$

where $0 < x < a,\ b > 0$ and $\mathrm{Re}\,(s) > 0$.

$$
H_\nu(f;\, \zeta) = H_\nu\{x^{\nu-\mu+1/2} J_\mu(bx);\, \zeta\}. \tag{34}
$$

Using the result (24) in (34), we get the result which is obtained by replacing a by b in (25).
Now, using the results (25) and (33) in (13), we get

$$
\begin{aligned}
&\int_0^\infty \zeta^{\nu-s/2+1}(b^2-\zeta^2)^{\mu-\nu-1} S_{\nu+s/2-1,\, s/2-\nu}(a\zeta)\, d\zeta \\
&= 2^{-1} a^{\nu-\mu} b^{\mu-s/2} [\Gamma(\mu)]^{-1} \Gamma(\nu)\Gamma(\mu-\nu) S_{\mu+s/2-1,\, s/2-\mu}(ab),
\end{aligned}
\tag{35}
$$

where $a > 0,\ \mathrm{Re}\,(\nu) > 0,\ 0 < \zeta < b$ and $\mathrm{Re}\,(\mu-\nu) > 0$.

In this section, four integral formulae (20), (26), (31) and (35) involving special functions have been obtained. Several other integral formulae extending the results given in [8, 9] may be obtained with the help of the theorems established in this paper and Mellin transforms available in [7].

Acknowledgement. The author is extremely thankful to the worthy referee for the valuable suggestions given for the improvement of the present paper.

REFERENCES

[1] A. ERDÉLYI, W. MAGNUS, F. OBERHETTINGER AND F. G. TRICOMI, *Higher Transcendental Functions, Vol. I*, McGraw-Hill, New York, Toronto and London 1953.

[2] A. ERDÉLYI, W. MAGNUS, F. OBERHETTINGER AND F. G. TRICOMI, *Higher Transcendental Functions, Vol. II*, McGraw-Hill, New York, Toronto and London 1953.

[3] A. ERDÉLYI, W. MAGNUS, F. OBERHETTINGER AND F. G. TRICOMI, *Tables of Integral Transforms, Vol. I*, McGraw-Hill, New York, Toronto and London 1954.

[4] A. ERDÉLYI, W. MAGNUS, F. OBERHETTINGER AND F. G. TRICOMI, *Tables of Integral Transforms, Vol. II*, McGraw-Hill, New York, Toronto and London 1954.

[5] L. LANDAU, *Monotonicity and bounds for Bessel functions*, Proceedings of the Symposium on Mathematical Physics and Quantum Field Theory (Berkeley, California: June 11-13, 1999)(Warchall. H, Editor), 147–154(2000); Electron. J. Differential Equations, Southwest Texas State University, San Marcos, Texas, 4 (2002).

[6] L. J. LANDAU, *Bessel functions: Monotonicity and bounds*, Journal of the London Mathematical Society, **61**, 1 (2000), 197–215.

[7] F. OBERHETTINGER, *Tables of Mellin Transforms*, Springer-Verlag, Berlin, Germany, 1974.

[8] A. P. PRUDNIKOV, YU. A. BRYCHKOV AND O. I. MARICHEV, *Integrals and series. Volume 2. Elementary functions*, Gordon and Breach science publishers, New York, 1986.

[9] A. P. PRUDNIKOV, YU. A. BRYCHKOV AND O. I. MARICHEV, *Integrals and series. Volume 2. Special functions*, Gordon and Breach science publishers, New York, 1986.
[10] I. N. SNEDDON, *Fourier Transforms*, McGraw-Hill, New York, 1951.

(Received November 18, 2019)

Virendra Kumar
Formerly Scientist-B
Defence Research and Development Organization
New Delhi, India
Address for correspondence:
D-436, Shastri Nagar, Ghaziabad-201002, India
e-mail: vkumar10147@yahoo.com